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CERES and Fast Cut-Elimination

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Introduction

Cut elimination

- Cut-elimination is a proof transformation that removes all cut rules from a proof.
- The cut-elimination theorem was proved by Gerhard Gentzen in 1934.
- For the systems, that have a cut-elimination theorem, it is easy to prove consistency.
- Cut-elimination is nonelementary in general, i.e. there is no elementary bound on the size of cut-free proof w.r.t the original one.

Sequent calculus LK

- A sequent is an expression of the form $\Gamma \vdash \Delta$, where Γ and Δ are multisets of formulas.
- A rule is an inference of a lower sequent from an upper sequent.
- A derivation is a directed tree with nodes as sequences and edges as inferences.
- A proof of the sequence S is a derivation of S with axioms as leaf nodes.

The *cut* rule

- The *cut* rule:

$$\frac{\Gamma \vdash \Delta, A \quad A, \Pi \vdash \Lambda}{\Gamma, \Pi \vdash \Delta, \Lambda}\, cut$$

- The *cut* rule is the only rule such that its upper sequents may contain formulas that do not appear in the lower sesuents.
- The *cut* rule is the only rule that may produce an empty sequent $\vdash$ (inconsistency).
- The upper sequents of a *cut* rule corresponds to the lemmas into the proof.

The resolution calculus

- Clauses are atomic sequents.
- The resolution rule is a cut rule on clauses, where cut-formulas A and B can be unified with m.g.u σ.
- The factorization rule is a contraction rule on clauses, where contracted formulas can be unified with m.g.u σ.
- The resolution deduction is a derivation tree having clauses as nodes and resolution, factorization and weakening rules as edges.
- The resolution refutation is a resolution derivation of the empty clause $\vdash$.

Tools

Decidable subclasses of FOL

- Herbrand class: $(Q\vec{x})(L_1 \wedge \ldots \wedge L_m)$.
- Bernays - Schönfinkel class: $(\exists\vec{x})(\forall\vec{y})M$.
- Ackermann class: $(\exists\vec{x})(\forall y)(\exists\vec{z})M$.
- One-variable class: $|Var(F)| \leq 1$.
- Monadic class: formulas contain only unary predicate symbols.

Idea

- Identify classes of **LK**-proofs, whose characteristic clause set falls into one of the decidable subclasses of first-order logic.
- Use CERES method as a tool to prove fast cut-elimination.

Fast cut-elimination classes

Class UIE

- All inferences that go into the end-sequent are unary.

Class AXDC

- Different axioms are variable disjoint.

Class UILM

- Only one monotone cut.
- All inferences in the left cut-derivation that go into the end-sequent are unary.

$$2^{2^n}$$

Class UIRM

- Only one monotone cut.
- All inferences in the right cut-derivation that go into the end-sequent are unary.

Class MC

- All function and predicate symbols appearing in cut-formulas are monadic.

Methods of Cut Elimination

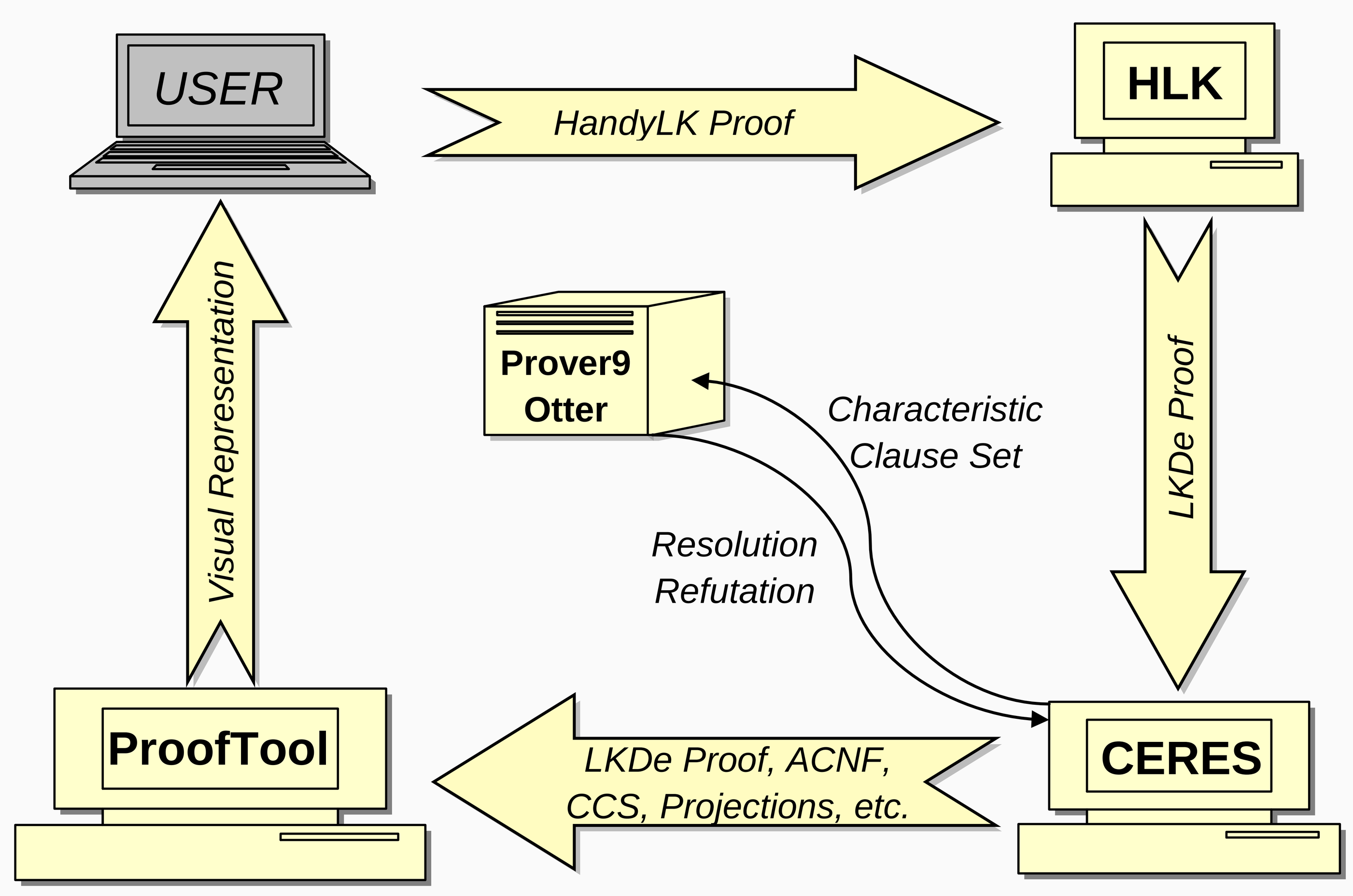
Gentzen's method

- Gentzen's method of cut-elimination is reductive, i.e. proof rewriting system is defined which is terminating and its normal form is a cut-free proof.
- Rewriting rules are divided into two parts: grade reduction and rank reduction rules.
- Grade of a cut rule is the number of logical symbols in the cut-formula.
- Rank of a cut rule is the number of sequents in the left cut-derivation, where cut-formula occurs in its succedent plus the number of sequents in the right cut-derivation, where the cut-formula occurs in its antecedent.

CERES method

- CERES is a cut-elimination method by resolution.
- CERES method radically differs from reductive methods.
- CERES method consists of the following steps:
  1. Skolemization of the proof.
  2. Computation of the characteristic clause set.
  3. Refutation of the characteristic clause set.
  4. Computation of the proof projections and construction of the atomic cut normal form.
- CERES is fast on the subclass of **LK**-proofs iff resolution complexity of the characteristic clause set is bound by an elementary function.

Cut-elimination system *CERES*



Results

Class G-UILM

- All cuts are monotone.
- All inferences in all left cut-derivation that go into the end-sequent are unary.
- No binary rule, that goes into the end sequent, connects two cuts.

$$2^{2^n}$$

Class G-UIRM

- All cuts are monotone.
- All inferences in all right cut-derivation that go into the end-sequent are unary.
- No binary rule, that goes into the end sequent, connects two cuts.

Class ONEQ

- All cut-formulas have at most one quantifier.

Conclusion

- Proof transformation, in particular cut-elimination, is one of the key techniques of proof theory.
- Cut-elimination is nonelementary but we use CERES method as a tool to identify classes where it is elementary.
- We proved fast cut-elimination for new classes G-UILM, G-UIRM and ONEQ.