CERES for Propositional Proof Schemata

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Introduction

Basic Conventions

- Set of index variables is a set of variables over natural numbers.
- Linear arithmetic expression: built on the signature $0, s, +$ and on a set of index variables.
- We denote:
  - Linear arithmetic expressions: by $a, b, \ldots$,
  - Natural numbers: by $\alpha, \beta, \ldots$,
  - Bound index variables: by $i, j, l, \ldots$,
  - Parameters (free index variables): by $k, m, n, \ldots$.

Basic Conventions (ctd.)

- Indexed proposition is an expression of the form p_a, where a is a linear arithmetic expression.
- Propositional variable is an indexed proposition p_a, where $a \in \mathbb{N}$.
- Formula schema: are built as usual and denoted by $A, B, \ldots$.
- The notation $A(k)$: indicate a parameter k in A. Then $A(a)$ is $A\{k \leftarrow a\}$.

Schematic LK

Basic Notions

- Sequent Schema is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are multisets of formula schemata.
- Initial Sequent Schema is an expression of the form $A \vdash A$, where A is an indexed proposition.
- Proof Link is an expression $\frac{(\varphi(a))}{S}$ (dashed line).

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules:

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: $\wedge$ introduction:

$$\frac{A, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge: l1 \qquad \frac{B, \Gamma \vdash \Delta}{A \wedge B, \Gamma \vdash \Delta} \wedge: l2$$

$$\frac{\Gamma \vdash \Delta, A \qquad \Pi \vdash \Lambda, B}{\Gamma, \Pi \vdash \Delta, \Lambda, A \wedge B} \wedge: r$$

Equivalences: $A_0 \equiv \bigwedge_{i=0}^{0} A_i$ and $(\bigwedge_{i=0}^{n} A_i) \wedge A_{n+1} \equiv \bigwedge_{i=0}^{n+1} A_i$

Calculus LKS

- Axioms: initial sequent schemata or proof links.
- Rules: $\vee$ introduction:

$$\frac{A, \Gamma \vdash \Delta \qquad B, \Pi \vdash \Lambda}{A \vee B, \Gamma, \Pi \vdash \Delta, \Lambda} \vee : l$$

$$\frac{\Gamma \vdash \Delta, A}{\Gamma \vdash \Delta, A \vee B} \vee : r1 \qquad \frac{\Gamma \vdash \Delta, B}{\Gamma \vdash \Delta, A \vee B} \vee : r2$$

Equivalences: $A_0 \equiv \bigvee_{i=0}^{0} A_i$ and $(\bigvee_{i=0}^{n} A_i) \vee A_{n+1} \equiv \bigvee_{i=0}^{n+1} A_i$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: $\neg$ introduction:

$$\frac{\Gamma \vdash \Delta, A}{\neg A, \Gamma \vdash \Delta}\ \neg : l \qquad \frac{A, \Gamma \vdash \Delta}{\Gamma \vdash \Delta, \neg A}\ \neg : r$$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: Weakening rules:

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\, w: l \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\, w: r$$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: Contraction rules:

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\, c: l \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\, c: r$$

Calculus **LKS**

- Axioms: initial sequent schemata or proof links.
- Rules: Cut rule:

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\ cut$$

LKS-proof

- Derivation is a directed tree with nodes as sequences and edges as rules.
- **LKS**-proof of the sequence S is a derivation of S with axioms as leaf nodes.
- An **LKS**-proof is called ground if it does not contain parameters and proof links.

Proof Schemata

- Let $\psi_1, \ldots, \psi_\alpha$ be proof symbols and $S_1(n), \ldots, S_\alpha(n)$ be sequents.
- Proof schema Ψ of a sequent $S_1(n)$ is a tuple of pairs

$$\langle(\pi_1(0), \nu_1(k+1)), \ldots, (\pi_\alpha(0), \nu_\alpha(k+1))\rangle$$

such that:

1. $\pi_\beta(0)$ is a ground **LKS**-proof of $S_\beta(0)$, for all $\beta = 1, \ldots, \alpha$,
2. $\nu_\beta(k+1)$ is an **LKS**-proof of $S_\beta(k+1)$ such that $\nu_\beta(k+1)$ contains only one parameter k and proof links of the form:

$$\frac{(\psi_\beta(k))}{S_\beta(k)} \quad \text{and/or} \quad \frac{(\psi_\gamma(a))}{S_\gamma(a)}$$

for $\beta < \gamma$ and a arbitrary.

An Example

$\Psi = (\psi(0), \psi(k+1))$, where:

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg: l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee: l$$

- $\psi(k+1)$:

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg: l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ \vee: l}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ cut}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}}\ \wedge: l$$

Evaluation of Proof Schema

- The rewrite rules for proof links:
  - $\dfrac{(\psi_\beta(0))}{S} \rightarrow \pi_\beta(0)$, and
  - $\dfrac{(\psi_\beta(k+1))}{S} \rightarrow \nu_\beta(k+1)$, for all $\beta = 1, \ldots, \alpha$.
- $\psi_\beta\downarrow_\gamma$ is a normal form of $\dfrac{(\psi_\beta(\gamma))}{S(\gamma)}$, and
- $\Psi\downarrow_\gamma = \psi_1\downarrow_\gamma$.

Evaluation of Proof Schema (ctd.)

Proposition (Soundness)

For every $\gamma \in \mathbb{N}$ *and* $1 \leq \beta \leq \alpha$, $\psi_\beta \downarrow_\gamma$ *is a ground* **LKS**-*proof with end-sequent* $S_\beta(\gamma)$. *Hence* $\Psi \downarrow_\gamma$ *is a ground* **LKS**-*proof with end-sequent* $S(\gamma)$.

Proof.

By induction. □

An Example (ctd.)

- $\Psi\downarrow_0$ is just $\psi(0)$,
- $\Psi\downarrow_1$ is the following proof:

$$\dfrac{\dfrac{\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\neg:l \quad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\vee:l \quad \dfrac{\dfrac{p_1 \vdash p_1}{\neg p_1, p_1 \vdash}\neg:l \quad p_2 \vdash p_2}{p_1, \neg p_1 \vee p_2 \vdash p_2}\vee:l}{p_0, \neg p_0 \vee p_1, \neg p_1 \vee p_2 \vdash p_2}cut}{p_0, \bigwedge_{i=0}^{1}(\neg p_i \vee p_{i+1}) \vdash p_2}\wedge:l$$

Schematic Characteristic Clause Set

Motivation Example

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee : l$$

- $\psi(k+1)$:

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ \vee : l}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ cut}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}}\ \wedge : l$$

Motivation Example

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee : l$$

- $\psi(k+1)$:

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ \vee : l}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ cut}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}}\ \wedge : l$$

Motivation Example

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \lor p_1 \vdash p_1}\ \lor : l$$

- $\psi(k+1)$:

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \lor p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \lor p_{k+2} \vdash p_{k+2}}\ \lor : l}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \lor p_{i+1}), \neg p_{k+1} \lor p_{k+2} \vdash p_{k+2}}\ cut}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \lor p_{i+1}) \vdash p_{k+2}}\ \land : l$$

Basic Notions

- Cut-configuration Ω of ψ is a set of formula occurrences from the end-sequent of ψ.
- $\text{cl}^{\Omega,\psi}$ is an unique indexed proposition symbol for all proof symbols ψ and cut-configurations Ω.
- The intended semantics of $\text{cl}_a^{\Omega,\psi}$ will be "the characteristic clause set of $\psi(a)$, with the cut-configuration Ω".

Characteristic Clause Term

$\Theta_\rho(\pi, \Omega)$ is defined inductively:

- if ρ is an axiom of the form $\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta$, then

$$\Theta_\rho(\pi, \Omega) = \Gamma_\Omega, \Gamma_C \vdash \Delta_\Omega, \Delta_C.$$

- if ρ is a proof link of the form

$$\frac{(\psi(a))}{\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta}$$

then

$$\Theta_\rho(\pi, \Omega) = \vdash \mathrm{cl}_a^{\Omega', \psi}$$

where Ω' is a set of formula occurrences from $\Gamma_\Omega, \Gamma_C \vdash \Delta_\Omega, \Delta_C$.

Characteristic Clause Term (ctd.)

- if ρ is an unary rule with immediate predecessor ρ', then

$$\Theta_{\rho}(\pi, \Omega) = \Theta_{\rho'}(\pi, \Omega).$$

- if ρ is a binary rule with immediate predecessors ρ_1, ρ_2, then either

$$\Theta_{\rho}(\pi, \Omega) = \Theta_{\rho_1}(\pi, \Omega) \oplus \Theta_{\rho_2}(\pi, \Omega)$$

or

$$\Theta_{\rho}(\pi, \Omega) = \Theta_{\rho_1}(\pi, \Omega) \otimes \Theta_{\rho_2}(\pi, \Omega).$$

- $\Theta(\pi, \Omega) = \Theta_{\rho_0}(\pi, \Omega)$, where ρ_0 is the last inference of π.

An Example

$\Psi = (\psi(0), \psi(k+1))$ of $p_0, \bigwedge_{i=0}^{n}(\neg p_i \lor p_{i+1}) \vdash p_{n+1}$, where:

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \lor p_1 \vdash p_1}\ \lor : l$$

- $\psi(k+1)$:

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \lor p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \lor p_{k+2} \vdash p_{k+2}}\ \lor : l}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \lor p_{i+1}), \neg p_{k+1} \lor p_{k+2} \vdash p_{k+2}}\ cut}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \lor p_{i+1}) \vdash p_{k+2}}\ \land : l$$

An Example (ctd.)

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\vee : l$$

An Example (ctd.)

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee : l$$

$$\Theta(\psi(0), \emptyset) = \vdash \otimes \vdash$$

An Example (ctd.)

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\vee : l$$

An Example (ctd.)

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee : l$$

$$\Theta(\psi(0), \{p_{n+1}\}) = \vdash \otimes \vdash p_1$$

An Example (ctd.)

- $\psi(k+1)$:

$$
\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ \vee : l}{\dfrac{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}}\ \wedge : l}\ cut
$$

An Example (ctd.)

- $\psi(k+1)$:

$$\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \quad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash} \neg : l \quad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}} \vee : l}{\dfrac{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}} \wedge : l} cut$$

$$\Theta(\psi(k+1), \emptyset) = \vdash \mathrm{cl}_k^{\{p_{n+1}\},\psi} \oplus (p_{k+1} \vdash \otimes \vdash)$$

An Example (ctd.)

- $\psi(k+1)$:

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\,\neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\,\vee : l}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\,cut}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}}\,\wedge : l$$

An Example (ctd.)

- $\psi(k+1)$:

$$\dfrac{\dfrac{\overset{(\psi(k))}{\cdots}}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \quad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash} \neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}} \vee : l}{\dfrac{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}} \wedge : l} cut$$

$$\Theta(\psi(k+1), \{p_{n+1}\}) = \vdash \mathrm{cl}_k^{\{p_{n+1}\},\psi} \oplus (p_{k+1} \vdash \otimes \vdash p_{k+2})$$

Evaluation of Clause Term

- The rewrite rules for clause term symbols:
  - $\vdash \mathrm{cl}_0^{\Omega,\psi_\beta} \to \Theta(\pi_\beta(0), \Omega)$, and
  - $\vdash \mathrm{cl}_{k+1}^{\Omega,\psi_\beta} \to \Theta(\nu_\beta(k+1), \Omega)$, for all $\beta = 1, \ldots, \alpha$.
- $\Theta(\Psi, \Omega) = \Theta(\psi_1, \Omega)$, and
- $\Theta(\Psi) = \Theta(\Psi, \emptyset)$.

Evaluation of Clause Term (ctd.)

Proposition (Soundness)

Let $\gamma \in \mathbb{N}$ *and* Ω *be a cut-configuration, then* $\Theta(\psi_\beta, \Omega)\downarrow_\gamma$ *is a ground clause term for all* $1 \leq \beta \leq \alpha$*. Hence* $\Theta(\Psi)\downarrow_\gamma$ *is a ground clause term.*

Proof.

By induction. □

Evaluation of Clause Term (ctd.)

Proposition (Commutativity)

Let Ω *be a cut-configuration and* $\gamma \in \mathbb{N}$. *Then* $\Theta(\Psi \downarrow_\gamma, \Omega) = \Theta(\Psi, \Omega) \downarrow_\gamma$.

Proof.

By double induction. □

Term to Set Transformation

- Let $\Gamma \vdash \Delta$ and $\Pi \vdash \Lambda$ be sequents, then $\Gamma \vdash \Delta \times \Pi \vdash \Lambda = \Gamma, \Pi \vdash \Delta, \Lambda$ and $P \times Q = \{S_P \times S_Q \mid S_P \in P, S_Q \in Q\}$.
- Let Θ be a clause term, then we define $|\Theta|$ as:
  - $|\vdash \mathrm{cl}_a^{\Omega',\psi}| = \mathcal{C}_{\Theta(\psi,\Omega')}(a)$, where $\mathcal{C}_{\Theta(\psi,\Omega')}$ is a clause set symbol assigned to $\Theta(\psi,\Omega')$,
  - $|\Gamma \vdash \Delta| = \{\Gamma \vdash \Delta\}$,
  - $|\Theta_1 \otimes \Theta_2| = |\Theta_1| \times |\Theta_2|$,
  - $|\Theta_1 \oplus \Theta_2| = |\Theta_1| \cup |\Theta_2|$.

Characteristic Clause Set Schemata

- Let $\Psi = \langle(\pi_1(0), \nu_1(k+1)), \ldots, (\pi_\alpha(0), \nu_\alpha(k+1))\rangle$, then assign each pair of terms, $\Theta(\pi_\beta, \Omega)$ and $\Theta(\nu_\beta, \Omega)$, a unique symbol $\mathcal{C}_\gamma$ and define:
  - $\mathcal{C}_\gamma(0) = |\Theta(\pi_\beta, \Omega)|$,
  - $\mathcal{C}_\gamma(k+1) = |\Theta(\nu_\beta, \Omega)|$.
- The characteristic clause set schema

$$\mathrm{CL}(\Psi) = \langle(\mathcal{C}_1(0), \mathcal{C}_1(k+1)), \ldots\rangle$$

where $\mathcal{C}_1$ is assigned to the pair of terms $\Theta(\pi_1, \emptyset)$ and $\Theta(\nu_1, \emptyset)$.

An Example (ctd.)

- $\mathrm{CL}(\Psi) = \langle (\mathcal{C}(0), \mathcal{C}(k+1)), (\mathcal{D}(0), \mathcal{D}(k+1)) \rangle$, where:
  - $\mathcal{C}(0) = |\Theta(\psi(0), \emptyset)| = \{\vdash\}$
  - $\mathcal{C}(k+1) = |\Theta(\psi(k+1), \emptyset)| = \mathcal{D}(k) \cup \{p_{k+1} \vdash\}$
  - $\mathcal{D}(0) = |\Theta(\psi(0), \{p_{n+1}\})| = \{\vdash p_1\}$
  - $\mathcal{D}(k+1) = |\Theta(\psi(k+1), \{p_{n+1}\})| = \mathcal{D}(k) \cup \{p_{k+1} \vdash p_{k+2}\}$

An Example (ctd.)

- $\mathrm{CL}(\Psi)\downarrow_0$:
  (1) $\vdash$
- $\mathrm{CL}(\Psi)\downarrow_1$:
  (1) $\vdash p_1$
  (2) $p_1 \vdash$
- $\mathrm{CL}(\Psi)\downarrow_2$:
  (1) $\vdash p_1$
  (2) $p_1 \vdash p_2$
  (3) $p_2 \vdash$
- $\mathrm{CL}(\Psi)\downarrow_3$:
  (1) $\vdash p_1$
  (2) $p_1 \vdash p_2$
  (3) $p_2 \vdash p_3$
  (4) $p_3 \vdash$
- $\mathrm{CL}(\Psi)\downarrow_4$:
  (1) $\vdash p_1$
  (2) $p_1 \vdash p_2$
  (3) $p_2 \vdash p_3$
  (4) $p_3 \vdash p_4$
  (5) $p_4 \vdash$

Schematic Projections

Basic Notions

- $pr^{\Omega,\psi}$ is an unique proof symbol, called projection symbol.
- The intended semantics of $pr^{\Omega,\psi}(a)$ will be "the set of characteristic projections of $\psi(a)$, with the cut-configuration Ω".

Characteristic Projection Term

$\Xi_\rho(\pi, \Omega)$ is defined inductively:

- if ρ is an axiom S, then $\Xi_\rho(\pi, \Omega) = S$.
- if ρ is a proof link of the form

$$\frac{(\psi(a))}{\Gamma_\Omega, \Gamma_C, \Gamma \vdash \Delta_\Omega, \Delta_C, \Delta}$$

then

$$\Xi_\rho(\pi, \Omega) = pr^{\Omega', \psi}(a)$$

where Ω' is a set of formula occurrences from $\Gamma_\Omega, \Gamma_C \vdash \Delta_\Omega, \Delta_C$.

Characteristic Projection Term (ctd.)

- If ρ is an unary inference with immediate predecessor ρ', then either

$$\Xi_\rho(\pi, \Omega) = \Xi_{\rho'}(\pi, \Omega)$$

or

$$\Xi_\rho(\pi, \Omega) = \rho(\Xi_{\rho'}(\pi, \Omega)).$$

Characteristic Projection Term (ctd.)

- If ρ is a binary inference with immediate predecessors ρ_1 and ρ_2, then either

$$\Xi_\rho(\pi,\Omega) = w^{\Gamma_2 \vdash \Delta_2}(\Xi_{\rho_1}(\pi,\Omega)) \oplus w^{\Gamma_1 \vdash \Delta_1}(\Xi_{\rho_2}(\pi,\Omega))$$

  or

$$\Xi_\rho(\pi,\Omega) = \Xi_{\rho_1}(\pi,\Omega) \otimes_\rho \Xi_{\rho_2}(\pi,\Omega)$$

- $\Xi(\pi,\Omega) = \Xi_{\rho_0}(\pi,\Omega)$, where ρ_0 is the last inference of π.

An Example

$\Psi = (\psi(0), \psi(k+1))$ of $p_0, \bigwedge_{i=0}^{n}(\neg p_i \vee p_{i+1}) \vdash p_{n+1}$, where:

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg: l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee: l$$

- $\psi(k+1)$:

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg: l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ \vee: l}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ cut}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}}\ \wedge: l$$

An Example (ctd.)

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee : l$$

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$$\Xi(\psi(0), \emptyset) = \neg_l(p_0 \vdash p_0) \otimes_{\vee_l} p_1 \vdash p_1$$

An Example (ctd.)

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash {\color{blue} p_1}}{p_0, \neg p_0 \vee p_1 \vdash {\color{blue} p_1}}\ \vee : l$$

An Example (ctd.)

- $\psi(0)$:

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash {\color{blue}p_1}}{p_0, \neg p_0 \vee p_1 \vdash {\color{blue}p_1}}\ \vee : l$$

$$\Xi(\psi(0), \{p_{n+1}\}) = \neg_l(p_0 \vdash p_0) \otimes_{\vee_l} p_1 \vdash p_1$$

An Example (ctd.)

- $\psi(k+1)$:

$$
\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \quad \dfrac{\dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg : l \quad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ \vee : l}{\dfrac{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}}\ \wedge : l}\ cut
$$

An Example (ctd.)

- $\psi(k+1)$:

$$
\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \quad \dfrac{\dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash} \neg : l \quad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}} \vee : l}{}}{\dfrac{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}} \wedge : l} cut
$$

$$
\Xi(\psi(k+1), \emptyset) = \wedge_l(w^{\neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}(pr^{\{p_{n+1}\},\psi}(k)) \oplus w^{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash}(\neg_l(p_{k+1} \vdash p_{k+1}) \otimes_{\vee_l} p_{k+2} \vdash p_{k+2}))
$$

An Example (ctd.)

- $\psi(k+1)$:

$$
\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash} \neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}} \vee : l}{\dfrac{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}} \wedge : l} cut
$$

An Example (ctd.)

- $\psi(k+1)$:

$$\dfrac{\dfrac{\dfrac{(\psi(k))}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash p_{k+1}} \qquad \dfrac{\dfrac{p_{k+1} \vdash p_{k+1}}{\neg p_{k+1}, p_{k+1} \vdash}\ \neg : l \qquad p_{k+2} \vdash p_{k+2}}{p_{k+1}, \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ \vee : l}{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}), \neg p_{k+1} \vee p_{k+2} \vdash p_{k+2}}\ cut}{p_0, \bigwedge_{i=0}^{k+1}(\neg p_i \vee p_{i+1}) \vdash p_{k+2}}\ \wedge : l$$

$$\Xi(\psi(k+1), \{p_{n+1}\}) = \wedge_l(w^{\neg p_{k+1} \vee p_{k+2} \vdash}(pr^{\{p_{n+1}\},\psi}(k)) \oplus$$
$$w^{p_0, \bigwedge_{i=0}^{k}(\neg p_i \vee p_{i+1}) \vdash}(\neg_l(p_{k+1} \vdash p_{k+1}) \otimes_{\vee_l} p_{k+2} \vdash p_{k+2}))$$

Evaluation of Clause Term

- The rewrite rules for clause term symbols:
  - $pr^{\Omega,\psi_\beta}(0) \to \Xi(\pi_\beta(0), \Omega)$, and
  - $pr^{\Omega,\psi_\beta}(k+1) \to \Xi(\nu_\beta(k+1), \Omega)$, for all $\beta = 1, \ldots, \alpha$.
- $\Xi(\Psi, \Omega) = \Xi(\psi_1, \Omega)$, and
- $\Xi(\Psi) = \Xi(\Psi, \emptyset)$.

Evaluation of Clause Term (ctd.)

Proposition (Soundness)

Let $\gamma \in \mathbb{N}$ and Ω be a cut-configuration, then $\Xi(\psi_\beta, \Omega) \downarrow_\gamma$ is a ground projection term for all $1 \leq \beta \leq \alpha$. Hence $\Xi(\Psi) \downarrow_\gamma$ is a ground projection term.

Proof.

By induction. □

Evaluation of Clause Term (ctd.)

Proposition (Commutativity)

Let Ω *be a cut-configuration and* $\gamma \in \mathbb{N}$. *Then*
$\Xi(\Psi \downarrow_\gamma, \Omega) = \Xi(\Psi, \Omega) \downarrow_\gamma$.

Proof.

By double induction. □

Term to Set Transformation

- Let ρ be an unary and σ a binary rule. Let ϕ, ψ be **LKS**-proofs, then $\rho(\phi)$ is the **LKS**-proof obtained from the ϕ by applying ρ, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs ϕ and ψ by applying σ.

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$$\phi = p_0 \vdash p_0$$

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$$\neg_l(\phi) = \frac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash} \neg : l$$

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Term to Set Transformation

- Let ρ be an unary and σ a binary rule. Let ϕ, ψ be **LKS**-proofs, then $\rho(\phi)$ is the **LKS**-proof obtained from the ϕ by applying ρ, and $\sigma(\phi, \psi)$ is the proof obtained from the proofs ϕ and ψ by applying σ.

$$\vee_l(\neg_l(\phi), \psi) = \cfrac{\cfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash} \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1} \vee : l$$

Term to Set Transformation (ctd.)

- $P^{\Gamma \vdash \Delta} = \{\psi^{\Gamma \vdash \Delta} \mid \psi \in P\}$, where $\psi^{\Gamma \vdash \Delta}$ is ψ followed by weakenings adding $\Gamma \vdash \Delta$.

Term to Set Transformation (ctd.)

- $P^{\Gamma \vdash \Delta} = \{\psi^{\Gamma \vdash \Delta} \mid \psi \in P\}$, where $\psi^{\Gamma \vdash \Delta}$ is ψ followed by weakenings adding $\Gamma \vdash \Delta$.

$$\psi = \quad \dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash} \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1} \vee : l$$

Term to Set Transformation (ctd.)

- $P^{\Gamma \vdash \Delta} = \{\psi^{\Gamma \vdash \Delta} \mid \psi \in P\}$, where $\psi^{\Gamma \vdash \Delta}$ is ψ followed by weakenings adding $\Gamma \vdash \Delta$.

$$\psi^{\Gamma \vdash \Delta} = \frac{\dfrac{\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee : l}{p_0, \neg p_0 \vee p_1, \Gamma \vdash p_1}\ w : l*}{p_0, \neg p_0 \vee p_1, \Gamma \vdash \Delta, p_1}\ w : r*$$

Term to Set Transformation (ctd.)

- P, Q: sets of **LKS**-proofs.
- $P \times_\sigma Q = \{\sigma(\phi, \psi) \mid \phi \in P, \psi \in Q\}$.

Term to Set Transformation (ctd.)

- P, Q: sets of **LKS**-proofs.
- $P \times_\sigma Q = \{\sigma(\phi, \psi) \mid \phi \in P, \psi \in Q\}$.

$$P = \left\{ \frac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash} \neg : l \quad , \quad \frac{q_0 \vdash q_0}{\neg p_0, q_0 \vdash q_0} w : l \right\}$$

$$Q = \left\{ p_1 \vdash p_1 \quad , \quad \frac{q_1 \vdash q_1}{p_1, q_1 \vdash q_1} w : l \right\}$$

Term to Set Transformation (ctd.)

- $P \times_{\vee_l} Q =$

$$\left\{ \dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\vee : l \quad , \right.$$

$$\dfrac{\dfrac{q_0 \vdash q_0}{\neg p_0, q_0 \vdash q_0} w : l \qquad p_1 \vdash p_1}{q_0, \neg p_0 \vee p_1 \vdash q_0, p_1}\vee : l \quad ,$$

$$\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\neg : l \qquad \dfrac{q_1 \vdash q_1}{p_1, q_1 \vdash q_1} w : l}{p_0, q_1, \neg p_0 \vee p_1 \vdash q_1}\vee : l \quad ,$$

$$\left. \dfrac{\dfrac{q_0 \vdash q_0}{\neg p_0, q_0 \vdash q_0} w : l \qquad \dfrac{q_1 \vdash q_1}{p_1, q_1 \vdash q_1} w : l}{q_0, q_1, \neg p_0 \vee p_1 \vdash q_0, q_1}\vee : l \right\}$$

Term to Set Transformation (ctd.)

- Let Ξ be a ground projection term, then we define $|\Xi|$ as:
  - $|\Gamma \vdash \Delta| = \Gamma \vdash \Delta$,
  - $|\rho(\Xi)| = \rho(|\Xi|)$ for unary rule symbols ρ,
  - $|w^{\Gamma \vdash \Delta}(\Xi)| = |\Xi|^{\Gamma \vdash \Delta}$,
  - $|\Xi_1 \oplus \Xi_2| = |\Xi_1| \cup |\Xi_2|$,
  - $|\Xi_1 \otimes_\sigma \Xi_2| = |\Xi_1| \times_\sigma |\Xi_2|$ for binary rule symbols σ.

Term to Set Transformation (ctd.)

- For ground **LKS**-proofs π and cut-configurations Ω, define $PR(\pi, \Omega) = |\Xi(\pi, \Omega)|$ and $PR(\pi) = PR(\pi, \emptyset)$.
- $PR(\Psi) \downarrow_\gamma = |\Xi(\Psi) \downarrow_\gamma |$.

An Example (ctd.)

- $PR(\Psi)\downarrow_0$:

$$\left\{ \dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee : l \right\}$$

An Example (ctd.)

- $PR(\Psi)\downarrow_1$:

$$\left\{ \dfrac{\dfrac{\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg: l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \vee p_1 \vdash p_1}\ \vee: l}{p_0, \neg p_0 \vee p_1, \neg p_1 \vee p_2 \vdash p_2, p_1}\ w: l, r}{p_0, \bigwedge_{i=0}^{1} \neg p_i \vee p_{i+1} \vdash p_2, p_1}\ \wedge: l \right.$$

$$\left. \dfrac{\dfrac{\dfrac{\dfrac{p_1 \vdash p_1}{\neg p_1, p_1 \vdash}\ \neg: l \qquad p_2 \vdash p_2}{p_1, \neg p_1 \vee p_2 \vdash p_2}\ \vee: l}{p_1, p_0, \bigwedge_{i=0}^{0}(\neg p_i \vee p_{i+1}), \neg p_1 \vee p_2 \vdash p_2}\ w: l}{p_1, p_0, \bigwedge_{i=0}^{1}(\neg p_i \vee p_{i+1}) \vdash p_2}\ \wedge: l \right\}$$

An Example (ctd.)

- $PR(\Psi)\downarrow_2$:

$$\Bigg\{$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p_0 \vdash p_0}{\neg p_0, p_0 \vdash}\ \neg : l \qquad p_1 \vdash p_1}{p_0, \neg p_0 \lor p_1 \vdash p_1}\ \lor : l}{p_0, \neg p_0 \lor p_1, \neg p_1 \lor p_2 \vdash p_1}\ w : l}{p_0, \bigwedge_{i=0}^{1} \neg p_i \lor p_{i+1} \vdash p_1}\ \land : l}{\dfrac{p_0, \bigwedge_{i=0}^{1} \neg p_i \lor p_{i+1}, \neg p_2 \lor p_3 \vdash p_3, p_1}{p_0, \bigwedge_{i=0}^{2} \neg p_i \lor p_{i+1} \vdash p_3, p_1}\ \land : l}\ w : l, r$$

$$\ldots \qquad \Bigg\}$$

An Example (ctd.)

- $PR(\Psi)\downarrow_2$:

$$
\left\{ \quad \ldots \right.
$$

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{p_1 \vdash p_1}{\neg p_1, p_1 \vdash}\ \neg: l \qquad p_2 \vdash p_2}{p_1, \neg p_1 \lor p_2 \vdash p_2}\ \lor: l}{p_1, p_0, \bigwedge_{i=0}^{0}(\neg p_i \lor p_{i+1}), \neg p_1 \lor p_2 \vdash p_2}\ w: l}{p_1, p_0, \bigwedge_{i=0}^{1}(\neg p_i \lor p_{i+1}) \vdash p_2}\ \land: l}{p_1, p_0, \bigwedge_{i=0}^{1}(\neg p_i \lor p_{i+1}), \neg p_2 \lor p_3 \vdash p_3, p_2}\ w: l, r}{p_1, p_0, \bigwedge_{i=0}^{2}(\neg p_i \lor p_{i+1}) \vdash p_3, p_2}\ \land: l
$$

$$
\left. \ldots \quad \right\}
$$

An Example (ctd.)

- $PR(\Psi)\downarrow_2$:

$$\Bigg\{ \quad \ldots$$

$$\dfrac{\dfrac{\dfrac{\dfrac{p_2 \vdash p_2}{\neg p_2, p_2 \vdash}\ \neg : l \qquad p_3 \vdash p_3}{p_2, \neg p_2 \vee p_3 \vdash p_3}\ \vee : l}{p_2, p_0, \bigwedge_{i=0}^{1}(\neg p_i \vee p_{i+1}), \neg p_2 \vee p_3 \vdash p_3}\ w : l}{p_2, p_0, \bigwedge_{i=0}^{2}(\neg p_i \vee p_{i+1}) \vdash p_3}\ \wedge : l$$

$$\Bigg\}$$

Term to Set Transformation (ctd.)

Proposition (Soundness)

Let π be a ground **LKS**-*proof with end-sequent S, then for all clauses $C \in \mathrm{CL}(\pi)$, there exists a ground* **LKS**-*proof $\pi \in PR(\pi)$ with end-sequent $S \circ C$.*

Proposition (Commutativity)

Let $\gamma \in \mathbb{N}$, then $PR(\Psi\downarrow_\gamma) = PR(\Psi)\downarrow_\gamma$.

Proposition (Correctness)

Let $\gamma \in \mathbb{N}$, then for every clause $C \in \mathrm{CL}(\Psi)\downarrow_\gamma$ there exists a ground **LKS**-*proof $\pi \in PR(\Psi)\downarrow_\gamma$ with end-sequent $C \circ S(\gamma)$.*

Resolution Schemata

Clause Schemata

- We define s-clause as:
  - clause variables, denoted with $X, Y, \ldots$, are s-clauses,
  - clauses are s-clauses,
  - if s_1, s_2 are s-clauses, then $s_1 \circ s_2$ is an s-clause.
- A clause schema is a term $t(a, X_1, \ldots, X_\alpha)$ w.r.t a rewrite system $\mathcal{R}$:
  - $t(0, X_1, \ldots, X_\alpha) \to s_0$,
  - $t(k+1, X_1, \ldots, X_\alpha) \to t(k, s_1, \ldots, s_\alpha)$, for $s_0, \ldots, s_\alpha$ being s-clauses with clause variables in $\{X_1, \ldots, X_\alpha\}$.
- Example: consider $t(n, X)$ w.r.t
  - $t(0, X) \to (\vdash p_0) \circ X$,
  - $t(k+1, X) \to t(k, (\vdash p_{k+1}) \circ X)$,

  then $t(\alpha, \vdash q_0) \downarrow$ are $\vdash q_0, p_0, \ldots, p_\alpha$ for all $\alpha \geq 0$.

Resolution Term

- We define resolution terms inductively:
  - s-clauses are resolution terms,
  - clause schemata are resolution terms,
  - if r_1, r_2 are resolution terms w.r.t. $\mathcal{R}_1$ and $\mathcal{R}_2$, then $r(r_1;\ r_2;\ p_a)$ is a resolution term w.r.t. $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$.
- A resolution term r based on a set of clause schemata $\mathcal{C}$ is a resolution term s.t. all s-clauses and clause schemata in r are also in $\mathcal{C}$.
- Example: $r(r(t(n, X);\ p_n \vdash;\ p_n);\ q_0 \vdash;\ q_0)$ is a resolution term.

Resolution Deduction

- Let $\Gamma \vdash \Delta$ and $\Pi \vdash \Lambda$ be clauses. If p_a occurs in Δ and Π, then $res(\Gamma \vdash \Delta, \Pi \vdash \Lambda, p_a) = \Gamma, \Pi \backslash p_a \vdash \Delta \backslash p_a, \Lambda$ is called resolvent.
- We define resolution deduction inductively:
  - if C is a clause, then C is a resolution deduction and $ES(C) = C$,
  - if δ_1 and δ_2 are resolution deductions, $ES(\delta_1) = C_1$, $ES(\delta_2) = C_2$ and $res(C_1, C_2, p_a) = D$, then $r(\delta_1, \delta_2, p_a)$ is a resolution deduction and $ES(r(\delta_1, \delta_2, p_a)) = D$.
- δ is called resolution refutation, if $ES(\delta) = \vdash$.
- Examples:
  - $r(r(\vdash q_0, p_0, p_1;\ p_1 \vdash;\ p_1);\ q_0 \vdash;\ q_0)$ is a resolution deduction.
  - $r(r(\vdash q_0, p_0;\ p_0 \vdash;\ p_0);\ q_0 \vdash;\ q_0)$ is a resolution refutation.

Tree Transformation

- Let δ be a resolution deduction. If:
  - $\delta = C$, then $T(\delta) = C$,
  - $\delta = r(\delta_1; \delta_2; p_a), ES(\delta_1) = C_1, ES(\delta_2) = C_2$ and $res(C_1, C_2, p_a) = C$, then $T(\delta) =$

$$\frac{\begin{array}{cc} (T(\delta_1)) & (T(\delta_2)) \\ C_1 & C_2 \end{array}}{C}$$

- Example: $T(r(r(\vdash q_0, p_0, p_1;\ p_1 \vdash;\ p_1);\ q_0 \vdash;\ q_0))$ is:

$$\frac{\dfrac{\vdash q_0, p_0, p_1 \qquad p_1 \vdash}{\vdash q_0, p_0} \qquad q_0 \vdash}{\vdash p_0}$$

Resolution Refutation Schema

- A resolution proof schema with clause variables $X_1, \ldots, X_\beta$ is a structure $R = ((\varrho_1, \ldots, \varrho_\alpha), \mathcal{R}, \mathcal{D}, \mathcal{R}')$ where the ϱ_i denote resolution terms, $\mathcal{D}$ is a finite set of clause schemata w.r.t. $\mathcal{R}'$ and $\mathcal{R} = \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_\alpha$, where the $\mathcal{R}_i$ (for $0 \leq i \leq \alpha$) are defined as follows:
  - $\varrho_i(0, X_1, \ldots, X_\beta) \rightarrow s_i$,
  - $\varrho_i(k{+}1, X_1, \ldots, X_\beta) \rightarrow t_i[\varrho_i(k, \bar{s}^i_0), \varrho_{l_1}(a^i_1, \bar{s}^i_1), \ldots, \varrho_{l_{j(i)}}(a^i_{j(i)}, \bar{s}^i_{j(i)})]$, where
    - s_i is a resolution term containing some of $X_1, \ldots, X_\beta$,
    - $a^i_1, \ldots, a^i_{j(i)}$ are arithmetic terms,
    - $\bar{s}^i_0, \ldots, \bar{s}^i_{j(i)}$ are vectors of clause schemata over $X_1, \ldots, X_\beta$,
    - the $t_i[\varrho_i(k, \bar{s}^i_0), \varrho_{l_1}(a^i_1, \bar{s}^i_1), \ldots, \varrho_{l_{j(i)}}(a^i_{j(i)}, \bar{s}^i_{j(i)})]$ are resolution terms based on $\mathcal{D}$ after replacement of some clause schemata by the terms $\varrho_i(k, \bar{s}^i_0), \varrho_{l_1}(a^i_1, \bar{s}^i_1), \ldots, \varrho_{l_r}(a^i_{j(i)}, \bar{s}^i_{j(i)})$ where $i < \min\{l_1, \ldots, l_{j(i)}\}$ and $\max\{l_1, \ldots, l_{j(i)}\} \leq \alpha$.

Resolution Refutation Schema (ctd.)

- A resolution proof schema is called a resolution refutation schema of a clause schema $\mathcal{C}(n)$ if there exist clauses $C_1, \ldots, C_\alpha$ s.t. $\varrho_1(\beta, C_1, \ldots, C_\alpha)\downarrow$ is a resolution refutation of $\mathcal{C}(\beta)\downarrow$.

- Example: We define the resolution refutation schema

$$R = ((\varrho, \delta), \mathcal{R}, \emptyset, \emptyset)$$

where $\mathcal{R}$ is:

- $\varrho(0) \rightarrow \vdash$
- $\varrho(k+1) \rightarrow r(\delta(k);\ p_{k+1} \vdash;\ p_{k+1})$,
- $\delta(0) \rightarrow \vdash p_1$,
- $\delta(k+1) \rightarrow r(\delta(k);\ p_{k+1} \vdash p_{k+2};\ p_{k+1})$.

Atomic Cut Normal Form

Theorem (ACNF)

Let Ψ be a proof schema with end-sequent $S(n)$, and let R be a resolution refutation schema of $\mathrm{CL}(\Psi)$. Then for all $\alpha \in \mathbb{N}$ there exists a ground **LKS***-proof π of $S(\alpha)$ with at most atomic cuts such that its size $l(\pi)$ is polynomial in $l(R{\downarrow_\alpha}) \cdot l(PR(\Psi){\downarrow_\alpha})$.*

The Adder Example

Formula Definitions

- We introduce the following "shortcuts" for formulas:

$$
\begin{aligned}
A \oplus B &=_{def} (A \land \neg B) \lor (\neg A \land B) \\
A \Leftrightarrow B &=_{def} (\neg A \lor B) \land (\neg B \lor A) \\
\hat{\mathbf{S}}_i &=_{def} S_i \Leftrightarrow (A_i \oplus B_i) \oplus C_i \\
\hat{\mathbf{S}}'_i &=_{def} S'_i \Leftrightarrow (B_i \oplus A_i) \oplus C'_i \\
\hat{\mathbf{C}}_i &=_{def} C_{i+1} \Leftrightarrow (A_i \land B_i) \lor (C_i \land A_i) \lor (C_i \land B_i) \\
\hat{\mathbf{C}}'_i &=_{def} C'_{i+1} \Leftrightarrow (B_i \land A_i) \lor (C'_i \land B_i) \lor (C'_i \land A_i) \\
Adder_n &=_{def} \bigwedge_{i=0}^{n} \hat{\mathbf{S}}_i \land \bigwedge_{i=0}^{n} \hat{\mathbf{C}}_i \land \neg C_0 \\
Adder'_n &=_{def} \bigwedge_{i=0}^{n} \hat{\mathbf{S}}'_i \land \bigwedge_{i=0}^{n} \hat{\mathbf{C}}'_i \land \neg C'_0 \\
EqC_n &=_{def} \bigwedge_{i=0}^{n} (C_i \Leftrightarrow C'_i) \\
EqS_n &=_{def} \bigwedge_{i=0}^{n} (S_i \Leftrightarrow S'_i)
\end{aligned}
$$

The Adder Proof

- The proof schema Ψ is:

$$\langle(\psi(0),\psi(k+1)),(\varphi(0),\varphi(k+1)),(\phi(0),\phi(k+1)),(\chi(0),\chi(k+1))\rangle,$$

where $\psi(k)$ is:

$$\dfrac{\dfrac{\dfrac{(\varphi(k))}{\neg C_0,\neg C'_0,\bigwedge_{i=0}^k \hat{\mathbf{C}}_i,\bigwedge_{i=0}^k \hat{\mathbf{C}}'_i \vdash EqC_k} \quad \dfrac{(\chi(k))}{EqC_k,\bigwedge_{i=0}^k \hat{\mathbf{S}}_i,\bigwedge_{i=0}^k \hat{\mathbf{S}}'_i \vdash EqS_k}}{\neg C_0,\neg C'_0,\bigwedge_{i=0}^k \hat{\mathbf{C}}_i,\bigwedge_{i=0}^k \hat{\mathbf{C}}'_i,\bigwedge_{i=0}^k \hat{\mathbf{S}}_i,\bigwedge_{i=0}^k \hat{\mathbf{S}}'_i \vdash EqS_k}\ cut}{Adder_k, Adder'_k \vdash EqS_k}\ \wedge: l*$$

The Adder Proof (ctd.)

- $\varphi(k+1)$ is:

$$\dfrac{\dfrac{\dfrac{(\varphi(k))}{\neg C_0, \neg C'_0, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}_i, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}'_i \vdash EqC_k} \qquad \dfrac{(\phi(k))}{\neg C_0, \neg C'_0, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}_i, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}'_i \vdash C_{k+1} \Leftrightarrow C'_{k+1}}}{\neg C_0, \neg C'_0, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}_i, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}'_i \vdash EqC_{k+1}}\ \wedge: r, c: l*}{\neg C_0, \neg C'_0, \bigwedge_{i=0}^{k+1} \hat{\mathbf{C}}_i, \bigwedge_{i=0}^{k+1} \hat{\mathbf{C}}'_i \vdash EqC_{k+1}}\ \wedge: l*$$

- $\phi(k+1)$ is:

$$\dfrac{\dfrac{\dfrac{(\phi(k))}{\neg C_0, \neg C'_0, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}_i, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}'_i \vdash C_{k+1} \Leftrightarrow C'_{k+1}} \qquad \dfrac{\vdots}{C_{k+1} \Leftrightarrow C'_{k+1}, \hat{\mathbf{C}}_{k+1}, \hat{\mathbf{C}}'_{k+1} \vdash C_{k+2} \Leftrightarrow C'_{k+2}}}{\neg C_0, \neg C'_0, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}_i, \bigwedge_{i=0}^{k} \hat{\mathbf{C}}'_i, \hat{\mathbf{C}}_{k+1}, \hat{\mathbf{C}}'_{k+1} \vdash C_{k+2} \Leftrightarrow C'_{k+2}}\ cut}{\neg C_0, \neg C'_0, \bigwedge_{i=0}^{k+1} \hat{\mathbf{C}}_i, \bigwedge_{i=0}^{k+1} \hat{\mathbf{C}}'_i \vdash C_{k+2} \Leftrightarrow C'_{k+2}}\ \wedge: l*$$

The Adder Proof (ctd.)

- Finally, $\chi(k+1)$ is:

$$
\dfrac{\dfrac{\dfrac{(\chi(k))}{EqC_k, \bigwedge_{i=0}^{k} \hat{\mathbf{S}}_i, \bigwedge_{i=0}^{k} \hat{\mathbf{S}}'_i \vdash EqS_k} \qquad \dfrac{\vdots}{C_{k+1} \Leftrightarrow C'_{k+1}, \hat{\mathbf{S}}_{k+1}, \hat{\mathbf{S}}'_{k+1} \vdash S_{k+1} \Leftrightarrow S'_{k+1}}}{EqC_k, \bigwedge_{i=0}^{k} \hat{\mathbf{S}}_i, \bigwedge_{i=0}^{k} \hat{\mathbf{S}}'_i, C_{k+1} \Leftrightarrow C'_{k+1}, \hat{\mathbf{S}}_{k+1}, \hat{\mathbf{S}}'_{k+1} \vdash EqS_{k+1}} \wedge: r}{EqC_{k+1}, \bigwedge_{i=0}^{k+1} \hat{\mathbf{S}}_i, \bigwedge_{i=0}^{k+1} \hat{\mathbf{S}}'_i \vdash EqS_{k+1}} \wedge: l*
$$

Characteristic Clause Set

- We get the following schema:

$$\mathrm{CL}(\Psi) = \langle (\mathcal{C}_1(0), \mathcal{C}_1(k+1)), \ldots, (\mathcal{C}_4(0), \mathcal{C}_4(k+1)) \rangle$$

where:

- $\mathcal{C}_1(k) = \mathcal{C}_2(k) \cup \mathcal{C}_4(k)$,
- $\mathcal{C}_2(0) = \{$
  $C_0 \vdash$;
  $C_0' \vdash$
  $\}$,
- $\mathcal{C}_2(k+1) = \mathcal{C}_2(k) \cup \mathcal{C}_3(k)$

Characteristic Clause Set (ctd.)

- $\mathcal{C}_3(0) = \{$
  $C_1 \vdash C'_1$;
  $C'_1 \vdash C_1$
  $\}$,

- $\mathcal{C}_3(k+1) = \mathcal{C}_3(k) \cup \{$
  $C_{k+1} \vdash C'_{k+1}, C_k$;
  $C'_{k+1} \vdash C_{k+1}, C'_k$;
  $C'_k, C_{k+1} \vdash C'_{k+1}$;
  $C_k, C'_{k+1} \vdash C_{k+1}$
  $\}$

Characteristic Clause Set (ctd.)

- $\mathcal{C}_4(0) = \{$
  $\vdash C_0, C'_0$;
  $C_0, C'_0 \vdash$
  $\}$,
- $\mathcal{C}_4(k+1) = \mathcal{C}_4(k) \circ \{\vdash C_{k+1}, C'_{k+1}\} \cup \mathcal{C}_4(k) \circ \{C_{k+1}, C'_{k+1} \vdash\}$.

Characteristic Clause Set (ctd.)

- $\mathrm{CL}(\Psi)\downarrow_0$:
  (1) $C_0 \vdash$
  (2) $C'_0 \vdash$
  (3) $\vdash C_0, C'_0$
  (4) $C_0, C'_0 \vdash$

- $\mathrm{CL}(\Psi)\downarrow_1$:
  (1) $C_0 \vdash$
  (2) $C'_0 \vdash$
  (3) $C_1 \vdash C'_1$
  (4) $C'_1 \vdash C_1$
  (5) $\vdash C_0, C'_0, C_1, C'_1$
  (6) $C'_0, C_0 \vdash C_1, C'_1$
  (7) $C'_1, C_1 \vdash C_0, C'_0$
  (8) $C_0, C'_0, C_1, C'_1 \vdash$

- $\mathrm{CL}(\Psi)\downarrow_2$:
  (1) $C_0 \vdash$
  (2) $C'_0 \vdash$
  (3) $C_1 \vdash C'_1$
  (4) $C'_1 \vdash C_1$
  (5) $C_2 \vdash C'_2, C_1$
  (6) $C'_2 \vdash C_2, C'_1$
  (7) $C'_1, C_2 \vdash C'_2$
  (8) $C_1, C'_2 \vdash C_2$
  (9) $\vdash C_2, C_0, C'_0, C_1, C'_1, C'_2$
  (10) $C'_2, C_2 \vdash C_1, C'_1, C_0, C'_0$
  (11) $C'_1, C_1 \vdash C_2, C'_2, C_0, C'_0$
  (12) $C'_2, C_2, C'_1, C_1 \vdash C_0, C'_0$
  (13) $C'_0, C_0 \vdash C_2, C'_2, C_1, C'_1$
  (14) $C'_2, C_2, C'_0, C_0 \vdash C_1, C'_1$
  (15) $C'_1, C_1, C'_0, C_0 \vdash C_2, C'_2$
  (16) $C_2, C_0, C'_0, C_1, C'_1, C'_2 \vdash$

Refutation Schema

- A resolution refutation schema of $\mathrm{CL}(\Psi)$ is

$$R = ((\varrho, \delta, \eta), \mathcal{R}, \emptyset, \emptyset)$$

where:

- $\varrho(0, X) \to r(r((\vdash C_0, C'_0) \circ X;\ C_0 \vdash;\ C_0);\ C'_0 \vdash;\ C'_0),$
- $\varrho(k+1, X) \to r($

$$r(\varrho(k, (\vdash C_{k+1}, C'_{k+1}) \circ X);\ \eta(k);\ C'_{k+1});$$
$$r(\delta(k);\ \varrho(k, (C_{k+1}, C'_{k+1} \vdash) \circ X);\ C'_{k+1});$$
$$C_{k+1}).$$

Refutation Schema (ctd.)

- and
  - $\delta(0) \rightarrow C_1 \vdash C'_1$,
  - $\delta(k+1) \rightarrow r($

    $C_{k+2} \vdash C'_{k+2}, C_{k+1};$
    $r(\delta(k);\ C'_{k+1}, C_{k+2} \vdash C'_{k+2};\ C'_{k+1});$
    $C_{k+1}).$

  - $\eta(0) \rightarrow C'_1 \vdash C_1$,
  - $\eta(k+1) \rightarrow r($

    $C'_{k+2} \vdash C_{k+2}, C'_{k+1};$
    $r(\eta(k);\ C_{k+1}, C'_{k+2} \vdash C_{k+2};\ C_{k+1});$
    $C'_{k+1}).$

- Finally, refutation of $\mathrm{CL}(\Psi) \downarrow_\alpha$ is defined by $\varrho(\alpha, \vdash) \downarrow$.

Refutation Schema (ctd.)

- $T(\varrho(0, \vdash) \downarrow)$ is:

$$\dfrac{\dfrac{\vdash C_0, C'_0 \qquad C_0 \vdash}{\vdash C'_0} \qquad C'_0 \vdash}{\vdash}$$

- $T(\varrho(1, \vdash) \downarrow)$ is:

$$\dfrac{\dfrac{\overset{(\varrho(0, \vdash C_1, C'_1) \downarrow)}{\vdash C_1, C'_1} \qquad C'_1 \vdash C_1}{\vdash C_1} \qquad \dfrac{C_1 \vdash C'_1 \qquad \overset{(\varrho(0, C_1, C'_1 \vdash) \downarrow)}{C_1, C'_1 \vdash}}{C_1 \vdash}}{\vdash}$$

Refutation Schema (ctd.)

- $T(\varrho(0, \vdash C_1, C_1') \downarrow)$ is:

$$\dfrac{\dfrac{\vdash C_0, C_0', C_1, C_1' \qquad C_0 \vdash}{\vdash C_0', C_1, C_1'} \qquad C_0' \vdash}{\vdash C_1, C_1'}$$

- $T(\varrho(1, \vdash) \downarrow)$ is:

$$\dfrac{\dfrac{\begin{array}{c}(\varrho(0, \vdash C_1, C_1') \downarrow) \\ \vdash C_1, C_1'\end{array} \qquad C_1' \vdash C_1}{\vdash C_1} \qquad \dfrac{C_1 \vdash C_1' \qquad \begin{array}{c}(\varrho(0, C_1, C_1' \vdash) \downarrow) \\ C_1, C_1' \vdash\end{array}}{C_1 \vdash}}{\vdash}$$

Refutation Schema (ctd.)

- $T(\varrho(0, C_1, C_1' \vdash) \downarrow)$ is:

$$\dfrac{\dfrac{C_1, C_1' \vdash C_0, C_0' \qquad C_0 \vdash}{C_1, C_1' \vdash C_0'} \qquad C_0' \vdash}{C_1, C_1' \vdash}$$

- $T(\varrho(1, \vdash) \downarrow)$ is:

$$\dfrac{\dfrac{\begin{array}{c}(\varrho(0, \vdash C_1, C_1') \downarrow)\\ \vdash C_1, C_1'\end{array} \qquad C_1' \vdash C_1}{\vdash C_1} \qquad \dfrac{C_1 \vdash C_1' \qquad \begin{array}{c}(\varrho(0, C_1, C_1' \vdash) \downarrow)\\ C_1, C_1' \vdash\end{array}}{C_1 \vdash}}{\vdash}$$

Questions?