First-Order Gödel Logics

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Abstract

First-order Gödel logics are a family of infinite-valued logics where the sets of truth values V are closed subsets of $[0,1]$ containing both 0 and 1. Different such sets V in general determine different Gödel logics $\mathbf{G}_V$ (sets of those formulas which evaluate to 1 in every interpretation into V). It is shown that $\mathbf{G}_V$ is axiomatizable iff V is finite, V is uncountable with 0 isolated in V, or every neighborhood of 0 in V is uncountable. Complete axiomatizations for each of these cases are given. The r.e. prenex, negation-free, and existential fragments of all first-order Gödel logics are also characterized.

Contents

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1 Introduction

1.1 Motivation

The logics we investigate in this paper, first-order Gödel logics, can be characterized in a rough-and-ready way as follows: The language is a standard first-order language. The logics are many-valued, and the sets of truth values considered are closed subsets of $[0,1]$ which contain both 0 and 1. 1 is the "designated value," i.e., a formula is valid if it receives the value 1 in every interpretation. The truth functions of conjunction and disjunction are minimum and maximum, respectively, and quantifiers are defined by infimum and supremum over subsets of the set of truth values. The characteristic operator of Gödel logics, the Gödel conditional, is defined by $a \to b = 1$ if $a \leq b$ and $= b$ if $a > b$. Because the truth values are ordered (indeed, in many cases, densely ordered), the semantics of Gödel logics is suitable for formalization of *comparisons*. It is related in this respect to a more widely known many-valued logic, Łukasiewicz (or "fuzzy") logic—yet the truth function of the Łukasiewicz conditional is defined not just using comparison, but also addition. In contrast to Łukasiewicz logic, which might be considered a logic of *absolute* or *metric comparison*, Gödel logics are logics of *relative comparison*. This alone makes Gödel logics an interesting subject for logical investigations.

There are other reasons why the study of Gödel logics is important. As noted, Gödel logics are related to other many-valued logics of recognized importance. Indeed, Gödel logic is one of the three basic t-norm based logics which have received increasing attention in the last 15 or so years [Háj98] (the others are Łukasiewicz and product logic). Yet Gödel logic is also closely related to intuitionistic logic: it is the logic of linearly-ordered Heyting algebras. In the propositional case, infinite-valued Gödel logic can be axiomatized by the intuitionistic propositional calculus extended by the axiom schema $(A \to B) \vee (B \to A)$. This connection extends also to Kripke semantics for intuitionistic logic: Gödel logics can also be characterized as logics of (classes of) linearly ordered and countable intuitionistic Kripke structures with constant domains [BP].

One of the surprising facts about Gödel logics is that whereas there is only one infinite-valued propositional Gödel logic, there are infinitely many different infinite-valued first-order Gödel logics depending on the choice of the set of truth values. This is also the case when one considers the propositional consequence relation, and likewise when the language is extended to include quantification over propositions. For both quantified propositional and first-order Gödel logics, different sets of truth values with different order-theoretic properties result in different sets of valid formulas. Hence it is necessary to consider truth value sets other than the standard unit interval.

In the light of the result of Scarpellini [Sca62] on non-axiomatizability of infinite-valued first-order Łukasiewicz logic which can be extended to almost all linearly ordered infinite-valued logics, it is also surprising that some infinite-valued Gödel logics are recursively enumerable. Our main aim in this paper is to characterize those sets of truth values which give rise to axiomatizable Gödel logics, and those whose sets of validities are not r.e. We show that a set V of truth values determines an axiomatizable first-order Gödel logic if, and only if, V is finite, V is uncountable and 0 is isolated, or every neighborhood of 0 in V is uncountable. These cases also determine different sets of validities: the finite-valued Gödel logics $\mathbf{G}_n$, the logic $\mathbf{G}^0$, and the "standard" infinite-valued Gödel logic $\mathbf{G}_\mathbb{R}$ (based on the truth value set $[0,1]$).

1.2 History of Gödel logics

Gödel logics are one of the oldest families of many-valued logics. Propositional finite-valued Gödel logics were introduced by Gödel in [Göd33] to show that intuitionistic logic does not have a characteristic finite matrix. They provide the first examples of intermediate logics (intermediate, that is, in strength between classical and intuitionistic logics). Dummett [Dum59] was the first to study infinite valued propositional Gödel logics, axiomatizing the set of tautologies over infinite truth-value sets by intuitionistic logic extended by the linearity axiom $(A \to B) \vee (B \to A)$. Hence, infinite-valued propositional Gödel logic is also sometimes called Gödel-Dummett logic or Dummett's LC. In terms of Kripke semantics, the characteristic linearity axiom picks out those accessibility relations which are linear orders.

Standard first-order Gödel logic $\mathbf{G}_\mathbb{R}$—the one based on the full interval $[0,1]$—has been discovered and studied by several people independently. Alfred Horn was probably the first: He discussed this logic under the name *logic with truth values in a linearly ordered Heyting algebra* [Hor69], and gave an axiomatization and the first completeness proof. Takeuti and Titani [TT84] called $\mathbf{G}_\mathbb{R}$ *intuitionistic fuzzy logic*, and also gave an axiomatization for which they proved the completeness. This system incorporates the density rule

$$\frac{\Gamma \vdash A \vee (C \to p) \vee (p \to B)}{\Gamma \vdash A \vee (C \to B)}$$

(where p is any propositional variable not occurring in the lower sequent.) The rule is redundant for an axiomatization of $\mathbf{G}_{\mathbb{R}}$, as was shown by Takano [Tak87], who gave a streamlined completeness proof of Takeuti-Titani's system without the rule. (A syntactical proof of the elimination of the density rule was later given in [BZ00]. Other proof-theoretic investigations of Gödel logics can be found in [BC02] and [BFC03].) The density rule is nevertheless interesting: It forces the truth value set to be dense in itself (in the sense that, if the truth value set isn't dense in itself, the rule does not preserve validity). This contrasts with the expressive power of formulas: no formula is valid only for truth value sets which are dense in themselves.

First-order Gödel logics other than $\mathbf{G}_{\mathbb{R}}$ were first considered in [BLZ96b], where it was shown that $\mathbf{G}_{\downarrow}$, based on the truth value set $V_{\downarrow} = \{1/k : k \in \mathbb{N}\} \cup \{0\}$ is not r.e. Hájek [Háj05] has recently improved this result, and showed that not only is the set of validities not r.e., it is not even arithmetical. Hájek also showed that the Gödel logic $\mathbf{G}_{\uparrow}$ based on $V_{\uparrow} = \{1 - 1/k : k \in \mathbb{N}\} \cup \{1\}$ is Π_2-complete. Results preliminary to the results of the present paper were reported in [BPZ03, Pre02, Pre03].

1.3 Overview of the results

We begin with a preliminary discussion of the syntax and semantics of Gödel logics, including a discussion of some of the more interesting special cases of first-order Gödel logics and their relationships (Section 2). In Section 3, we present some relevant results regarding the topology of truth-value sets.

The main results of the paper are contained in Sections 4–6. We provide a complete classification of the axiomatizability of first order Gödel logics. The main results are, that a logic based on a truth value set V is axiomatizable if and only if

1. V is finite (Section 6), or
2. V is uncountable and 0 is contained in the perfect kernel (Section 5.1), or
3. V is uncountable and 0 is isolated (Section 5.2).

In all other cases, i.e., logics with countable truth value set (Section 4) and those where there is a countable neighborhood of 0 and 0 is not isolated (Section 5.3), the respective logics are not r.e.

In Section 7, we investigate the complexity of fragments of first-order Gödel logic, specifically, the prenex fragments (Section 7.1), the $\perp$-free fragments (Section 7.2), and the existential ($\forall$-free) fragments (Section 7.3). We show that the prenex fragment of a Gödel logic is axiomatizable if and only if the truth value set is finite or uncountable. This means that there are truth-value sets where the prenex fragment of the corresponding logic is r.e. even though the full logic is not. Moreover, there all axiomatizable prenex fragments coincide. This is also the case for $\perp$-free and existential fragments, but in these cases only those truth value sets determine r.e. $\perp$-free and existential fragments for which also the full logic is r.e., viz., truth value sets which are finite, uncountable with 0 isolated, and those where every neighborhood of 0 is uncountable.

2 Preliminaries

2.1 Syntax and Semantics

In the following we fix a standard first-order language $\mathscr{L}$ with finitely or countably many predicate symbols P and finitely or countably many function symbols f for every finite arity k. In addition to the two quantifiers $\forall$ and $\exists$ we use the connectives $\vee$, $\wedge$, $\rightarrow$ and the constant $\perp$ (for 'false'); other connectives are introduced as abbreviations, in particular we let $\neg A \equiv (A \rightarrow \perp)$.

Gödel logics are usually defined using the single truth value set $[0,1]$. For propositional logic the choice of any infinite subset of $[0,1]$ leads to the same propositional logic (set of tautologies). In the first order case, where quantifiers will be interpreted as infima and suprema, a closed subset of $[0,1]$ is necessary.

Definition 1 (Gödel set)**.** A *Gödel set* is a closed set $V \subseteq [0,1]$ which contains 0 and 1.

The semantics of Gödel logics, with respect to a fixed Gödel set as truth value set and a fixed language $\mathscr{L}$ of predicate logic, is defined using the extended language $\mathscr{L}^U$, where U is the universe of the interpretation $\mathfrak{I}$. $\mathscr{L}^U$ is $\mathscr{L}$ extended with constant symbols for each element of U.

Definition 2 (Semantics of Gödel logic)**.** Fix a Gödel set V. An *interpretation* $\mathfrak{I}$ into V consists of

1. a nonempty set $U = U^{\mathfrak{I}}$, the 'universe' of $\mathfrak{I}$,
2. for each k-ary predicate symbol P, a function $P^{\mathfrak{I}} : U^k \rightarrow V$,
3. for each k-ary function symbol f, a function $f^{\mathfrak{I}} : U^k \rightarrow U$.
4. for each variable v, a value $v^{\mathfrak{I}} \in U$.

Given an interpretation $\mathfrak{I}$, we can naturally define a value $t^{\mathfrak{I}}$ for any term t and a truth value $\mathfrak{I}(A)$ for any formula A of $\mathscr{L}^U$. For a terms $t = f(u_1,\dots,u_k)$ we define $\mathfrak{I}(t) = f^{\mathfrak{I}}(u_1^{\mathfrak{I}},\dots,u_k^{\mathfrak{I}})$. For atomic formulas $A \equiv P(t_1,\dots,t_n)$, we define $\mathfrak{I}(A) = P^{\mathfrak{I}}(t_1^{\mathfrak{I}},\dots,t_n^{\mathfrak{I}})$. For composite formulas A we define $\mathfrak{I}(A)$ by:

$$\mathfrak{I}(\perp) = 0 \tag{1}$$
$$\mathfrak{I}(A \wedge B) = \min(\mathfrak{I}(A), \mathfrak{I}(B)) \tag{2}$$
$$\mathfrak{I}(A \vee B) = \max(\mathfrak{I}(A), \mathfrak{I}(B)) \tag{3}$$
$$\mathfrak{I}(A \rightarrow B) = \begin{cases} 1 & \mathfrak{I}(A) \leq \mathfrak{I}(B) \\ \mathfrak{I}(B) & \text{otherwise} \end{cases} \tag{4}$$
$$\mathfrak{I}(\forall x A(x)) = \inf\{\mathfrak{I}(A(u)) : u \in U\} \tag{5}$$
$$\mathfrak{I}(\exists x A(x)) = \sup\{\mathfrak{I}(A(u)) : u \in U\} \tag{6}$$

(Here we use the fact that every Gödel sets V is a *closed* subset of $[0,1]$ in order to be able to interpret $\forall$ and $\exists$ as inf and sup in V.)

If $\mathfrak{I}(A) = 1$, we say that $\mathfrak{I}$ *satisfies* A, and write $\mathfrak{I} \models A$.

Definition 3 (Gödel logics based on V)**.** For a Gödel set V we define the *first order Gödel logic* $\mathbf{G}_V$ as the set of all formulas of $\mathscr{L}$ such that $\mathfrak{I} \models A$ for all V-interpretations $\mathfrak{I}$.

It should be noted that for Gödel logics with 0 isolated, the notion of *satisfiability* for sets of formulas is not particularly interesting, since a set of formulas Γ is satisfiable (in the sense that there is an $\mathfrak{I}$ so that $\mathfrak{I} \models A$ for all $A \in \Gamma$) iff it is satisfiable classically. For this reason, we take *entailment* to be the fundamental model-theoretic notion.

Definition 4. If Γ is a set of formulas (possibly infinite), we say that Γ entails A in $\mathbf{G}_V$, $\Gamma \models_V A$ iff for all $\mathfrak{I}$ into V,

$$\inf\{\mathfrak{I}(B) : B \in \Gamma\} \leq \mathfrak{I}(A);$$

and Γ 1-*entails* A in $\mathbf{G}_V$, $\Gamma \Vdash_V A$, iff, for all $\mathfrak{I}$ into V, whenever $\mathfrak{I}(B) = 1$ for all $B \in \Gamma$, then $\mathfrak{I}(A) = 1$.

Notation 5. We will write $\Gamma \models A$ instead of $\Gamma \models_V A$ in case it is obvious which truth value set V is meant. We will sometimes write $\Gamma \models \Delta \in \mathbf{G}_V$, by which we mean that $\Gamma \models_V \Delta$. The notation $\mathbf{G}_V \models A$ stands for $\emptyset \models_V A$, or $A \in \mathbf{G}_V$.

Whether or not a formula A evaluates to 1 under an interpretation $\mathfrak{I}$ depends only on the *relative ordering* of the truth values of the atomic formulas (in $\mathscr{L}^{\mathfrak{I}}$), and not directly on the set V or on the *values* of the atomic formulas. If $V \subseteq W$ are both Gödel sets, and $\mathfrak{I}$ is an interpretation into V, then $\mathfrak{I}$ can be seen also as a interpretation into W, and the values $\mathfrak{I}(A)$, computed recursively using (1)–(6), do not depend on whether we view $\mathfrak{I}$ as a V-interpretation or a W-interpretation. Consequently, if $V \subseteq W$, there are more interpretations into W than into V. Hence, if $\Gamma \models_W A$ then also $\Gamma \models_V A$ and $\mathbf{G}_W \subseteq \mathbf{G}_V$.

This can be generalized to embeddings between Gödel sets other than inclusion. First, we make precise which formulas are involved in the computation of the truth-value of a formula A in an interpretation $\mathfrak{I}$:

Definition 6. The only subformula of an atomic formula P in $\mathscr{L}^U$ is P itself. The subformulas of $A \star B$ for $\star \in \{\to, \land, \lor\}$ are the subformulas of A and of B, together with $A \star B$ itself. The subformulas of $\forall x A(x)$ and $\exists x A(x)$ with respect to a universe U are all subformulas of all $A(u)$ for $u \in U$, together with $\forall x A(x)$ (or, $\exists x A(x)$, respectively) itself.

The set of truth-values of subformulas of A under a given interpretation $\mathfrak{I}$ is denoted by

$$\mathrm{Val}(\mathfrak{I}, A) = \{\mathfrak{I}(B) : B \text{ subformula of } A \text{ w.r.t. } U^{\mathfrak{I}}\} \cup \{0, 1\}$$

If Γ is a set of formulas, then $\mathrm{Val}(\mathfrak{I}, \Gamma) = \bigcup\{\mathrm{Val}(\mathfrak{I}, A) : A \in \Gamma\}$.

Lemma 7. *Let $\mathfrak{I}$ be a V-interpretation, and let $h\colon \mathrm{Val}(\mathfrak{I}, \Gamma) \to W$ be a mapping satisfying the following properties:*

1. $h(0) = 0$, $h(1) = 1$;

2. *h is strictly monotonic, i.e., if $a < b$, then $h(a) < h(b)$;*

3. *for every $X \subseteq \mathrm{Val}(\mathfrak{I}, \Gamma)$, $h(\inf X) = \inf h(X)$ and $h(\sup X) = \sup h(X)$ (provided $\inf X$, $\sup X \in \mathrm{Val}(\mathfrak{I}, \Gamma)$).*

Then the W-interpretation $\mathfrak{I}_h$ with universe $U^{\mathfrak{I}}$, $f^{\mathfrak{I}_h} = f^{\mathfrak{I}}$, and for atomic $B \in \mathscr{L}^{\mathfrak{I}}$,

$$\mathfrak{I}_h(B) = \begin{cases} h(\mathfrak{I}(B)) & \text{if } \mathfrak{I}(B) \in \operatorname{dom} h \\ 1 & \text{otherwise} \end{cases}$$

satisfies $\mathfrak{I}_h(A) = h(\mathfrak{I}(A))$ for all $A \in \Gamma$.

Proof. By induction on the complexity of A. If $A \equiv \bot$, the claim follows from (1). If A is atomic, it follows from the definition of $\mathfrak{I}_h$. For the propositional connectives the claim follows from the strict monotonicity of h (2). For the quantifiers, it follows from property (3). □

Remark. Note that the construction of $\mathfrak{I}_h$ and the proof of Lemma 7 also goes through without the condition $h(0) = 0$, provided that the formulas in Γ do not contain $\bot$, and goes through without the requirement that existing inf's be preserved ($h(\inf X) = \inf h(X)$ if $\inf X \in \mathrm{Val}(\mathfrak{I}, \Gamma)$) provided they do not contain $\forall$.

Definition 8. A **G**-*embedding* $h\colon V \to W$ is a strictly monotonic, continuous mapping between Gödel sets which preserves 0 and 1.

Lemma 9. *Suppose $h\colon V \to W$ is a **G**-embedding. (a) If $\mathfrak{I}$ is a V-interpretation, and $\mathfrak{I}_h$ is the interpretation induced by $\mathfrak{I}$ and h, then $\mathfrak{I}_h(A) = h(\mathfrak{I}(A))$. (b) If $\Gamma \models_W A$ then $\Gamma \models_V A$ (and hence $\mathbf{G}_W \subseteq \mathbf{G}_V$). (c) If h is bijective, then $\Gamma \models_W A$ iff $\Gamma \models_V A$ (and hence, $\mathbf{G}_V = \mathbf{G}_W$).*

Proof. (a) h satisfies the conditions of Lemma 7, for Γ the set of all formulas. (b) If $\Gamma \not\models_V A$, then for some $\mathfrak{I}$, $\mathfrak{I}(B) = 1$ for all $B \in \Gamma$ and $\mathfrak{I}(A) < 1$. By Lemma 7, $\mathfrak{I}_h(B) = 1$ for all $B \in \Gamma$ and $\mathfrak{I}_h(A) < 1$ (by strict monotonicity of h). Thus $\Gamma \not\models_W A$. (c) If h is bijective then h^{-1} is also a **G**-embedding. □

Definition 10 (Submodel, elementary submodel)**.** Let $\mathfrak{I}_1$, $\mathfrak{I}_2$ be interpretations. We write $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ ($\mathfrak{I}_2$ extends $\mathfrak{I}_1$) iff $U^{\mathfrak{I}_1} \subseteq U^{\mathfrak{I}_2}$, and for all k, all k-ary predicate symbols P in $\mathscr{L}$, and all k-ary function symbols f in $\mathscr{L}$ we have

$$P^{\mathfrak{I}_1} = P^{\mathfrak{I}_2} \restriction (U^{\mathfrak{I}_1})^k \quad f^{\mathfrak{I}_1} = f^{\mathfrak{I}_2} \restriction (U^{\mathfrak{I}_1})^k$$

or in other words, if $\mathfrak{I}_1$ and $\mathfrak{I}_2$ agree on closed atomic formulas.

We write $\mathfrak{I}_1 \prec \mathfrak{I}_2$ if $\mathfrak{I}_1 \subseteq \mathfrak{I}_2$ and $\mathfrak{I}_1(A) = \mathfrak{I}_2(A)$ for all $\mathscr{L}^{U^{\mathfrak{I}_1}}$-formulas A.

Proposition 11 (Downward Löwenheim-Skolem)**.** *For any interpretation $\mathfrak{I}$ with $U^{\mathfrak{I}}$ infinite, there is an interpretation $\mathfrak{I}' \prec \mathfrak{I}$ with a countable universe $U^{\mathfrak{I}'}$.*

Proof sketch. The proof is an easy generalization of the construction for the classical case. We construct a sequence of countable subsets $U_1 \subseteq U_2 \subseteq \cdots$ of $U^{\mathfrak{I}}$: U_1 simply contains $t^{\mathfrak{I}}$ for all closed terms of the original language. U_{i+1} is constructed from U_i by adding, for each of the (countably many) formulas of the form $\exists x A(x)$ and $\forall x A(x)$ in the language $\mathscr{L}^{U_i}$, a countable sequence a_j of elements of $U^{\mathfrak{I}}$ so that $(\mathfrak{I}(A(a_j)))_j \to \mathfrak{I}(\exists x A(x))$ or $\to \mathfrak{I}(\forall x A(x))$, respectively. $U^{\mathfrak{I}'} = \bigcup_i U_i$. □

Lemma 12. *Let $\mathfrak{I}$ be a interpretation into V, $w \in [0,1]$, and let $\mathfrak{I}_w$ be defined by*

$$\mathfrak{I}_w(B) = \begin{cases} \mathfrak{I}(B) & \text{if } \mathfrak{I}(B) < w \\ 1 & \text{otherwise} \end{cases}$$

for atomic formulas B in $\mathscr{L}^U$. Then $\mathfrak{I}_w$ is an interpretation into V. If $w \notin \mathrm{Val}(\mathfrak{I}, A)$, then $\mathfrak{I}_w(A) = \mathfrak{I}(A)$ if $\mathfrak{I}(A) < w$, and $\mathfrak{I}_w(A) = 1$ otherwise.

Proof. Let $h_w(a) = a$ if $a < w$ and $= 1$ otherwise. By induction on the complexity of formulas B it is easily shown that $\mathfrak{I}'(B) = h_w(\mathfrak{I}(B))$ for all subformulas B of A w.r.t. $U^{\mathfrak{I}}$. □

Proposition 13. $\Gamma \models A$ *iff* $\Gamma \Vdash A$

Proof. Only if: obvious. If: Suppose that $\Gamma \not\models A$, i.e., there is a V-interpretation $\mathfrak{I}$ so that $\inf\{\mathfrak{I}(B) : B \in \Gamma\} > \mathfrak{I}(A)$. By Proposition 11, we may assume that $U^{\mathfrak{I}}$ is countable. Hence, there is some w with $\mathfrak{I}(A) < w < \inf\{\mathfrak{I}(B) : B \in \Gamma\}$ and $w \notin \mathrm{Val}(\mathfrak{I}, \Gamma \cup \{A\})$. Let $\mathfrak{I}_w$ be as in Lemma 12. Then $\mathfrak{I}_w(B) = 1$ for all $B \in \Gamma$ and $\mathfrak{I}_w(A) < 1$. □

The coincidence of the two consequence relations is a unique feature of Gödel logics. Proposition 13 does not hold in Łukasiewicz logic, for instance. There, $A, A \to_{Ł} B \Vdash B$ but $A, A \to_{Ł} B \not\models B$. In what follows, we will use $\models$ when semantic consequence is at issue; the preceding propositions shows that the results we obtain for $\models$ hold for $\Vdash$ as well.

Lemma 14 (Semantic deduction theorem)**.**

$$\Gamma, A \models B \quad \textit{iff} \qquad \Gamma \models A \to B.$$

Proof. Immediate consequence of the definition of $\models$ and the semantics for $\to$. □

We want to conclude this part with two interesting observations:

Relation to residuated algebras If one considers the truth value set as a Heyting algebra with $a \wedge b = \min(a,b)$, $a \vee b = \max(a,b)$, and

$$a \to b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{otherwise} \end{cases}$$

then $\to$ and $\wedge$ are residuated, i.e.,

$$(a \to b) = \sup\{x : (x \wedge a) \leq b\}.$$

The Gödel conditional A large class of many-valued logics can be developed from the theory of t-norms [Háj98]. The class of t-norm based logics includes not only (standard) Gödel logic, but also Łukasiewicz- and product logic. In these logics, the conditional is defined as the residuum of the respective t-norm, and the logics differ only in the definition of their t-norm and the respective residuum, i.e., the conditional. The truth function for the Gödel conditional is of particular interest as it can be 'deduced' from simple properties of the evaluation and the entailment relation, a fact which was first observed by G. Takeuti.

Lemma 15. *Suppose we have a standard language containing a 'conditional' $\twoheadrightarrow$ interpreted by a truth-function into $[0,1]$. Suppose further that*

1. *a conditional evaluates to* 1 *if the truth value of the antecedent is less or equal to the truth value of the consequent, i.e., if $\mathfrak{I}(A) \leq \mathfrak{I}(B)$, then $\mathfrak{I}(A \twoheadrightarrow B) = 1$;*
2. *$\models$ is defined as above, i.e., if $\Gamma \models B$, then $\min\{\mathfrak{I}(A) : A \in \Gamma\} \leq \mathfrak{I}(B)$;*
3. *the deduction theorem holds, i.e., $\Gamma \cup \{A\} \models B \Leftrightarrow \Gamma \models A \twoheadrightarrow B$.*

Then $\twoheadrightarrow$ is the Gödel conditional.

Proof. From (1), we have that $\mathfrak{I}(A \twoheadrightarrow B) = 1$ if $\mathfrak{I}(A) \leq \mathfrak{I}(B)$. Since $\models$ is reflexive, $B \models B$. Since it is monotonic, $B, A \models B$. By the deduction theorem, $B \models A \twoheadrightarrow B$. By (2),

$$\mathfrak{I}(B) \leq \mathfrak{I}(A \twoheadrightarrow B).$$

From $A \twoheadrightarrow B \models A \twoheadrightarrow B$ and the deduction theorem, we get $A \twoheadrightarrow B, A \models B$. By (2),

$$\min\{\mathfrak{I}(A \twoheadrightarrow B), \mathfrak{I}(A)\} \leq \mathfrak{I}(B).$$

Thus, if $\mathfrak{I}(A) > \mathfrak{I}(B)$, $\mathfrak{I}(A \twoheadrightarrow B) \leq \mathfrak{I}(B)$. □

Note that all usual conditionals (Gödel, Łukasiewicz, product conditionals) satisfy condition (1). So, in some sense, the Gödel conditional is the only many-valued conditional which validates both directions of the deduction theorem for $\models$. For instance, for the Łukasiewicz conditional $\to_{\text{Ł}}$ the right-to-left direction fails: $A \to_{\text{Ł}} B \models A \to_{\text{Ł}} B$, but $A \to_{\text{Ł}} B, A \not\models B$. (With respect to $\Vdash$, the left-to-right direction of the deduction theorem fails for $\to_{\text{Ł}}$.)

2.2 Axioms and deduction systems

In this section we introduce certain axioms and deduction systems for Gödel logics, and we will show completeness of these deduction systems subsequently. We will use a Hilbert style proof system:

Definition 16. A formula A is derivable from formulas Γ in a system $\mathscr{A}$ consisting of the axioms and the rules iff there are formulas $A_0, \ldots, A_n = A$ such that for each $0 \leq i \leq n$ either $A_i \in \Gamma$, or A_i is an instance of an axiom in $\mathscr{A}$, or there are indices $j_1, \ldots, j_l < i$ and a rule in $\mathscr{A}$ such that $A_{j_1}, \ldots, A_{j_l}$ are the premises and A_i is the conclusion of the rule. In this case we write $\Gamma \vdash_{\mathscr{A}} A$.

We will denote by IL the following complete axiom system for intuitionistic logic (taken from [Tro77]). Rules are written as $A_1, \dots, A_n \vdash A$.

(I1)	$A, A \to B \vdash B$	(I2)	$A \to B, B \to C \vdash A \to C$
(I3)	$A \vee A \to A, A \to A \wedge A$	(I4)	$A \to A \vee B, A \wedge B \to A$
(I5)	$A \vee B \to B \vee A, A \wedge B \to B \wedge A$	(I6)	$A \to B \vdash C \vee A \to C \vee B$
(I7)	$A \wedge B \to C \vdash A \to (B \to C)$	(I8)	$A \to (B \to C) \vdash A \wedge B \to C$
(I9)	$\bot \to A$		
(I10)	$B^{(x)} \to A(x) \vdash B^{(x)} \to \forall x A(x)$	(I11)	$\forall x A(x) \to A(t)$
(I12)	$A(t) \to \exists x A(x)$	(I13)	$A(x) \to B^{(x)} \vdash \exists x A(x) \to B^{(x)}$

(where $B^{(x)}$ means that x is not free in B).

The following axioms will play an important rôle (QS stands for 'quantifier shift', LIN for 'linearity', ISO_0 for 'isolation axiom of 0', and $\text{FIN}(n)$ for 'finite with n elements'):

$$\text{QS} \quad \forall x(C^{(x)} \vee A(x)) \to (C^{(x)} \vee \forall x A(x))$$
$$\text{LIN} \quad (A \to B) \vee (B \to A)$$
$$\text{ISO}_0 \quad \forall x \neg\neg A(x) \to \neg\neg\forall x A(x)$$
$$\text{FIN}(n) \quad (\top \to A_1) \vee (A_1 \to A_2) \vee \ldots \vee (A_{n-2} \to A_{n-1}) \vee (A_{n-1} \to \bot)$$

Notation 17. $\mathbf{H}$ denotes the axiom system $\text{IL} + \text{QS} + \text{LIN}$.

$\mathbf{H}_n$ for $n \geq 2$ denotes the axiom system $\mathbf{H} + \text{FIN}(n)$.

$\mathbf{H}_0$ denotes the axiom system $\mathbf{H} + \text{ISO}_0$.

Theorem 18 (Soundness)**.** *Suppose Γ contains only closed formulas, and all axioms of $\mathscr{A}$ are valid in $\mathbf{G}_V$. Then, if $\Gamma \vdash_{\mathscr{A}} A$ then $\Gamma \models_V A$.*

Proof. By induction on the complexity of proofs. By assumption, all axioms of $\mathscr{A}$ are valid in $\mathbf{G}_V$, hence $\Gamma \models_V A_i$ if A_i is an axiom. If $A_i \in \Gamma$, then obviously $\Gamma \models_V A_i$. It remains to show that the rules of inference preserve consequence. We show this for modus ponens (I1) and existential generalization (I13), the other cases are analogous.

Suppose $\Gamma \models_V A$ and $\Gamma \models_V A \to B$ and consider a V-interpretation $\mathfrak{I}$. Let $v = \inf\{\mathfrak{I}(C) : C \in \Gamma\}$. If $\mathfrak{I}(A) \leq \mathfrak{I}(B)$, then we have $v \leq \mathfrak{I}(B)$ because $v \leq \mathfrak{I}(A)$. If $\mathfrak{I}(A) > \mathfrak{I}(B)$, then $v \leq \mathfrak{I}(B)$ because $\mathfrak{I}(B) = \mathfrak{I}(A \to B)$.

Suppose $\Gamma \models_V A(x) \to B$ and x does not occur free in B. Let $\mathfrak{I}$ be a V-interpretation, and let $w = \sup\{\mathfrak{I}(A(u)) : u \in U^{\mathfrak{I}}\}$, and let $\mathfrak{I}_u$ be the interpretation resulting from $\mathfrak{I}$ by assigning u to x. Since the formulas in Γ are all closed and B does not contain x free, $\mathfrak{I}_u(C) = \mathfrak{I}(C)$ for all $C \in \Gamma \cup \{B\}$ and $u \in U^{\mathfrak{I}}$. Now suppose $w > \mathfrak{I}(\exists x A(x) \to B)$. In this case, $\mathfrak{I}(\exists x A(x)) > \mathfrak{I}(B)$. But then, for some $u \in U^{\mathfrak{I}}$, $\mathfrak{I}_u(A(x)) > \mathfrak{I}(B)$ and we'd have $w > \mathfrak{I}_u(A(x) \to B)$, contradicting $\Gamma \models_V A(x) \to B$. The case for (I10) is analogous. $\square$

Note that the restriction to closed formulas in Γ is essential: $A(x) \vdash_{\mathbf{H}} \forall x A(x)$ but obviously $A(x) \not\models_V \forall x A(x)$.

2.3 Relationships between Gödel logics

The relationships between finite and infinite valued *propositional* Gödel logics are well understood. Any choice of an infinite set of truth-values results in the same propositional Gödel logic, viz., Dummett's **LC**. **LC** was defined using the set of truth-values $V_\downarrow$ (see below). Furthermore, we know that **LC** is the intersection of all finite-valued propositional Gödel logics, and that it is axiomatized by intuitionistic propositional logic IPL plus the schema $(A \to B) \vee (B \to A)$. IPL is contained in all Gödel logics.

In the first-order case, the relationships are somewhat more interesting. First of all, let us note the following fact corresponding to the end of the previous paragraph:

Proposition 19. *Intuitionistic predicate logic IL is contained in all first-order Gödel logics.*

Proof. The axioms and rules of IL are sound for the Gödel truth functions. □

As a consequence of this proposition, we will be able to use any intuitionistically sound rule and intuitionistically true formula when working in any of the Gödel logics.

We can consider special truth value sets which will act as prototypes for other logics. This is due to the fact that the logic is defined extensionally as the set of formulas valid in this truth value set, so the Gödel logics on different truth value sets may coincide.

$$\begin{aligned}
V_\mathbb{R} &= [0,1] \\
V_\downarrow &= \{1/k : k \geq 1\} \cup \{0\} \\
V_\uparrow &= \{1 - 1/k : k \geq 1\} \cup \{1\} \\
V_m &= \{1 - 1/k : 1 \leq k \leq m-1\} \cup \{1\}
\end{aligned}$$

The corresponding Gödel logics are $\mathbf{G}_\mathbb{R}$, $\mathbf{G}_\downarrow$, $\mathbf{G}_\uparrow$, $\mathbf{G}_m$. $\mathbf{G}_\mathbb{R}$ is the *standard* Gödel logic.

The logic $\mathbf{G}_\downarrow$ also turns out to be closely related to some temporal logics [BLZ96b, BLZ96a]. $\mathbf{G}_\uparrow$ is the intersection of all finite-valued first-order Gödel logics as shown in Theorem 23.

Proposition 20. $\mathbf{G}_\mathbb{R} = \bigcap_V \mathbf{G}_V$*, where V ranges over all Gödel sets.*

Proof. If $\mathbf{G}_V \models A$ for every V, then also for $V = [0,1]$. Conversely, if there is some Gödel set V and a V-interpretation $\mathfrak{I}$ with $\mathfrak{I} \not\models A$, then $\mathfrak{I}$ is also a $[0,1]$-interpretation and hence $\mathbf{G}_\mathbb{R} \not\models A$. □

Proposition 21. *The following strict containment relationships hold:*

1. $\mathbf{G}_m \supsetneq \mathbf{G}_{m+1}$,
2. $\mathbf{G}_m \supsetneq \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\mathbb{R}$,
3. $\mathbf{G}_m \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R}$.

Proof. The only non-trivial part is proving that the containments are strict. For this note that

$$(A_1 \to A_2) \vee \ldots \vee (A_m \to A_{m+1})$$

is valid in $\mathbf{G}_m$ but not in $\mathbf{G}_{m+1}$. Furthermore, let

$$C_\uparrow = \exists x(A(x) \to \forall y A(y)) \text{ and}$$
$$C_\downarrow = \exists x(\exists y A(y) \to A(x)).$$

$C_\downarrow$ is valid in all $\mathbf{G}_m$ and in $\mathbf{G}_\uparrow$ and $\mathbf{G}_\downarrow$; $C_\uparrow$ is valid in all $\mathbf{G}_m$ and in $\mathbf{G}_\uparrow$, but not in $\mathbf{G}_\downarrow$; neither is valid in $\mathbf{G}_\mathbb{R}$ ([BLZ96b], Corollary 2.9). □

The formulas $C_\uparrow$ and $C_\downarrow$ are of some importance in the study of first-order infinite-valued Gödel logics. $C_\uparrow$ expresses the fact that every infimum in the set of truth values is a minimum, and $C_\downarrow$ states that every supremum (except possibly 1) is a maximum. The intuitionistically admissible quantifier shifting rules are given by the following implications and equivalences:

$$\begin{array}{rcl}
(\forall x A(x) \wedge B) & \equiv & \forall x(A(x) \wedge B) \\
(\exists x A(x) \wedge B) & \equiv & \exists x(A(x) \wedge B) \\
(\forall x A(x) \vee B) & \to & \forall x(A(x) \vee B) \\
(\exists x A(x) \vee B) & \equiv & \exists x(A(x) \vee B) \\
(B \to \forall x A(x)) & \equiv & \forall x(B \to A(x)) \\
(B \to \exists x A(x)) & \leftarrow & \exists x(B \to A(x)) \\
(\forall x A(x) \to B) & \leftarrow & \exists x(A(x) \to B) \\
(\exists x A(x) \to B) & \equiv & \forall x(A(x) \to B)
\end{array}$$

The remaining three are:

$$\begin{array}{rclr}
(\forall x A(x) \vee B) & \leftarrow & \forall x(A(x) \vee B) & (S_1) \\
(B \to \exists x A(x)) & \to & \exists x(B \to A(x)) & (S_2) \\
(\forall x A(x) \to B) & \to & \exists x(A(x) \to B) & (S_3)
\end{array}$$

Of these, S_1 is valid in any Gödel logic. S_2 and S_3 imply and are implied by $C_\downarrow$ and $C_\uparrow$, respectively (take $\exists y A(y)$ and $\forall y A(y)$, respectively, for B). S_2 and S_3 are, respectively, both valid in $\mathbf{G}_\uparrow$, invalid and valid in $\mathbf{G}_\downarrow$, and both invalid in $\mathbf{G}_\mathbb{R}$. Thus we obtain

Corollary 22. $\mathbf{G}_\uparrow$ *is the only Gödel logic where every formula is equivalent to a prenex formula with the same propositional matrix.*

We now also know that $\mathbf{G}_\uparrow \neq \mathbf{G}_\downarrow$. In fact, we have $\mathbf{G}_\downarrow \subsetneq \mathbf{G}_\uparrow$; this follows from the following theorem.

Theorem 23.

$$\mathbf{G}_\uparrow = \bigcap_{m \geq 2} \mathbf{G}_m$$

Proof. By Proposition 21, $\mathbf{G}_\uparrow \subseteq \bigcap_{m \geq 2} \mathbf{G}_m$. We now prove the reverse inclusion. Assume that there is an interpretation $\mathfrak{I}$ such that $\mathfrak{I} \not\models A$, we want to give an interpretation $\mathfrak{I}'$ such that $\mathfrak{I}' \not\models A$ and $\mathfrak{I}'$ is a $\mathbf{G}_m$ interpretation for some m.

Suppose there is an interpretation $\mathfrak{I}$ such that $\mathfrak{I} \not\models A$, let $\mathfrak{I}(A) = 1 - 1/k$. Let w be somewhere between $1 - 1/k$ and $1 - 1/(k+1)$. Then the interpretation $\mathfrak{I}_w$ given in Lemma 12 also is a counterexample for A. Since there are only finitely many truth values below w in $V_\uparrow$, $\mathfrak{I}_w$ is a $\mathbf{G}_{k+1}$ interpretation with $\mathfrak{I}_w \not\models A$. This completes the proof of the theorem. □

Corollary 24. $\mathbf{G}_m \supsetneq \bigcap_m \mathbf{G}_m = \mathbf{G}_\uparrow \supsetneq \mathbf{G}_\downarrow \supsetneq \mathbf{G}_\mathbb{R}$

As we will see later, the axioms FIN(n) axiomatize exactly the finite-valued Gödel logics. In these logics the quantifier shift axiom QS is not necessary. Furthermore, all quantifier shift rules are valid in the finite valued logics. Since $\mathbf{G}_\uparrow$ is the intersection of all the finite ones, all quantifier shift rules are valid in $\mathbf{G}_\uparrow$. Moreover, any infinite-valued Gödel logic other than $\mathbf{G}_\uparrow$ is defined by some V which either contains an infimum which is not a minimum, or a supremum (other than 1) which is not a maximum. Hence, in V either $C_\uparrow$ or $C_\downarrow$ will be invalid, and therewith either S_3 or S_2. We have:

Corollary 25. $\mathbf{G}_\uparrow$ *is the only Gödel logic with infinite truth value set which admits all quantifier shift rules.*

3 Topology and Order

3.1 Perfect sets

All the following notations, lemmas, theorems are carried out within the framework of Polish spaces, which are separable, completely metrizable topological spaces. For our discussion it is only necessary to know that $\mathbb{R}$ and all its closed subsets are Polish spaces (hence, every Gödel set is a Polish space). For a detailed exposition see [Mos80, Kec95].

Definition 26 (limit point, perfect space, perfect set)**.** A *limit point* of a topological space is a point that is not isolated, i.e. for every open neighborhood U of x there is a point $y \in U$ with $y \neq x$. A space is *perfect* if all its points are limit points. A set $P \subseteq \mathbb{R}$ is *perfect* if it is closed and together with the topology induced from $\mathbb{R}$ is a perfect space.

It is obvious that all (non-trivial) closed intervals are perfect sets, also all countable unions of (non-trivial) intervals. But all these sets generated from closed intervals have the property that they are 'everywhere dense', i.e., contained in the closure of their inner component. There is another very famous set which is perfect but is nowhere dense, the Cantor set:

Example (Cantor Set)*.* The set of all numbers in the unit interval which can be expressed in triadic notation only by digits 0 and 2 is called *Cantor set* $\mathbb{D}$.

A more intuitive way to obtain this set is to start with the unit interval, take out the open middle third and restart this process with the lower and the upper third. Repeating this you get exactly the Cantor set because the middle third always contains the numbers which contain the digit 1 in their triadic notation.

This set has a lot of interesting properties, the most important one is that it is a perfect set:

Proposition 27. *The Cantor set is perfect.*

It is possible to embed the Cauchy space into any perfect space, yielding the following proposition:

Proposition 28. *If X is a nonempty perfect Polish space, then the cardinality of X is $2^{\aleph_0}$ and therefore, all nonempty perfect subsets, too, have cardinality of the continuum.*

It is possible to obtain the following characterization of perfect sets (see [Win99]):

Proposition 29 (Characterization of perfect sets in $\mathbb{R}$)**.** *For any perfect subset of $\mathbb{R}$ there is a unique partition of the real line into countably many intervals such that the intersections of the perfect set with these intervals are either empty, the full interval or isomorphic to the Cantor set.*

So we see that intervals and Cantor sets are prototypical for perfect sets and the basic building blocks of more complex perfect sets.

Every Polish space can be partitioned into a perfect kernel and a countable rest. This is the well known Cantor-Bendixon Theorem:

Theorem 30 (Cantor-Bendixon)**.** *Let X be a Polish space. Then X can be uniquely written as $X = P \cup C$, with P a perfect subset of X and C countable and open. The subset P is called the* perfect kernel *of X (denoted with V^∞).*

As a corollary we obtain that any uncountable Polish space contains a perfect set, and therefore, has cardinality $2^{\aleph_0}$.

3.2 Relation to Gödel logics

The following lemma was originally proved in [Pre03], where it was used to extend the proof of recursive axiomatizability of 'standard' Gödel logics (those with $V = [0,1]$) to Gödel logics with a truth value set containing a perfect set in the general case. The following more simple proof is inspired by [BGP]:

Lemma 31. *Suppose that $M \subseteq [0,1]$ is countable and $P \subseteq [0,1]$ is perfect. Then there is a strictly monotone continuous map $h\colon M \to P$ (i.e., infima and suprema already existing in M are preserved). Furthermore, if $\inf M \in M$, then one can choose h such that $h(\inf M) = \inf P$.*

Proof. Let σ be the mapping which scales and shifts M into $[0,1]$, i.e. the mapping $x \to (x - \inf M)/(\sup M - \inf M)$ (assuming that M contains more than one point). Let w be an injective monotone map from $\sigma(M)$ into 2^ω, i.e. $w(m)$ is a fixed binary representation of m. For dyadic rational numbers (i.e. those with different binary representations) we fix one possible.

Let i be the natural bijection from 2^ω (the set of infinite $\{0,1\}$-sequences, ordered lexicographically) onto $\mathbb{D}$, the Cantor set. i is an order preserving homeomorphism. Since P is perfect, we can find a continuous strictly monotone map c from the Cantor set $\mathbb{D} \subseteq [0,1]$ into P, and c can be chosen so that $c(0) = \inf P$.

Now $h = c \circ i \circ w \circ \sigma$ is also a strictly monotone map from M into P, and $h(\inf M) = \inf P$, if $\inf M \in M$. Since c is continuous, existing infima and suprema are preserved. $\square$

Corollary 32. *A Gödel set V is uncountable iff it contains a non-trivial dense linear subordering.*

Proof. If: Every countable non-trivial dense linear order has order type η, $\mathbf{1}+\eta$, $\eta+\mathbf{1}$, or $\mathbf{1}+\eta+\mathbf{1}$ [Ros82, Corollary 2.9], where η is the order type of $\mathbb{Q}$. The completion of any ordering of order type η has order type λ, the order type of $\mathbb{R}$ [Ros82, Theorem 2.30], thus the truth value set must be uncountable.

Only if: By Theorem 30, V^∞ is non-empty. Take $M = \mathbb{Q} \cap [0,1]$ and $P = V^\infty$ in Lemma 31. The image of M under h is a non-trivial dense linear subordering in V. $\square$

Theorem 33. *Suppose V is a truth value set with non-empty perfect kernel P, and let $W = V \cup [\inf P, 1]$. Then $\models_V = \models_W$, i.e. $\Gamma \models_V A$ iff $\Gamma \models_W A$. Thus also the logics induced by V and W are the same, i.e., $\mathbf{G}_V = \mathbf{G}_W$.*

Proof. As $V \subseteq W$ we have $\models_W \subseteq \models_V$ (cf. the Remark preceding Definition 3). Now assume that $\mathfrak{I}$ is a W-interpretation which shows that $\Gamma \models_W A$ does *not* hold, i.e., $\inf\{\mathfrak{I}(B) : B \in \Gamma\} > \mathfrak{I}(A)$. By Proposition 11, we may assume that $U^{\mathfrak{I}}$ is countable. The set $\mathrm{Val}(\mathfrak{I}, \Gamma \cup A)$ has cardinality at most $\aleph_0$, thus there is a $b \in [0,1]$ such that $b \notin \mathrm{Val}(\mathfrak{I}, \Gamma \cup A)$ and $\mathfrak{I}(A) < b < 1$. By Lemma 12, $\mathfrak{I}_b(A) < b < 1$. Now consider $M = \mathrm{Val}(\mathfrak{I}_b, \Gamma \cup A)$: these are all the truth values from $W = V \cup [\inf P, 1]$ required to compute $\mathfrak{I}_b(A)$ and $\mathfrak{I}_b(B)$ for all $B \in \Gamma$. We have to find some way to map them to V so that the induced interpretation is a counterexample to $\Gamma \models_V A$.

Let $M_0 = M \cap [0, \inf P)$ and $M_1 = (M \cap [\inf P, b]) \cup \{\inf P\}$. By Lemma 31 there is a strictly monotone continuous (i.e. preserving all existing infima and suprema) map h from M_1 into P. Furthermore, we can choose h such that $h(\inf M_1) = \inf P$.

We define a function g from $\mathrm{Val}(\mathfrak{I}_b, \Gamma \cup A)$ to V as follows:

$$g(x) = \begin{cases} x & 0 \le x \le \inf P \\ h(x) & \inf P \le x \le b \\ 1 & x = 1 \end{cases}$$

Note that there is no $x \in \mathrm{Val}(\mathfrak{I}_b, \Gamma \cup A)$ with $b < x < 1$. This function has the following properties: $g(0) = 0$, $g(1) = 1$, g is strictly monotonic and preserves existing infima and suprema. Using Lemma 7 we obtain that $\mathfrak{I}_g$ is a V-interpretation with $\mathfrak{I}_g(C) = g(\mathfrak{I}_b(C))$ for all $C \in \Gamma \cup A$, thus also $\inf\{\mathfrak{I}_g(B) : B \in \Gamma\} > \mathfrak{I}_g(A)$. $\square$

4 Countable Gödel sets

In this section we show that the first-order Gödel logics where the set of truth values does not contain a dense subset are not axiomatizable. We establish this result by reducing the classical validity of a formula in all finite models to the validity of a formula in Gödel logic (the set of these formulas is not r.e. by Trakhtenbrot's Theorem).

Definition 34. A formula is called *crisp* if all occurrences of atomic formulas are either negated or double-negated.

Lemma 35. *If A and B are crisp and classically equivalent, then also $\mathbf{G}_{\mathbb{R}} \models A \leftrightarrow B$. Specifically, if $A(x)$ and B are crisp, then*

$$\begin{array}{l} \models \forall x A(x) \to B \leftrightarrow \exists x(A(x) \to B) \quad \textit{and} \\ \models B \to \exists x A(x) \leftrightarrow \exists x(B \to A(x)). \end{array}$$

Proof. Given an interpretation $\mathfrak{I}$, define $\mathfrak{I}'(C) = 1$ if $\mathfrak{I}(C) > 0$ and $= 0$ if $\mathfrak{I}(C) = 0$ for atomic C. It is easily seen that if A, B are crisp, then $\mathfrak{I}(A) = \mathfrak{I}'(A)$ and $\mathfrak{I}(B) = \mathfrak{I}'(B)$. But $\mathfrak{I}'$ is a classical interpretation, so by assumption $\mathfrak{I}'(A) = \mathfrak{I}'(B)$. □

Theorem 36. *If V is countably infinite, then $\mathbf{G}_V$ is not recursively enumerable.*

Proof. By Theorem 32, V is countably infinite iff it is infinite and does not contain a non-trivial densely ordered subset. We show that for every sentence A there is a sentence A^g s.t. A^g is valid in $\mathbf{G}_V$ iff A is true in every finite (classical) first-order structure.

We define A^g as follows: Let P be a unary and L be a binary predicate symbol not occurring in A and let $Q_1, \ldots, Q_n$ be all the predicate symbols in A. We use the abbreviations $x \in y \equiv \neg\neg L(x,y)$ and $x \prec y \equiv (P(y) \to P(x)) \to P(y)$. Note that for any interpretation $\mathfrak{I}$, $\mathfrak{I}(x \in y)$ is either 0 or 1, and as long as $\mathfrak{I}(P(x)) < 1$ for all x (in particular, if $\mathfrak{I}(\exists z P(z)) < 1$), we have $\mathfrak{I}(x \prec y) = 1$ iff $\mathfrak{I}(P(x)) < \mathfrak{I}(P(y))$. Let $A^g \equiv$

$$\left\{ \begin{array}{l} S \wedge c_1 \in 0 \wedge c_2 \in 0 \wedge c_2 \prec c_1 \wedge \\ \quad \forall i \big[\forall x, y \forall j \forall k \exists z D \vee \forall x \neg(x \in s(i))\big] \end{array} \right\} \to (A' \vee \exists u P(u)) \tag{7}$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$D \equiv \begin{array}{l} (j \leq i \wedge x \in j \wedge k \leq i \wedge y \in k \wedge x \prec y) \to \\ \quad \to (z \in s(i) \wedge x \prec z \wedge z \prec y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(i) \equiv \exists x(x \in i)$.

Intuitively, L is a predicate that divides a subset of the domain into levels, and $x \in i$ means that x is an element of level i. If the antecendent is true, then the true standard axioms S force the domain to be a model of PA, which could be either a standard model (isomorphic to $\mathbb{N}$) or a non-standard model ($\mathbb{N}$ followed by copies of $\mathbb{Z}$). P orders the elements of the domain which fall into one of the levels in a subordering of the truth values.

The idea is that for any two elements in a level $\leq i$ there is an element in a not-empty level $j \geq i$ which lies strictly between those two elements in the ordering given by $\prec$. If this condition cannot be satisfied, the levels above i are empty. Clearly, this condition can be satisfied in an interpretation $\mathfrak{I}$ only for finitely many levels if V does not contain a dense subset, since if more than finitely many levels are non-empty, then $\bigcup_i \{\mathfrak{I}(P(d)) : \mathfrak{I} \models d \in i\}$ gives a dense subset. By relativizing the quantifiers in A to the indices of non-empty levels, we in effect relativize to a finite subset of the domain. We make this more precise:

Suppose A is classically false in some finite structure $\mathfrak{I}$. W.l.o.g. we may assume that the domain of this structure is the naturals $0, \ldots, n$. We extend $\mathfrak{I}$ to a $\mathbf{G}_V$-interpretation $\mathfrak{I}^g$ with domain $\mathbb{N}$ as follows: Since V contains infinitely many values, we can choose c_1, c_2, L and P so that $\exists x(x \in i)$ is true for $i = 0, \ldots, n$ and false otherwise, and so that $\mathfrak{I}^g(\exists x P(x)) < 1$. The number-theoretic symbols receive their natural interpretation. The antecedent of A^g clearly receives the value 1, and the consequent receives $\mathfrak{I}_g(\exists x P(x)) < 1$, so $\mathfrak{I}^g \not\models A^g$.

Now suppose that $\mathfrak{I} \not\models A^g$. Then $\mathfrak{I}(\exists x P(x)) < 1$. In this case, $\mathfrak{I}(x \prec y) = 1$ iff $\mathfrak{I}(P(x)) < \mathfrak{I}(P(y))$, so $\prec$ defines a strict order on the domain of $\mathfrak{I}$. It is easily seen that in order for the value of the antecedent of A^g under $\mathfrak{I}$ to be greater than that of the consequent, it must be $= 1$ (the values of all subformulas are either $\leq \mathfrak{I}(\exists x P(x))$ or $= 1$). For this to happen, of course, what the antecedent is intended to express must actually be true in $\mathfrak{I}$, i.e., that $x \in i$ defines a series of levels and any level $i > 0$ is either empty, or for all x, and y occuring in some smaller level there is a z with $x \prec z \prec y$ and $z \in i$.

To see this, consider the relevant part of the antecedent, $B = \forall i \big[\forall x, y \forall j \forall k \exists z D \vee \forall x \neg(x \in i)\big]$. If $\mathfrak{I}(B) = 1$, then for all i, either $\mathfrak{I}(\forall x, y \forall j \forall k \exists z D) = 1$ or $\mathfrak{I}(\forall x \neg(x \in i)) = 1$. In the first case, we have $\mathfrak{I}(\exists z D) = 1$ for all x, y, j, and k. Now suppose that for all z, $\mathfrak{I}(D) < 1$, yet $\mathfrak{I}(\exists z D) = 1$. Then for at least some z the value of that formula would have to be $> \mathfrak{I}(\exists z P(z))$, which is impossible. Thus, for every x, y, j, k, there is a z such that $\mathfrak{I}(D) = 1$. But this means that for all x, y s.t. $x \in j$, $y \in k$ with $j, k \leq i$ and $x \prec y$ there is a z with $x \prec z \prec y$ and $z \in i + 1$.

In the second case, where $\mathfrak{I}(\forall x \neg(x \in i)) = 1$, we have that $\mathfrak{I}(\neg(x \in i)) = 1$ for all x, hence $\mathfrak{I}(x \in i) = 0$ and level i is empty.

Note that the non empty levels can be distributed over the whole range of the non-standard model, but since V contains no dense subset, the total number of non empty levels is finite. Thus, A is false in the classical interpretation $\mathfrak{I}^c$ obtained from $\mathfrak{I}$ by restricting $\mathfrak{I}$ to the domain $\{i : \exists x(x \in i)\}$ and $\mathfrak{I}^c(Q) = \mathfrak{I}(\neg\neg Q)$ for atomic Q. ☐

This shows that no infinite-valued Gödel logic whose set of truth values does not contain a dense subset, i.e., no countably infinite Gödel logic is axiomatizable. We strengthen this result in Section 7.1 to show that the prenex fragments are likewise not axiomatizable.

5 Uncountable Gödel sets

5.1 0 is contained in the perfect kernel

If V is uncountable, and 0 is contained in V^∞, then $\mathbf{G}_V$ is axiomatizable. Indeed, Theorem 33 showed that the sets of validities of all such V coincide. Thus, it is only necessary to establish completeness of the axioms system $\mathbf{H}$ with respect to $\mathbf{G}_\mathbb{R}$. This result has been shown by several people over the years. We give here a generalization of the proof of Takano [Tak87].

Theorem 37 (Strong completeness of Gödel logic [Tak87]). *If $\Gamma \models A$ in $\mathbf{G}_{\mathbb{R}}$, then $\Gamma \vdash_{\mathbf{H}} A$.*

Proof. Assume that $\Gamma \nvdash A$, we construct an interpretation $\mathfrak{I}$ in which $\mathfrak{I}(A) = 1$ for all $B \in \Gamma$ and $\mathfrak{I}(A) < 1$. Let $y_1, y_2, \ldots$ be a sequence of free variables which do not occur in $\Gamma \cup \Delta$, let $\mathscr{T}$ be the set of all terms in the language of $\Gamma \cup \Delta$ together with the new variables $y_1, y_2, \ldots$, and let $\mathscr{F} = \{F_1, F_2, \ldots\}$ be an enumeration of the formulas in this language in which y_i does not appear in $F_1, \ldots, F_i$ and in which each formula appears infinitely often.

If Δ is a set of formulas, we write $\Gamma \Rightarrow \Delta$ if for some $A_1, \ldots, A_n \in \Gamma$, and some $B_1, \ldots, B_m \in \Delta$, $\vdash_{\mathbf{H}} (A_1 \wedge \ldots \wedge A_n) \to (B_1 \vee \ldots \vee B_m)$ (and $\not\Rightarrow$ if this is not the case). We define a sequence of sets of formulas Γ_n, Δ_n such that $\Gamma_n \not\Rightarrow \Delta_n$ by induction. First, $\Gamma_0 = \Gamma$ and $\Delta_0 = \{A\}$. By the assumption of the theorem, $\Gamma_0 \not\Rightarrow \Delta_0$.

If $\Gamma_n \Rightarrow \Delta_n \cup \{F_n\}$, then $\Gamma_{n+1} = \Gamma_n \cup \{F_n\}$ and $\Delta_{n+1} = \Delta_n$. In this case, $\Gamma_{n+1} \not\Rightarrow \Delta_{n+1}$, since otherwise we would have $\Gamma_n \Rightarrow \Delta_n \cup \{F_n\}$ and $\Gamma_n \cup \{F_n\} \Rightarrow \Delta_n$. But then, we'd have that $\Gamma_n \Rightarrow \Delta_n$, which contradicts the induction hypothesis (note that $\vdash_{\mathbf{H}} (A \to B \vee F) \to ((A \wedge F \to B) \to (A \to B))$).

If $\Gamma_n \not\Rightarrow \Delta_n \cup \{F_n\}$, then $\Gamma_{n+1} = \Gamma_n$ and $\Delta_{n+1} = \Delta_n \cup \{F_n, B(y_n)\}$ if $F_n \equiv \forall x B(x)$, and $\Delta_{n+1} = \Delta_n \cup \{F_n\}$ otherwise. In the latter case, it is obvious that $\Gamma_{n+1} \not\Rightarrow \Delta_{n+1}$. In the former, observe that by $I10$ and QS, if $\Gamma_n \Rightarrow \Delta_n \cup \{\forall x B(x), B(y_n)\}$ then also $\Gamma_n \Rightarrow \Delta_n \cup \{\forall x B(x)\}$ (note that y_n does not occur in Γ_n or Δ_n).

Let $\Gamma^* = \bigcup_{i=0}^{\infty} \Gamma_i$ and $\Delta^* = \bigcup_{i=0}^{\infty} \Delta_i$. We have:

1. $\Gamma^* \not\Rightarrow \Delta^*$, for otherwise there would be a k so that $\Gamma_k \Rightarrow \Delta_k$.
2. $\Gamma \subseteq \Gamma^*$ and $\Delta \subseteq \Delta^*$ (by construction).
3. $\Gamma^* = \mathscr{F} \setminus \Delta^*$, since each F_n is either in Γ_{n+1} or Δ_{n+1}, and if for some n, $F_n \in \Gamma^* \cap \Delta^*$, there would be a k so that $F_n \in \Gamma_k \cap \Delta_k$, which is impossible since $\Gamma_k \not\Rightarrow \Delta_k$.
4. If $\Gamma^* \Rightarrow B_1 \vee \ldots \vee B_n$, then $B_i \in \Gamma^*$ for some i. For suppose not, then for $i = 1, \ldots, n$, $B_i \notin \Gamma^*$, and hence, by (3), $B_i \in \Delta^*$. But then $\Gamma^* \Rightarrow \Delta^*$, contradicting (1).
5. If $B(t) \in \Gamma^*$ for every $t \in \mathscr{T}$, then $\forall x B(x) \in \Gamma^*$. Otherwise, by (3), $\forall x B(x) \in \Delta^*$ and so there is some n so that $\forall x B(x) = F_n$ and Δ_{n+1} contains $\forall x B(x)$ and $B(y_n)$. But, again by (3), then $B(y_n) \notin \Gamma^*$.
6. Γ^* is closed under provable implication, since if $\Gamma^* \Rightarrow A$, then $A \notin \Delta^*$ and so, again by (3), $A \in \Gamma^*$. In particular, if $\vdash_{\mathbf{H}} A$, then $A \in \Gamma^*$.

Define relations $\preceq$ and $\equiv$ on $\mathscr{F}$ by

$$B \preceq C \Leftrightarrow B \to C \in \Gamma^* \quad \text{and} \quad B \equiv C \Leftrightarrow B \preceq C \wedge C \preceq B.$$

Then $\preceq$ is reflexive and transitive, since for every B, $\vdash_{\mathbf{H}} B \to B$ and so $B \to B \in \Gamma^*$, and if $B \to C \in \Gamma^*$ and $C \to D \in \Gamma^*$ then $B \to D \in \Gamma^*$, since $B \to C, C \to D \Rightarrow B \to D$ (recall (6) above). Hence, $\equiv$ is an equivalence relation on $\mathscr{F}$. For every B in $\mathscr{F}$ we let $|B|$ be the equivalence class under $\equiv$ to which B belongs, and $\mathscr{F}/\!\equiv$ the set of all equivalence classes. Next we define the relation $\leq$ on $\mathscr{F}/\!\equiv$ by

$$|B| \leq |C| \Leftrightarrow B \preceq C \Leftrightarrow B \to C \in \Gamma^*.$$

Obviously, $\leq$ is independent of the choice of representatives A, B.

Lemma 38. *$\langle \mathscr{F}/{\equiv}, \leq \rangle$ is a countably linearly ordered structure with distinct maximal element $|\top|$ and minimal element $|\bot|$.*

Proof. Since $\mathscr{F}$ is countably infinite, $\mathscr{F}/{\equiv}$ is countable. For every B and C, $\vdash_{\mathbf{H}} (B \to C) \vee (C \to B)$ by LIN, and so either $B \to C \in \Gamma^*$ or $C \to B \in \Gamma^*$ (by (4)), hence $\leq$ is linear. For every B, $\vdash_{\mathbf{H}} B \to \top$ and $\vdash_{\mathbf{H}} \bot \to B$, and so $B \to \top \in \Gamma^*$ and $\bot \to B \in \Gamma^*$, hence $|\top|$ and $|\bot|$ are the maximal and minimal elements, respectively. Pick any A in Δ^*. Since $\top \to \bot \Rightarrow A$, and $A \notin \Gamma^*$, $\top \to \bot \notin \Gamma^*$, so $|\top| \neq |\bot|$. ☐

We abbreviate $|\top|$ by $\mathbf{1}$ and $|\bot|$ by $\mathbf{0}$.

Lemma 39. *The following properties hold in $\langle \mathscr{F}/{\equiv}, \leq \rangle$:*

1. $|B| = \mathbf{1} \Leftrightarrow B \in \Gamma^*$.
2. $|B \wedge C| = \min\{|B|, |C|\}$.
3. $|B \vee C| = \max\{|B|, |C|\}$.
4. $|B \to C| = \mathbf{1}$ *if* $|B| \leq |C|$, $|B \to C| = |C|$ *otherwise.*
5. $|\neg B| = \mathbf{1}$ *if* $|B| = \mathbf{0}$; $|\neg B| = \mathbf{0}$ *otherwise.*
6. $|\exists x B(x)| = \sup\{|B(t)| : t \in \mathscr{T}\}$.
7. $|\forall x B(x)| = \inf\{|B(t)| : t \in \mathscr{T}\}$.

Proof. (1) If $|B| = \mathbf{1}$, then $\top \to B \in \Gamma^*$, and hence $B \in \Gamma^*$. And if $B \in \Gamma^*$, then $\top \to B \in \Gamma^*$ since $B \Rightarrow \top \to B$. So $|\top| \leq |B|$. It follows that $|\top| = |B|$ as also $|B| \leq |\top|$.

(2) From $\Rightarrow B \wedge C \to B$, $\Rightarrow B \wedge C \to C$ and $D \to B, D \to C \Rightarrow D \to B \wedge C$ for every D, it follows that $|B \wedge C| = \inf\{|B|, |C|\}$, from which (2) follows since $\leq$ is linear. (3) is proved analogously.

(4) If $|B| \leq |C|$, then $B \to C \in \Gamma^*$, and since $\top \in \Gamma^*$ as well, $|B \to C| = \mathbf{1}$. Now suppose that $|B| \not\leq |C|$. From $B \wedge (B \to C) \Rightarrow C$ it follows that $\min\{|B|, |B \to C|\} \leq |C|$. Because $|B| \not\leq |C|$, $\min\{|B|, |B \to C|\} \neq |B|$, hence $|B \to C| \leq |C|$. On the other hand, $\vdash C \to (B \to C)$, so $|C| \leq |B \to C|$.

(5) If $|B| = \mathbf{0}$, $\neg B = B \to \bot \in \Gamma^*$, and hence $|\neg B| = \mathbf{1}$ by (1). Otherwise, $|B| \not\leq |\bot|$, and so by (4), $|\neg B| = |B \to \bot| = \mathbf{0}$.

(6) Since $\vdash_{\mathbf{H}} B(t) \to \exists x B(x)$, $|B(t)| \leq |\exists x B(x)|$ for every $t \in \mathscr{T}$. On the other hand, for every D without x free,

	$\lvert B(t)\rvert \leq \lvert D\rvert$	for every $t \in \mathscr{T}$
$\Leftrightarrow$	$B(t) \to D \in \Gamma^*$	for every $t \in \mathscr{T}$
$\Rightarrow$	$\forall x(B(x) \to D) \in \Gamma^*$	by property (5) of Γ^*
$\Rightarrow$	$\exists x B(x) \to D \in \Gamma^*$	since $\forall x(B(x) \to D) \Rightarrow \exists x B(x) \to D$
$\Leftrightarrow$	$\lvert\exists x B(x)\rvert \leq \lvert D\rvert$.	

(7) is proved analogously. ☐

$\langle \mathscr{F}/{\equiv}, \leq \rangle$ is countable, let $\mathbf{0} = a_0, \mathbf{1} = a_1, a_2, \ldots$ be an enumeration. Define $h(\mathbf{0}) = 0$, $h(\mathbf{1}) = 1$, and define $h(a_n)$ inductively for $n > 1$: Let $a_n^- = \max\{a_i : i < n \text{ and } a_i < a_n\}$ and $a_n^+ = \min\{a_i : i < n \text{ and } a_i > a_n\}$, and define $h(a_n) = (h(a_n^-) + h(a_n^+))/2$ (thus, $a_2^- = \mathbf{0}$ and $a_2^+ = \mathbf{1}$ as $\mathbf{0} = a_0 < a_2 < a_1 = \mathbf{1}$, hence $h(a_2) = \frac{1}{2}$). Then $h\colon \langle \mathscr{F}/{\equiv}, \leq \rangle \to \mathbb{Q} \cap [0,1]$ is a strictly monotone map which preserves infs and sups. By Lemma 31 there exists a **G**-embedding h' from $\mathbb{Q} \cap [0,1]$ into $\langle [0,1], \leq \rangle$ which is also strictly monotone and preserves infs and sups. Put $\mathfrak{I}(B) = h'(h(|B|))$ for every atomic $B \in \mathscr{F}$ and we obtain a $V_{\mathbb{R}}$-interpretation.

Note that for every B, $\mathfrak{I}(B) = 1$ iff $|B| = \mathbf{1}$ iff $B \in \Gamma^*$. Hence, we have $\mathfrak{I}(B) = 1$ for all $B \in \Gamma$ while if $A \notin \Gamma^*$, then $\mathfrak{I}(A) < 1$, so $\Gamma \not\models A$. Thus we have proven that on the assumption that if $\Gamma \nvdash A$, then $\Gamma \not\models A$ □

As already mentioned we obtain from this completeness proof together with the soundness theorem (Theorem 18) and Theorem 33 the characterization of recursive axiomatizability:

Theorem 40. *Let V be a Gödel set with* 0 *contained in the perfect kernel of V. Suppose that Γ is a set of closed formulas. Then $\Gamma \models_V A$ iff $\Gamma \vdash_{\mathbf{H}} A$.*

Corollary 41 (Deduction theorem for Gödel logics)**.** *Suppose that Γ is a set of formulas, and A is a closed formula. Then*

$$\Gamma, A \vdash_{\mathbf{H}} B \quad \textit{iff} \quad \Gamma \vdash_{\mathbf{H}} A \rightarrow B.$$

Proof. Use the soundness theorem (Theorem 18), completeness theorem (Theorem 40) and the semantic deduction theorem 14. Another proof would be by induction on the length of the proof. See [Háj98], Theorem 2.2.18. □

5.2 0 is isolated

In the case where 0 is isolated, and thus also not contained in the perfect kernel, we will transform a counter example in $\mathbf{G}_{\mathbb{R}}$ for $\Gamma, \Pi \models A$, where Π is a set of sentences stating that every infimum is a minimum, into a counter example in $\mathbf{G}_V$ for $\Gamma \models A$.

Lemma 42. *Let $x, \bar{y}$ be the free variables in A.*

$$\vdash_{\mathbf{H}_0} \forall \bar{y}(\neg \forall x A(x, \bar{y}) \rightarrow \exists x \neg A(x, \bar{y}))$$

Proof. It is easy to see that in all Gödel logics the following weak form of the law of excluded middle is valid: $\neg\neg A(a) \vee \neg A(a)$. By quantification we obtain $\forall x \neg\neg A(x) \vee \exists x \neg A(x)$ and by valid quantifier shifting rules $\neg\neg\forall x A(x) \vee \exists \neg A(x)$. From the intuitionistically valid $\neg A \vee B \rightarrow (A \rightarrow B)$ we can prove $\neg \forall x A(x) \rightarrow \exists x \neg A(x)$. A final quantification of the free variables concludes the proof. □

Theorem 43. *Let V be an uncountable Gödel set where* 0 *is isolated. Suppose Γ is a set of closed formulas. Then $\Gamma \models_V A$ iff $\Gamma \vdash_{\mathbf{H}_0} A$.*

Proof. If: Follows from soundness (Theorem 18) and the observation that $\textsc{iso}_0$ is valid for any V where 0 is isolated.

Only if: We already know from Theorem 33 that the entailment relation of V and $V \cup [\inf P, 1]$ coincide, where P is the perfect kernel of V. So we may assume wthout loss of generality that V already is of this form, i.e. that $\lambda = \inf P$ and $V \cap [\lambda, 1] = [\lambda, 1]$. Let $V' = [0,1]$. Define

$$\Pi = \{\forall \bar{y}(\neg \forall x A(x, \bar{y}) \rightarrow \exists x \neg A(x, \bar{y})) : A(x, \bar{y}) \text{ formula}\}$$

where $A(x,\bar{y})$ ranges over *all* formulas with free variables x and $\bar{y}$. We consider the entailment relation in V'. Either $\Pi,\Gamma \models_{V'} A$ or $\Pi,\Gamma \not\models_{V'} A$. In the former case we know from the strong completeness of $\mathbf{H}$ for $\mathbf{G}_{\mathbb{R}}$ that there are finite subsets Π' and Γ' of Π and Γ, respectively, such that $\Pi',\Gamma' \vdash_{\mathbf{H}} A$. Since all the sentences in Π are provable in $\mathbf{H}_0$ (see Lemma 42) we obtain that $\Gamma' \vdash_{\mathbf{H}_0} A$. In the latter case there is an interpretation $\mathfrak{I}'$ such that

$$\inf\{\mathfrak{I}'(G) : G \in \Pi \cup \Gamma\} > \mathfrak{I}'(A).$$

It is obvious from the structure of the formulas in Π that their truth value will always be either 0 or 1. Combined with the above we know that for all $G \in \Pi$, $\mathfrak{I}'(G) = 1$. Next we define a function $f(x)$ which maps values from $\mathrm{Val}(\mathfrak{I}', \Gamma \cup \Pi \cup \{A\})$ into V:

$$f(x) = \begin{cases} 0 & x = 0 \\ \lambda + x/(1-\lambda) & x > 0 \end{cases}$$

We see that f satisfies conditions (1) and (2) of Lemma 7, but we cannot use Lemma 7 directly, as not all existing infima and suprema are necessarily preserved.

Consider as in Lemma 7 the interpretation $\mathfrak{I}_f(B) = f(\mathfrak{I}'(B))$ for atomic subformulas of $\Gamma \cup \Pi \cup \{A\}$. We want to show that the identity $\mathfrak{I}_f(B) = f(\mathfrak{I}'(B))$ extends to all subformulas of $\Gamma \cup \Pi \cup \{A\}$. For propositional connectives and the existentially quantified formulas this is obvious. The important case is $\forall x A(x)$. First assume that $\mathfrak{I}'(\forall x A(x)) > 0$. Then it is obvious that $\mathfrak{I}_f(\forall x A(x)) = f(\mathfrak{I}'(\forall x A(x)))$. In the case where $\mathfrak{I}'(\forall x A(x)) = 0$ we observe that $A(x)$ contains a free variable and therefore $\neg\forall x A(x) \to \exists x \neg A(x) \in \Pi$, thus $\mathfrak{I}'(\neg\forall x A(x) \to \exists x \neg A(x)) = 1$. This implies that there is a witness c such that $\mathfrak{I}'(A(c)) = 0$. Using the induction hypothesis we know that $\mathfrak{I}_f(A(c)) = 0$, too. We obtain that $\mathfrak{I}_f(\forall x A(x)) = 0$, concluding the proof.

Thus we have shown that $\mathfrak{I}_f$ is a counterexample to $\Gamma \models_V A$ which completes the proof of the theorem. □

5.3 0 not isolated but not in the perfect kernel

In the preceding sections, we gave axiomatizations for the logics based on those uncountably infinite Gödel sets V where 0 is either isolated or in the perfect kernel of V. It remains to determine whether logics based on uncountable Gödel sets where 0 is neither isolated nor in the perfect kernel are axiomatizable. The answer in this case is negative. If 0 is not isolated in V, 0 has a countably infinite neighborhood. Furthermore, any sequence $(a_n)_{n\in\mathbb{N}} \to 0$ is so that, for sufficiently large n, $V \cap [0, a_n]$ is countable and hence, by (the proof of) Theorem 32, contains no densely ordered subset. This fact is the basis for the following non-axiomatizability proof, which is a variation on the proof of Theorem 36.

Theorem 44. *If V is uncountable, 0 is not isolated in V, but not in the perfect kernel of V, then $\mathbf{G}_V$ is not axiomatizable.*

Proof. We show that for every sentence A there is a sentence A^h s.t. A^h is valid in $\mathbf{G}_V$ iff A is true in every finite (classical) first-order structure.

The definition of A^h mirrors the definition of A^g in the proof of Theorem 36, except that the construction there is carried out infinitely many times for $V \cap [0, a_n]$, where

$(a_n)_{n\in\mathbb{N}}$ is a strictly descending sequence, $a_n > 0$ for all n, which converges to 0. Let P be a binary and L be a ternary predicate symbol not occurring in A and let $R_1, \ldots, R_n$ be all the predicate symbols in A. We use the abbreviations $x \in_\ell y \equiv \neg\neg L(x,y,\ell)$ and $x \prec_\ell y \equiv (P(y,\ell) \to P(x,\ell)) \to P(y,\ell)$. As before, for a fixed ℓ, provided $\mathfrak{I}(\exists x P(x,\ell)) < 1$, $\mathfrak{I}(x \prec_\ell y) = 1$ iff $\mathfrak{I}(P(x,\ell)) < \mathfrak{I}(P(y,\ell))$, and $\mathfrak{I}(x \in_\ell y)$ is always either 0 or 1. We also need a binary predicate symbol $Q(\ell)$ to give us the descending sequence $(a_n)_{n\in\mathbb{N}}$: Note that $\mathfrak{I}(\neg\forall\ell Q(\ell)) = 1$ iff $\inf\{\mathfrak{I}(Q(d)) : d \in |\mathfrak{I}|\} = 0$ and $\mathfrak{I}(\exists\ell\neg Q(\ell)) = 1$ iff $0 \notin \{\mathfrak{I}(Q(d)) : d \in |\mathfrak{I}|\}$.

Let $A^h \equiv$

$$\left\{\begin{array}{l} S \wedge \forall\ell((Q(s(\ell)) \to Q(\ell)) \to Q(s(\ell)) \wedge \\ \quad \neg\forall\ell Q(\ell) \wedge \exists\ell\neg Q(\ell) \wedge \\ \quad \forall\ell\forall x((Q(\ell) \to P(x,\ell)) \to Q(\ell)) \wedge \\ \quad \forall\ell\exists x\exists y(x \in_\ell 0 \wedge y \in_\ell 0 \wedge x \prec_\ell y) \wedge \\ \qquad \forall\ell\forall i\big[\forall x,y\forall j\forall k\exists z E \vee \forall x\neg(x \in_\ell s(i))\big] \end{array}\right\} \to (A' \vee \exists\ell\exists u P(u,\ell) \vee \exists\ell Q(\ell)) \quad (8)$$

where S is the conjunction of the standard axioms for 0, successor and $\leq$, with double negations in front of atomic formulas,

$$E \equiv \begin{array}{c} (j \leq i \wedge x \in_\ell j \wedge k \leq i \wedge y \in_\ell k \wedge x \prec_\ell y) \to \\ \to (z \in_\ell s(i) \wedge x \prec_\ell z \wedge z \prec_\ell y) \end{array}$$

and A' is A where every atomic formula is replaced by its double negation, and all quantifiers are relativized to the predicate $R(\ell) \equiv \forall i\exists x(x \in_\ell i)$.

The idea here is that an interpretation $\mathfrak{I}$ will define a sequence $(a_n)_{n\in\mathbb{N}} \to 0$ by $a_n = \mathfrak{I}(Q(\bar{n}))$ where $a_n > a_{n+1}$, and $0 < a_n < 1$ for all n. Let $L^i_\ell = \{x : \mathfrak{I}(x \in_\ell i)\}$ be the i-th ℓ-level. $P(x,\ell)$ orders the set $\bigcup_i L^i_\ell = \{x : \mathfrak{I}(\exists i x \in_\ell i) = 1\}$ in a subordering of $V \cap [0,a_n]$: $x \prec_\ell y$ iff $\mathfrak{I}(x \prec_\ell y) = 1$. Again we force that whenever $x,y \in L^i_\ell$ with $x \prec_\ell y$, there is a $z \in L^{i+1}_\ell$ with $x \prec_\ell z \prec_\ell y$, or, if no possible such z exists, $L^{i+1}_\ell = \emptyset$. Let $r(\ell)$ be the least i so that L^i_ℓ is empty, or ∞ otherwise. If $r(\ell) = \infty$ then there is a densely ordered subset of $V \cap [0,a_\ell]$. So if 0 is not in the perfect kernel, for some sufficiently large L, $r(\ell) < \infty$ for all $\ell > L$. $\mathfrak{I}(R(\ell)) = 1$ iff $r(\ell) = \infty$ hence $\{\ell : \mathfrak{I}(R(\ell)) = 1\}$ is finite whenever the interpretations of P, L, and Q are as intended.

Now if A is classically false in some finite structure $\mathfrak{I}$, we can again choose a $\mathbf{G}_V$-interpretation $\mathfrak{I}^h$ in which the interpretations of P, Q, L are as intended, the number theoretic predicates and functions receive their standard interpretation, there are as many ℓ with $\mathfrak{I}^h(R(\ell)) = 1$ as there are elements in the domain of $\mathfrak{I}$, and the predicates of A behave on $\{\ell : \mathfrak{I}(R(\ell)) = 1\}$ just as they do on $\mathfrak{I}$. $\mathfrak{I}^h \not\models A^h$.

On the other hand, if $\mathfrak{I} \not\models A^h$, then the value of the consequent is < 1. Then as required, for all x, ℓ, $\mathfrak{I}(P(x,\ell)) < 1$ and $\mathfrak{I}(Q(\ell)) < 1$. Since the antecedent, as before, must be $= 1$, this means that $x \prec_\ell y$ expresses a strict ordering of the elements of L^i_ℓ and $\mathfrak{I}(((Q(s(\ell) \to Q(\ell)) \to Q(s(\ell))) = 1$ for all ℓ guarantees that $\mathfrak{I}(Q(s(\ell))) = a_{n+1} < a_n = \mathfrak{I}(Q(\ell))$. The other conditions are likewise seen to hold as intended, so that we can extract a finite countermodel for A based on the interpretation of the predicate symbols of A on $\{\ell : \mathfrak{I}(R(\ell)) = 1\}$, which must be finite. $\square$

6 Finite Gödel sets

Theorem 45. *Suppose* Γ *contains only closed formulas. Then* $\Gamma \models_{V_n} A$ *iff* $\Gamma \vdash_{\mathbf{H}_n} A$.

Proof. If: By Theorem 18, since every instance of $\mathrm{FIN}(n)$ is valid in $\mathbf{G}_n$.

Only if: Suppose $\Gamma \nvdash_{\mathbf{H}_n} A$.

Now consider the set Π of closed formulas of the form

$$\forall \bar{x}_1 \dots \bar{x}_{n+1} (A_1(\bar{x}_1) \to A_2(\bar{x}_2) \vee \dots \vee A_n(\bar{x}_n) \to A_{n+1}(\bar{x}_{n+1})),$$

where $A_1, \dots, A_{n+1}$ ranges over all sequences (with repetitions) of length $n+1$ where each A_i is either $\bot$, $\top$, or $P(\bar{x})$ for some predicate symbol P occurring in Γ or A.

Each formula in Π follows from an instance of $\mathrm{FIN}(n)$ by generalization. Hence, $\Gamma, \Pi \nvdash_{\mathbf{H}} A$. From the (strong) completeness (Theorem 40) of $\mathbf{H}$ for $\mathbf{G}_{\mathbb{R}}$ we know there is an interpretation $\mathfrak{I}_{\mathbb{R}}$ (into $[0,1]$) such that $\mathfrak{I}_{\mathbb{R}}(B) = 1$ for all $B \in \Gamma \cup \Pi$ and $\mathfrak{I}_{\mathbb{R}}(A) < 1$.

For sake of brevity let $\mathrm{Val}^a(\mathfrak{I}_{\mathbb{R}}, \Delta)$ for a set of formulas Δ be the set of all truth values of atomic subformulas of formulas in Δ, i.e., $\mathrm{Val}^a(\mathfrak{I}_{\mathbb{R}}, \Delta) = \{\mathfrak{I}_{\mathbb{R}}(P(\bar{c})) : \bar{c}$ constants from $\mathscr{L}^{\mathfrak{I}}\}$. We claim that $\mathrm{Val}^a(\mathfrak{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ contains at most n elements. To see this, assume that it contains more than n elements. Then there exist atomic subformulas $B_1, \dots, B_{n+1}$ of A or formulas in Γ w.r.t. $\mathfrak{I}$ such that $\mathfrak{I}_{\mathbb{R}}(B_i) > \mathfrak{I}_{\mathbb{R}}(B_{i+1})$ for $i = 1, \dots, n$. Thus, $\mathfrak{I}_{\mathbb{R}}(B_1 \to B_2 \vee \dots \vee B_n \to B_{n+1}) < 1$. But this formula is an instance of a formula $B \in \Gamma$, and so we have a contradiction with $\mathfrak{I}_{\mathbb{R}}(B) = 1$.

Now let $\mathrm{Val}^a(\mathfrak{I}_{\mathbb{R}}, \Gamma \cup \{A\}) = \{0, v_1, \dots, v_k, 1\}$ be sorted in increasing order, and let $h(0) = 0$, $h(1) = 1$, and $h(v_i) = 1 - 1/(i+1)$. Note that any truth value occuring in $\mathrm{Val}(\mathfrak{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ must be one of the elements of $\mathrm{Val}^a(\mathfrak{I}_{\mathbb{R}}, \Gamma \cup \{A\})$. This is easily seen by induction on the complexity of subformulas of $\Gamma \cup \{A\}$ w.r.t. $\mathfrak{I}_{\mathbb{R}}$, as the inf and sup of any subset of the finite set $\mathrm{Val}^a(\mathfrak{I}_{\mathbb{R}}, \Gamma \cup \{A\})$ is a member of the finite set. By Lemma 7, $\mathfrak{I}_h$ is a V_n-interpretation with $\mathfrak{I}_h(B) = h(\mathfrak{I}_{\mathbb{R}}) = 1$ for all $B \in \Gamma$ and $\mathfrak{I}_h(A) = h(\mathfrak{I}_{\mathbb{R}}) < 1$. □

6.1 Remarks on the propositional case and an alternative proof

In the finite case propositional Gödel logics exhibit some interesting properties which we want to mention here.

Proposition 46. *If a propositional formula contains less than* $n-2$ *variables, then it is valid in propositional* $\mathbf{G}_n$ *if and only if it is valid in propositional* $\mathbf{G}$ *(i.e. LC).*

Proof. Due to the projective nature of the the truth functions, every interpretation of a formula will only need at most n different intermediate values (the number of variables plus 0 and 1 for $\bot$ and $\top$). □

Definition 47. An *n-reduct* of a propositional formula A with more than n propositional variables is a formula obtained from A by identifying propositional variables so that the resulting formula contains exactly n variables. If the formula contains less than n propositional variables, the only n-reduct of A is A itself.

It is easy to verify the following propostions:

Proposition 48. *If A (propositional) is valid in $\mathbf{G}_n$, then all n-reducts of A are valid in $\mathbf{G}_n$.*

Proof. Every interpretation of an n-reduct of A can be extended to an interpretation of A. □

Proposition 49. *If all n-reducts of A are provable in $\mathbf{H}$, than A itself is provable in $\mathbf{H}_n$.*

Proof. Assume that the n-reducts of A are $\{A_1, \ldots, A_N\}$ and assume that for $1 \leq i \leq N$, $\vdash_{\mathbf{H}} A_i$. We assume for simplicity that only one variable q has to be identified. The general case is proven in the same way: From $\vdash_{\mathbf{H}} A_i$ we get $\vdash_{\mathbf{H}} (q \leftrightarrow p_i) \to A$, and from this $(q \leftrightarrow p_i) \vdash_{\mathbf{H}} A$, for every $1 \leq i \leq N$. Thus also

$$\bigvee_{i=1}^{N} (q \leftrightarrow p_i) \vdash_{\mathbf{H}} A.$$

Together with $\vdash_{\mathbf{H}_n} \bigvee_{i=1}^{N} (q \leftrightarrow p_i)$ and cut or modus ponens we obtain $\vdash_{\mathbf{H}_n} A$. □

The notion of n-reducts allows us to show the completeness propositional $\mathbf{H}_n$ with respect to finite-valued propositional Gödel logics.

Theorem 50. *A propositional formula is valid in propositional $\mathbf{G}_n$ iff it is provable in propositional $\mathbf{H}_n$.*

Proof. The right-to-left direction is done by induction on proofs. If A is valid in $\mathbf{G}_n$, then all n-reducts of A are valid in $\mathbf{G}_n$, too (Proposition 48). Thus, all the n-reducts are valid in $\mathbf{G}_{\mathbb{R}}$ (Proposition 46), and due to the completeness of $\mathbf{H}$ with respect to $\mathbf{G}_{\mathbb{R}}$, all n-reducts are provable in (propositional) $\mathbf{H}$. Using Proposition 49 we obtain the A is provable in $\mathbf{H}_n$. □

Together with the Herbrand Theorem for Gödel logics (Theorem 58) this allows us to give an alternative proof of Theorem 45, although only for the case of weak completeness.

Alternative proof for the weak variant of Theorem 45. Observe that in the finite case all quantifier shift rules are valid. Thus, we can transform A into a prenex formula A^*. Later on we will show that for prenex formulas in finite valued logics the Herbrand Theorem is valid, i.e., there is a propositional formula B such that A^* is valid iff B is valid (Theorem 58). Thus A is valid iff the propositional formula B is valid. Using the completeness theorem for the propositional case just proven we see that B is provable in propositional $\mathbf{H}_n$, and thus A is provable in first order $\mathbf{H}_n$.

Note that this proof only provide weak or normal completeness, i.e. a single formula is valid iff it is provable. □

7 Fragments

7.1 Prenex fragments

One interesting restriction of the axiomatizability problem is the question whether the prenex fragment of $\mathbf{G}_V$, i.e., the set of prenex formulas valid in $\mathbf{G}_V$, is axiomatizable. This is non-trivial, since in general in Gödel logics, arbitrary formulas are not equivalent to prenex formulas. Thus, so far the proofs of non-axiomatizability of the logics treated in Sections 4 and 5.3 do not establish the non-axiomatizability of their prenex fragments, nor do they exclude the possibility that the corresponding prenex fragments are r.e. We investigate this question in this section, and show that the prenex fragments of all finite and uncountable Gödel logics are r.e., and that the prenex fragments of all countably infinite Gödel logics are not r.e. The axiomatizability result is obtained from a version of Herbrand's Theorem for finite and uncountably-valued Gödel logics, which is of independent interest. The non-axiomatizability of countably infinite Gödel logics is obtained as a corrolary of Theorem 36.

Let V be a Gödel set which is either finite or uncountable. Let $\mathbf{G}_V$ be a Gödel logic with such a truth value set. We show how to effectively associate with each prenex formula A a quantifier-free formula A^* which is valid in $\mathbf{G}_V$ if and only if A is a tautology. The axiomatizability of the prenex fragment of $\mathbf{G}_V$ then follows from the axiomatizability of $\mathbf{LC}$ (in the infinite-valued case) and propositional $\mathbf{G}_m$ (in the finite-valued case).

Definition 51 (Herbrand form)**.** Given a prenex formula $A \equiv \mathsf{Q}_1 x_1 \dots \mathsf{Q}_n x_n B(\bar{x})$ (B quantifier free), the *Herbrand form* A^H of A is $\exists x_{i_1} \dots \exists x_{i_m} B(t_1, \dots, t_n)$, where $\{x_{i_j} : 1 \leq j \leq m\}$ is the set of existentially quantified variables in A, and t_i is x_{i_j} if $i = i_j$, or is $f_i(x_{i_1}, \dots, x_{i_k})$ if x_i is universally quantified and $k = \max\{j : i_j < i\}$. We will write $B(t_1, \dots, t_n)$ as $B^F(x_{i_1}, \dots, x_{i_m})$ if we want to emphasize the free variables.

Lemma 52. *If A is prenex and $\mathbf{G}_V \models A$, then $\mathbf{G}_V \models A^H$.*

Proof. Follows from the usual laws of quantification, which are valid in all Gödel logics. □

Our next main result will be Herbrand's theorem for $\mathbf{G}_V$ for V uncountable or finite. The *Herbrand universe* $\mathrm{HU}(B^F)$ of B^F is the set of all variable-free terms which can be constructed from the set of function symbols occurring in B^F. To prevent $\mathrm{HU}(B^F)$ from being finite or empty we add a constant and a function symbol of positive arity if no such symbols appear in B^F. The *Herbrand base* $\mathrm{HB}(B^F)$ is the set of atoms constructed from the predicate symbols in B^F and the terms of the Herbrand universe. In the next theorem we will consider the Herbrand universe of a formula $\exists \bar{x} B^F(\bar{x})$. We fix a non-repetitive enumeration $C_1, C_2, \dots$ of $\mathrm{HB}(B^F)$, and let $X_\ell = \{\bot, C_1, \dots, C_\ell, \top\}$ (we may take $\top$ to be a formula which is always $= 1$). $B^F(\bar{t})$ is an ℓ*-instance* of $B^F(\bar{x})$ if the atomic subformulas of $B^F(\bar{t})$ are in X_ℓ.

Definition 53. An ℓ*-constraint* is a non-strict linear ordering $\preceq$ of X_ℓ s.t. $\bot$ is minimal and $\top$ is maximal. An interpretation $\mathfrak{I}$ *fulfils* the constraint $\preceq$ provided for all $C, C' \in X_\ell$, $C \preceq C'$ iff $\mathfrak{I}(C) \leq \mathfrak{I}(C')$. We say that the constraint $\preceq'$ on $X_{\ell+1}$ *extends* $\preceq$ if for all $C, C' \in X_\ell$, $C \preceq C'$ iff $C \preceq' C'$.

Lemma 9 showed that if $h : V \to W$ is a **G**-embedding and $\mathfrak{I}$ is a V-interpretation, then $h(\mathfrak{I}(A)) = \mathfrak{I}_h(A)$ for any formula A. If no quantifiers are involved in A, this also holds without the requirement of continuity. For the following proof we need a similar notion. Let V be a Gödel sets, X a set of atomic formulas, and suppose there is an order-preserving, strictly monotone $h\colon \{\mathfrak{I}(C)\colon C \in X\} \to V$ which is so that $h(1) = 1$ and $h(0) = 0$. Call any such h a *truth value injection on* X. Now suppose B is a quantifier-free formula, and X its set of atomic subformulas. Two interpretations $\mathfrak{I}$, $\mathfrak{J}$ are *compatible on* X if $\mathfrak{I}(C) \leq \mathfrak{I}(C')$ iff $\mathfrak{J}(C) \leq \mathfrak{J}(C')$ for all $C \in X$.

Proposition 54. *Let B^F be a quantifier free formula, and X its set of atomic subformulas together with $\top$, $\bot$. If $\mathfrak{I}$, $\mathfrak{J}$ are compatible on X, then there is a truth value injection h on X with $h(\mathfrak{I}(C^F)) = \mathfrak{J}(C^F)$.*

Proof. Let $h(\mathfrak{I}(C)) = \mathfrak{J}(C)$ for $B \in X$. Since $\mathfrak{I}$, $\mathfrak{J}$ are compatible on X, $\mathfrak{I}(C) \leq \mathfrak{I}(C')$ iff $\mathfrak{J}(C) \leq \mathfrak{J}(C')$, and hence $\mathfrak{I}(C) \leq \mathfrak{I}(C')$ iff $h(\mathfrak{I}(C)) \leq h(\mathfrak{J}(C'))$ and h is strictly monotonic. The conditions $h(0) = 0$ and $h(1) = 1$ are satisfied by definition, since $\top$, $\bot \in X$. We get $h(\mathfrak{I}(B^F)) = \mathfrak{J}(B^F)$ by induction on the complexity of A. ☐

Proposition 55. *(a) If $\preceq'$ extends $\preceq$, then every $\mathfrak{I}$ which fulfills $\preceq'$ also fulfills $\preceq$. (b) If $\mathfrak{I}$, $\mathfrak{J}$ fulfill the ℓ-constraint $\preceq$, then there is a truth value injection h on X_ℓ with $h(\mathfrak{I}(B^F(\bar{t}))) = \mathfrak{J}(B^F(\bar{t}))$ for all ℓ-instances $B^F(\bar{t})$ of $B^F(\bar{x})$; in particular, $\mathfrak{I}(B^F(\bar{t})) = 1$ iff $\mathfrak{J}(B^F(\bar{t})) = 1$.*

Proof. (a) Obvious. (b) Follows from Proposition 54 together with the observation that $\mathfrak{I}$ and $\mathfrak{J}$ both fulfill $\preceq$ iff they are compatible on X_ℓ. ☐

Lemma 56. *Let B^F be a quantifier-free formula, and let V be a finite or uncountably infinite Gödel set. If $\mathbf{G}_V \models \exists \bar{x} B^F(\bar{x})$ then there are tuples $\bar{t}_1, \ldots \bar{t}_n$ of terms in $U(B^F)$, such that $\mathbf{G}_V \models \bigvee_{i=1}^n B^F(\bar{t}_i)$.*

Proof. Suppose first that V is uncountable. By Theorem 32, V contains a dense linear subordering. We construct a "semantic tree" T; i.e., a systematic representation of all possible order types of interpretations of the atoms C_i in the Herbrand base. T is a rooted tree whose nodes appear at levels. Each node at level ℓ is labelled with an ℓ-constraint.

T is constructed in levels as follows: At level 0, the root of T is labelled with the constraint $\bot < \top$. Let ν be a node added at level ℓ with label $\preceq$, and let T_ℓ be the set of terms occurring in X_ℓ. Let (*) be: For every interpretation $\mathfrak{I}$ which fulfils $\preceq$, there is some ℓ-instance $B^F(\bar{t})$ so that $\mathfrak{I}(B^F(\bar{t})) = 1$. If (*) obtains, ν is a leaf node of T, and no successor nodes are added at level $\ell + 1$.

Note that by Proposition 55(b), any two interpretations which fulfill $\preceq$ make the same ℓ-instances of $B^F(\bar{t})$ true; hence ν is a leaf node if and only if there is an ℓ-instance $A(\bar{t})$ s.t. $\mathfrak{I}(A(\bar{t})) = 1$ for all interpretations $\mathfrak{I}$ that fulfil $\preceq$.

If (*) does not obtain, for each $(\ell+1)$-constraint $\preceq'$ extending $\preceq$ we add a successor node ν' labelled with $\preceq'$ to ν at level $\ell + 1$.

We now have two cases:

(1) T is finite. Let $\nu_1, \ldots, \nu_m$ be the leaf nodes of T of levels ℓ_1, …, ℓ_m, each labelled with a constraint $\preceq_1$, …, $\preceq_m$. By (*), for each j there is an ℓ_j-instance $B^F(\bar{t}_j)$

with $\mathfrak{I}(B^F(\bar{t})) = 1$ for all $\mathfrak{I}$ which fulfill $\preceq_j$. It is easy to see that every interpretation fulfills at least one of the $\preceq_j$. Hence, for all $\mathfrak{I}$, $\mathfrak{I}(B^F(\bar{t}_1) \vee \ldots \vee B^F(\bar{t}_m)) = 1$, and so $\mathbf{G}_V \models \bigvee_{i=1}^m B^F(\bar{t}_i)$.

(2) T is infinite. By König's lemma, T has an infinite branch with nodes v_0, v_1, $v_2, \ldots$ where v_ℓ is labelled by $\preceq_\ell$ and is of level ℓ. Each $\preceq_{\ell+1}$ extends $\preceq_\ell$, hence we can form $\preceq = \bigcup_\ell \preceq_\ell$. Let $V' \subseteq V$ be a non-trivial densely ordered subset of V, let $V' \ni c < 1$, and let $V'' = V' \cap [0, c)$. V'' is clearly also densely ordered. Now let V_c be $V'' \cup \{0, 1\}$, and let $h : B(A(x)) \cup \{\bot, \top\} \to V_c$ be an injection which is so that, for all $A_i, A_j \in B(A(x))$, $h(A_i) \leq h(A_j)$ iff $A_i \preceq A_j$, $h(\bot) = 0$ and $h(\top) = 1$. We define an interpretation $\mathfrak{I}$ by: $f^{\mathfrak{I}}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$ for all n-ary function symbols f and $P^{\mathfrak{I}}(t_1, \ldots, t_n) = h(P(t_1, \ldots, t_n))$ for all n-ary predicate symbols P (clearly then, $\mathfrak{I}(A_i) = h(A_i)$). By definition, $\mathfrak{I}$ ℓ-fulfills $\preceq_\ell$ for all ℓ. By (*), $\mathfrak{I}(A(\bar{t})) < 1$ for all ℓ-instances $A(\bar{t})$ of $A(x)$, and by the definition of V_c, $\mathfrak{I}(A(\bar{t})) < c$. Since every $A(\bar{t})$ with $\bar{t} \in U(A(x))$ is an ℓ-instance of $A(x)$ for some ℓ, we have $\mathfrak{I}(\exists x A(\bar{x})) \leq c < 1$.This contradicts the assumption that $\mathbf{G}_V \models \exists \bar{x} A(\bar{x})$.

If V is finite, the proof is the similar, except simpler. Suppose $|V| = n$. Call a constraint $\preceq$ *n-admissible* if there is some V-interpretation $\mathfrak{I}$ which fulfills it. Such $\preceq$ have no more than n equivalence classes under the equivalence relation $C \sim C'$ iff $C \preceq C'$ and $C' \preceq C$. In the construction of the semantic tree above, replace each mention of ℓ-constraints by n-admissible ℓ-constraints. The argument in the case where the resulting tree is finite is the same. If T is infinite, then the resulting order $\preceq = \bigcup_\ell \preceq_\ell$ is n-admissible, since all $\preceq_\ell$ are. Let $c = \max\{b : b \in V, b < 1\}$ and $V_c = V$. The rest of the argument goes through without change. □

Lemma 57. *Let $\exists \bar{x} B^F(\bar{x})$ be the Herbrand form of the prenex formula $A \equiv \overline{\mathsf{Q}}_i B(\bar{y}_i)$, and let $\bar{t}_1, \ldots, \bar{t}_m$ be tuples of terms in* $\mathrm{HU}(B^F)$. *If $\mathbf{G}_V \models \bigvee_{i=1}^m B^F(\bar{t}_i)$, then $\mathbf{G}_V \models A$.*

Proof. For any Gödel set V, the following rules are valid in $\mathbf{G}_V$:

(1) $A \vee B \vdash B \vee A$.
(2) $(A \vee B) \vee C \vdash A \vee (B \vee C)$.
(3) $A \vee (B \vee B) \vdash A \vee B$.
(4) $A(y) \vdash \forall x A(x)$.
(5) $A(t) \vdash \exists x A(x)$.
(6) $\forall x (A(x) \vee B) \vdash \forall x A(x) \vee B$.
(7) $\exists x (A(x) \vee B) \vdash \exists x A(x) \vee B$.

(x is not free in B.) The result follows from [BCF01], Lemma 6, and are also easily verified directly. □

Theorem 58. *Let A be prenex, $\exists \bar{x} B^F(\bar{x})$ its Herbrand form, and let V be a finite or uncountably infinite Gödel set. Then $\mathbf{G}_V \models A$ iff there are tuples $\bar{t}_1, \ldots \bar{t}_m$ of terms in* $\mathrm{HU}(B^F)$, *such that $\mathbf{G}_V \models \bigvee_{i=1}^m B^F(\bar{t}_i)$.*

Proof. If: This is Lemma 57. Only if: By Lemma 52 and Lemma 56. □

Remark. An alternative proof of Herbrand's theorem can be obtained using the analytic calculus *HIF* ("Hypersequent calculus for Intuitionistic Fuzzy logic") [BZ00].

Theorem 59. *The prenex fragment of a Gödel logic based on a truth value set V which is either finite or uncountable infinite is axiomatizable. An axiomatization is given by the standard axioms and rules for* **LC** *extended by the rules (4)–(7) of the proof of Lemma 57. For the m-valued case add the characteristic axiom for* $\mathbf{G}_m$, $G_m \equiv \bigvee_{i=1}^{m} \bigvee_{j=i+1}^{m+1}((A_i \to A_j) \wedge (A_j \to A_i))$.

Proof. Completeness: Let $\overline{\mathsf{Q}}\overline{y}_i B(\overline{y})$ be a prenex formula valid in $\mathbf{G}_V$. By Theorem 58, a Herbrand disjunction $\bigvee_{i=1}^{n} B^F(\bar{t}_i)$ is a tautology in $\mathbf{G}_V$. Hence, it is provable in **LC** or $\mathbf{LC} + G_m$ [Got01, Chapter 10.1]. $\overline{\mathsf{Q}}\overline{y}B(\overline{y})$ is provable by Lemma 57.

Soundness: The rules in the proof of Lemma 57 are valid in $\mathbf{G}_V$. In particular, note that $\forall x(A(x) \vee B) \to (\forall x A(x) \vee B)$ with x not free in B is valid in all Gödel logics, and $\exists x(A(x) \vee B) \to \exists x A(x) \vee B$ is already intuitionistically valid. □

In Theorem 36, we showed that for every first-order formula A, there is a formula A^g which is valid in $\mathbf{G}_V$ for V countably infinite iff A is valid in every finite classical interpretation. We now strengthen this result to show that the prenex fragment of $\mathbf{G}_V$ (for V countably infinite) is likewise not axiomatizable. This is done by showing that if A is prenex, then there is a formula A^G which is also prenex and which is valid in $\mathbf{G}_V$ iff A^g is. Note that not all quantifier shifting rules are generally valid in Gödel logics, so we have to show that for the particular case of formulas of the form of A^g, there is a prenex formula which is valid in $\mathbf{G}_V$ iff A^g is.

Theorem 60. *If V is countably infinite, the prenex fragment of* $\mathbf{G}_V$ *is not r.e.*

Proof. By the proof of Theorem 36, a formula A is true in all finite models iff $\mathbf{G}_V \models A^g$. A^g is of the form $B \to (A' \vee \exists u P(u))$. We show that A^g is validity-equivalent in $\mathbf{G}_V$ to a prenex formula.

From Lemma 35 we see that each crisp formula is equivalent to a prenex formula; let A_0 be a prenex form of A'. Since all quantifier shifts for conjunctions are valid, the antecedent B of A^g is equivalent to a prenex formula $\mathsf{Q}_1 x_1 \ldots \mathsf{Q}_n x_n B_0(x_1, \ldots, x_n)$. Hence, A^g is equivalent to $\overline{\mathsf{Q}}\overline{x}B_0(\overline{x}) \to (A_0 \vee \exists u P(u))$.

Let Q_i' be $\exists$ if Q_i is $\forall$, and $\forall$ if Q_i is $\exists$, let $C \equiv A_0 \vee \exists u P(u)$, and $v = \mathfrak{I}(\exists u P(u)))$. We show that $\overline{\mathsf{Q}}\overline{x}B_0(\overline{x}) \to C$ is equivalent to $\overline{\mathsf{Q}}'\overline{x}(B_0(\overline{x}) \to C)$ by induction on n. Let $\overline{\mathsf{Q}}\overline{x}B_0 \equiv \mathsf{Q}_1 x_1 \ldots \mathsf{Q}_i x_i B_1(d_1, \ldots, d_{i-1}, x_i)$. Since quantifier shifts for $\exists$ in the antecent of a conditional are valid, we only have to consider the case $\mathsf{Q}_i = \forall$. Suppose $\mathfrak{I}(\forall x_i B_1(\overline{d}, x_i) \to C) \neq \mathfrak{I}(\exists x_i (B_1(\overline{d}, x_i) \to C)$. This can only happen if $\mathfrak{I}(\forall x_i B_1(\overline{d}, x_i)) = \mathfrak{I}(C) < 1$ but $\mathfrak{I}(B_1(\overline{d}, c)) > \mathfrak{I}(C) \geq v$ for all c. However, it is easy to see by inspecting B that $\mathfrak{I}(B_1(\overline{d}, c))$ is either $= 1$ or $\leq v$.

Now we show that $\mathfrak{I}(B_0(\overline{d}) \to (A_0 \vee \exists u P(u))) = \mathfrak{I}(\exists u(B_0(\overline{d}) \to (A_0 \vee P(u))))$. If $\mathfrak{I}(A_0) = 1$, then both sides equal $= 1$. If $\mathfrak{I}(A_0) = 0$, then $\mathfrak{I}(A_0 \vee \exists u P(u)) = v$. The only case where the two sides might differ is if $\mathfrak{I}(B_0(\overline{d})) = v$ but $\mathfrak{I}(A_0 \vee P(c)) = \mathfrak{I}(P(c)) < v$ for all c. But inspection of B_0 shows that $\mathfrak{I}(B_0(\bar{t})) = 1$ or $= \mathfrak{I}(P(e))$ for some $e \in \overline{d}$ (the only subformulas of $B_0(\overline{d})$ which do not appear negated are of the form $e' \prec e$). Hence, if $\mathfrak{I}(B_0(\overline{d})) = v$, then for some e, $\mathfrak{I}(P(e)) = v$.

Last we consider the quantifiers in $A_0 \equiv \overline{\mathsf{Q}}\overline{y}A_1$. Since A_0 is crisp, $\mathfrak{I}(B_0(\overline{d}) \to (A_0 \vee P(c))) = \mathfrak{I}(\overline{\mathsf{Q}}\overline{y}(B_0(\overline{d}) \to (A_1 \vee P(c))))$ for all $\overline{d}$, c. To see this, first note that shifting quantifiers across $\vee$, and shifting universal quantifiers out of the consequent of a conditional is always possible. Hence it suffices to consider the case of $\exists$. $\mathfrak{I}(\exists y A_2)$ is either $= 0$ or $= 1$. In the former case, both sides equal $\mathfrak{I}(B_0(\overline{d}) \to P(d))$, in the latter, both sides equal 1. $\square$

In summary, we obtain the following characterization of axiomatizability of prenex fragments of Gödel logics:

Theorem 61. *The prenex fragment of $\mathbf{G}_V$ is axiomatizable if and only if V is finite or uncountable. The prenex fragments of any two $\mathbf{G}_V$ where V is uncountable coincide.*

7.2 $\perp$-free fragments

In the following we will denote the $\perp$-free fragment of $\mathbf{G}_V$ with $\mathbf{G}_V^{\not\perp}$. $\mathbf{G}_V^{\not\perp}$ is the set of all $\mathbf{G}_V$-valid formulas which do not contain $\perp$ (and hence also no $\neg$). First we show that the only candidates for r.e. fragments are the $\perp$-free fragments of $\mathbf{G}_V$ where V is uncountable and either $0 \in V^\infty$ or 0 is isolated V.

Lemma 62. *If $\mathbf{G}_V$ is not r.e., then $\mathbf{G}_V^{\not\perp}$ is also not r.e.*

Define A^b as the formula obtained from A by replacing all occurences of $\perp$ with the new propositional variable b (a 0-place predicate symbol). Then define A^* as

$$A^* = \Big(\bigwedge_{P \in A} \forall \bar{x}(b \to P(\bar{x}))\Big) \to A^b$$

where $P \in A$ means that P ranges over all predicate symbols occuring in A. We will first prove a lemma relating A^* and A:

Lemma 63.

$$\mathbf{G}_V \models A \quad \textit{iff} \quad \mathbf{G}_V^{\not\perp} \models A^*$$

Proof. If: Replace b by $\perp$.

Only if: Suppose $\mathbf{G}_V^{\not\perp} \not\models A^*$. Thus, there is an interpretation $\mathfrak{I}_0$ such that $\mathfrak{I}_0(A^*) < 1$. By Proposition 11 and Lemma 12, there is an interpretation $\mathfrak{I}$ such that $\mathfrak{I}(A^b) < 1$ and $\mathfrak{I}((\bigwedge_{P \in A} \forall \bar{x}(b \to P(\bar{x})))) = 1$. Because of the latter, for every atomic subformula B of A, $\mathfrak{I}(B) \geq \mathfrak{I}(b) = v$. Define $\mathfrak{I}'(B)$ for atomic subformulas B of A by

$$\mathfrak{I}'(B) = \begin{cases} 0 & \mathfrak{I}(B) \leq v \\ \mathfrak{I}(B) & \mathfrak{I}(B) > v \end{cases}$$

(and arbitrary for other atomic formulas). It is easily seen by induction that $\mathfrak{I}'(B) = \mathfrak{I}(B)$ if $\mathfrak{I}(B) > v$, and if $\mathfrak{I}(B) = v$, then $\mathfrak{I}'(B) = v$ or $= 0$. In particular, $\mathfrak{I}'(A^b) < 1$. But, of course, $\mathfrak{I}'(b) = \mathfrak{I}'(\perp) = 0$, and hence $\mathfrak{I}'(A^b) = \mathfrak{I}'(A)$.

$\square$

Proof of Lemma 62. If $\mathbf{G}_V^{\not\perp}$ were recursively enumerable, then by Lemma 63, $\mathbf{G}_V$ would also be recursively enumerable. $\square$

Thus, by Theorem 36, we only have two candidates for axiomatizable $\perp$-free fragments: both truth-value sets have a non-empty perfect kernel P, and in the one case $0 \in P$ and in the other $0 \notin P$ but 0 is isolated. The prototypical Gödel sets for these cases are $V_1 = [0,1]$ and $V_2 = \{0\} \cup [1/2,1]$. We will show that the $\perp$-free fragments of these two logics coincide, thus in fact proving that there is only one axiomatizable $\perp$-free fragment.

Lemma 64. *Let $V_1 = [0,1]$ and $V_2 = \{0\} \cup [1/2,1]$. The $\perp$-free fragments of $\mathbf{G}_{V_1}$ and $\mathbf{G}_{V_2}$ coincide, i.e.*

$$\mathbf{G}_{V_1}^{\not\perp} \models A \quad \textit{iff} \quad \mathbf{G}_{V_2}^{\not\perp} \models A$$

Proof. Only if: obvious, since a counter-example in V_2 actually also is a counter-example in V_1.

If: Suppose that $\mathbf{G}_{V_1}^{\not\perp} \not\models A$, i.e., there is an $\mathfrak{I}_1$ such that $\mathfrak{I}_1(A) < 1$. Define $\mathfrak{I}_2$ for all atomic subformulas B of A by $\mathfrak{I}_2(B) = 1/2(1 + \mathfrak{I}_1(B))$. By Lemma 7 and the remark following it we see that the definition of $\mathfrak{I}_2$ extends to all formulas. □

Theorem 65. *The $\perp$-free fragment of $\mathbf{G}_V$ is recursively axiomatizable if and only if V is finite or uncountable and either 0 belongs to V^∞ or is isolated. The $\perp$-free fragment of any two such V coincide.*

Proof. From Lemma 62, Lemma 64 and Theorem 33 for the uncountable case. The finite case is obvious as the additional axioms $\textsc{fin}(n)$ do not contain $\perp$. □

7.3 $\forall$-free fragments

In the following we will denote the $\exists$-fragment of $\mathbf{G}_V$ with $\mathbf{G}_V^{\exists}$. It is the set of all formulas valid in $\mathbf{G}_V$ which do not contain $\forall$.

First we show, as in the case of the $\perp$-free fragment, that the only candidates for axiomatizable fragments are the two uncountable ones, $0 \in P$ and 0 isolated. We will do this by showing that the formulas used to reduce validity in the other cases to Trachtenbrodt's Theorem are validity-equivalent to $\forall$-free formulas.

Lemma 66. *If $A(x)$ and B are $\forall$-free, then*

$$\models \forall x A(x) \to B \quad \textit{iff} \quad \models \exists x (A(x) \to B)$$

Proof. If: This is a valid quantifier shift rule.

Only if: Suppose that $\not\models \exists x(A(x) \to B)$, i.e., there is an interpretation $\mathfrak{I}$ such that $\mathfrak{I}(\exists x(A(x) \to B) < 1$. But this implies that

$$\forall u \in U \quad \mathfrak{I}(A(u)) > \mathfrak{I}(B). \tag{9}$$

Now define $\mathfrak{I}'(Q)$ for atomic subformulas Q of A by

$$\mathfrak{I}'(Q) = \begin{cases} \mathfrak{I}(Q) & \text{if } \mathfrak{I}(Q) \leq \mathfrak{I}(B) \\ 1 & \text{if } \mathfrak{I}(Q) > \mathfrak{I}(B). \end{cases}$$

Then (i) If C is $\forall$-free and $\mathfrak{I}(C) > \mathfrak{I}(B)$, then $\mathfrak{I}'(C) = 1$, and if $\mathfrak{I}(C) \leq \mathfrak{I}(B)$, then $\mathfrak{I}'(C) = \mathfrak{I}(C)$; and (ii) $\mathfrak{I}'(\forall x A(x)) = 1$

(i) For atomic C this is the definition of $\mathfrak{I}'$. The cases for $\wedge$, $\vee$, and $\rightarrow$ are trivial. Now let $C \equiv \exists x D(x)$. If $\mathfrak{I}(\exists x D(x)) > \mathfrak{I}(B)$, then for some $u \in U^{\mathfrak{I}}$, $\mathfrak{I}(D(u)) > \mathfrak{I}(B)$. By induction hypthesis, $\mathfrak{I}'(D(u)) = 1$ and hence $\mathfrak{I}'(\exists x D(x)) = 1$. Otherwise, $\mathfrak{I}(\exists x D(x)) \leq \mathfrak{I}(B)$, in which case $\mathfrak{I}'(D(u)) = \mathfrak{I}'(D(u))$ for all u. (ii) By (9), for all $u \in U$, $\mathfrak{I}(A(u)) > \mathfrak{I}(B)$, hence, by (i), $\mathfrak{I}'(A(u)) = 1$.

By (i) and (ii) we have that $\mathfrak{I}'(\forall x A(x)) = 1$ and $\mathfrak{I}'(B) = \mathfrak{I}(B) < 1$, thus $\mathfrak{I}'(\forall x A(x) \rightarrow B) < 1$, i.e., $\not\models \forall x A(x) \rightarrow B$. □

Note that in the preceding Lemma we can replace the prefix of $A(x)$ by a string of universal quantifiers and the same proof will work.

Lemma 67. *If* $\mathbf{G}_V$ *is not recursively enumerable, then also* $\mathbf{G}_V^{\exists}$.

Proof. It is sufficient to show that Formula 7 for A^g as given on page 16 and Formula 8 for A^h as given on page 22 are validity-equivalent to $\forall$-free formulas.

If we only consider the quantifier structure of these formulas and apply valid quantifier shifting rules, including the shifting rule for crisp formulas given in Lemma 35, we obtain in both cases formulas which are of the form

$$\forall \bar{x} A(\bar{x}) \rightarrow B$$

where $A(\bar{x})$ and B are $\forall$-free. By to Lemma 66 we see that both formulas are validity equivalent to $\forall$-free formulas. □

As for the $\bot$-free fragments, it turns out that the two prototypical examples of Gödel sets create the same $\exists$-fragment:

Lemma 68. *Let* $V_1 = [0,1]$ *and* $V_2 = \{0\} \cup [1/2, 1]$. *The* $\exists$*-fragments of* $\mathbf{G}_{V_1}$ *and* $\mathbf{G}_{V_2}$ *coincide, i.e.*

$$\mathbf{G}_{V_1}^{\exists} \models A \quad \textit{iff} \quad \mathbf{G}_{V_2}^{\exists} \models A$$

Proof. Only if: obvious, since a counter-example in V_2 actually also is a counter-example in V_1.

If: Suppose that $\mathbf{G}_{V_1}^{\exists} \not\models A$, i.e., there is an $\mathfrak{I}_1$ such that $\mathfrak{I}_1(A) < 1$. Define $\mathfrak{I}_2$ for all atomic subformulas B of A by $\mathfrak{I}_2(B) = 1/2(1 + \mathfrak{I}_1(B))$ if $\mathfrak{I}_1(B) > 0$ and $= 0$ if $\mathfrak{I}_1(Q) = 0$. By Lemma 7 and the remark following it we see that the definition of $\mathfrak{I}_2$ extends to all formulas. □

Theorem 69. *The* $\exists$*-fragment of* $\mathbf{G}_V$ *is r.e. if and only if* V *is finite or uncountable and either* 0 *belongs to* V^{∞} *or is isolated. The* $\exists$*-fragment of any two such* V *coincide.*

Proof. From Lemma 67, Lemma 68 and Theorem 33 for the uncountable case. The finite case is obvious as the additional axioms $\textsc{fin}(n)$ do not contain universal quantifiers. □

8 Conclusion

In the preceding sections, we have given a complete characterization of the r.e. and non-r.e. first-order Gödel logics. Our main result is that there are two distinct r.e. infinite-valued Gödel logics, viz., $\mathbf{G}_{\mathbb{R}}$ and $\mathbf{G}^0$. What we have not done, however, is investigate how many *non*-r.e. Gödel logics there are. It is known that there are continuum-many different propositional consequence relations and continuum-many different propositional quantified Gödel logics [BV00]. In forthcoming work [BGP], it is shown that there are only countably many first-order Gödel logics. Although this result goes some way to clarifying the situation, a criterion of identity of Gödel logics using some topological property of the underlying truth value set is a desideratum. We have only given (Lemma 9) a sufficient condition: if there is a continuous bijection between V and V', then $\mathbf{G}_V = \mathbf{G}_{V'}$. But this condition is not necessary: any pair of non-isomorphic uncountable Gödel sets with 0 contained in the perfect kernel provides a quick counterexample (as any two such sets determine $\mathbf{G}_{\mathbb{R}}$ as their logic). Such a topological characterization of first-order infinite valued Gödel logics could then be used to obtain a more fine-grained analysis of the complexity of the non-r.e. Gödel logics. As noted already, these also differ in the degree to which they are non r.e. [Háj05].

Another avenue for future research would be to carry out the characterization offered here for extensions of the language. Candidates for such extensions are the addition of the projection modalities ($\triangle a = 0$ if $a = 1$ and $= 1$ if $a < 1$), of the globalization operator of [TT86], or of the involutive negation ($\sim a = 1 - a$). It is known that $\mathbf{G}_{\mathbb{R}}$ with the addition of these operators is still axiomatizable. The presence of the projection modality, in particular, disturbs many of the nice features we have been able to exploit in this paper, for instance, in the presence of $\triangle$ the crucial Lemma 12 and Proposition 13 no longer hold. Thus, not all of our results go through for the extended language and new methods will have to be developed.

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