On Proving and Characterizing Operational Termination of Deterministic Conditional Rewrite Systems

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1 Introduction and Overview

Conditional term rewriting systems (CTRSs) are a natural extension of unconditional such systems (TRSs) allowing rules to be guarded by conditions. Conditional rules tend to be very intuitive and easy to formulate and are therefore used in several programming and specification languages, such as Maude or ELAN. Besides, the particular class of *deterministic* (oriented) CTRSs (DCTRSs) has been used for instance in proofs of termination of (well-moded) logic programs, [2]. Recently, the notion of *operational termination* of DCTRSs was defined in [3], which guarantees finite reductions in existing rewrite engines. In [8], based on the idea of *unravelings* of [5], a transformation from DCTRSs into TRSs was proposed, such that termination of the transformed TRS implies *quasi-reductivity* of the DCTRS. In this extended abstract [1] we provide an alternative definition of *quasi-reductivity*, which allows us to prove the property for strictly more DCTRSs than Ohlebusch's transformation was able to handle, while preserving the important implications of quasi-reductivity. The key concept used is context-sensitive rewriting [4]. We define a transformation from DCTRSs into context-sensitive (unconditional) TRSs such that termination of the transformed context-sensitive TRS implies *context-sensitive quasi-reductivity* of the DCTRS. Furthermore we relate the latter notion to other known ones and give equivalent characterizations of operational termination.

2 Preliminaries

We assume familiarity with the basic concepts and notations of (context-free and context-sensitive) term rewriting (cf. e.g. [1], [4]). We are concerned with *oriented* 3-TRSs. Such systems consist of rules of the form $l \to r \Leftarrow c$, with c of the form $s_1 \to^* t_1, ..., s_n \to^* t_n$ such that l is not a variable and $Var(r) \subseteq Var(l) \cup Var(c)$. The conditional rewrite relation induced by a CTRS $\mathcal{R}$ is inductively defined as $\to_{\mathcal{R}} = \bigcup_{i \geq 0} R_i$ where $R_0 = \emptyset$ and $R_{i+1} = \{\sigma l \to \sigma r \mid l \to r \Leftarrow s_1 \to^* t_1, ..., s_n \to^* t_n \in \mathcal{R} \wedge \sigma s_i \to^*_{R_i} \sigma t_i$ for all $1 \leq i \leq n\}$. A deterministic CTRS (DCTRS) is an oriented 3-CTRS where for each rule $l \to r \Leftarrow s_1 \to^* t_1, ..., s_n \to^* t_n$ it holds that $Var(s_i) \subseteq l \cup \bigcup_{j=1}^{i-1} Var(t_j)$.

A DCTRS (Σ, R) is called *quasi-reductive*, cf. [8] [2], if there exists an extension Σ' of Σ and a well-founded partial order $\succ$ on $\mathcal{T}(\Sigma', V)$, which is monotonic, i.e., closed under contexts, such that for every rule $l \to r \Leftarrow s_1 \to^* t_1, ..., s_n \to^* t_n \in R$, every $\sigma : V \to \mathcal{T}(\Sigma', V)$ and every $i \in \{0, ..., n-1\}$:

- If $\sigma s_j \succeq \sigma t_j$ for every $1 \leq j \leq i$, then $\sigma l \succ_{st} \sigma s_{i+1}$.

[1] A long version of this paper with complete proofs is available at http://www.logic.at/people/schernhammer/papers/wst07-long.pdf.

– If $\sigma s_j \succeq \sigma t_j$ for every $1 \leq j \leq n$, then $\sigma l \succ \sigma r$.

Here $\succ_{st} = (\succ \cup \rhd)^+$ ($\rhd$ denotes the proper subterm relation).

A DCTRS $\mathcal{R} = (\Sigma, R)$ is *quasi-decreasing* [8] if there is a well-founded partial ordering $\succ$ on $\mathcal{T}(\Sigma, V)$, such that $\rightarrow_{\mathcal{R}} \subseteq \succ$; $\succ = \succ_{st}$; and for every rule $l \rightarrow r \Leftarrow s_1 \rightarrow^* t_1, ..., s_n \rightarrow^* t_n \in R$, every substitution σ and every $i \in \{0, ..., n-1\}$ it holds that $\sigma s_j \rightarrow^*_{\mathcal{R}} \sigma t_j$ for all $j \in \{1, ..., i\}$ implies $\sigma l \succ \sigma s_{i+1}$.

In [3] a notion of *operational termination* of (D)CTRSs is defined via the absence of infinite well-formed trees in a certain logical inference system. In the case of DCTRSs, this notion is shown to be equivalent to quasi-decreasingness [3]. The relation between the latter notions is ([8], [3]): Quasi-reductivity $\Rightarrow$ quasi-decreasingness $\Leftrightarrow$ operational termination $\Rightarrow$ effective termination.

3 Context-sensitive quasi-reductivity

Definition 1. *A DCTRS $\mathcal{R}$ ($\mathcal{R} = (\Sigma, R)$) is called* context-sensitively quasi-reductive *(*cs-quasi-reductive*) if there is an extension of the signature Σ' ($\Sigma' \supseteq \Sigma$), a replacement μ (s.t. $\mu(f) = \{1, ..., ar(f)\}$ for all $f \in \Sigma$) and a μ-monotonic, well-founded partial order $\succ_\mu$ on $\mathcal{T}(\Sigma', V)$ satisfying for every rule $l \rightarrow r \Leftarrow s_1 \rightarrow^* t_1, ..., s_n \rightarrow^* t_n$, every $\sigma\colon V \rightarrow \mathcal{T}(\Sigma, V)$ and every $i \in \{0, ..., n-1\}$:* [2]

– *If $\sigma s_j \succeq_\mu \sigma t_j$ for every $1 \leq j \leq i$ then $\sigma s_{i+1} \prec^{st}_\mu \sigma l$.*
– *If $\sigma s_j \succeq_\mu \sigma t_j$ for every $1 \leq j \leq n$ then $\sigma r \prec_\mu \sigma l$.*

The ordering $\prec^{st}_\mu$ is defined as $(\prec_\mu \cup \lhd_\mu)^+$ where $t \lhd_\mu s$ if and only if s is a proper subterm of t at some position $p \in Pos^\mu(t)$.

Lemma 1. *Quasi-reductivity implies cs-quasi-reductivity which in turn implies quasi-decreasingness.*

We will define a transformation from DCTRSs into CSRSs, such that μ-termination of the transformed CSRS implies cs-quasi-reductivity of the original DCTRS. The transformation is actually a variant of the one in [8].

Definition 2. [8] *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS. For every rule $\alpha : l \rightarrow r \Leftarrow s_1 \rightarrow^* t_1, ..., s_n \rightarrow^* t_n$ we use n new functions symbols U^α_i ($i \in \{1, ..., n\}$). Then α is transformed into a set of unconditional rules in the following way:*

$$\begin{aligned} l &\rightarrow U^\alpha_1(s_1, Var(l)) \\ U^\alpha_1(t_1, Var(l)) &\rightarrow U^\alpha_2(s_2, Var(l), \mathcal{E}Var(t_1)) \\ &\vdots \\ U^\alpha_n(t_n, Var(l), \mathcal{E}Var(t_1), ..., \mathcal{E}Var(t_{n-1})) &\rightarrow r \end{aligned}$$

Here $Var(s)$ denotes the sequence of variables in a term s rather than the set. The set $\mathcal{E}Var(t_i)$ is $Var(t_i) \setminus (Var(l) \cup \bigcup_{j=1}^{i-1} Var(t_j))$. Again, abusing notation, by $\mathcal{E}Var(t_i)$ we mean an arbitrary but fixed sequence of the variables in the set $\mathcal{E}Var(t_i)$. Any unconditional rule of $\mathcal{R}$ is transformed into itself (this degenerate case is missing in [8, Def. 7.2.48]). The transformed system $U(\mathcal{R}) = (U(\Sigma), U(R))$ is obtained by transforming each rule of $\mathcal{R}$ where $U(\Sigma)$ is Σ extended by all new function symbols.

[2] Note that – in contrast to the definition of quasi-reductivity – the substitution maps variables only into terms over the original signature. This restriction is crucial for some results (cf. Theorem 2 and Corollary 1).

This transformation is sound w.r.t. quasi-reductivity, i.e., whenever the transformed system $U(\mathcal{R})$ is terminating, the original DCTRS $\mathcal{R}$ is *quasi-reductive* [8]. The transformation is not complete in this respect, though.

Example 1. **[5]** Consider the deterministic CTRS $\mathcal{R} = (\Sigma, R)$ given by

$$\begin{array}{rlll} a \to c & & b \to c & c \to e \\ a \to d & & b \to d & c \to l \\ k \to l & & k \to m & d \to m \\ h(x,x) \to g(x,x,f(k)) & & g(d,x,x) \to A & \alpha : f(x) \to x \Leftarrow x \to^* e \\ A \to h(f(a), f(b)). & & & \end{array}$$

The system $U(\mathcal{R}) = (U(\Sigma), U(R))$ is given by $U(\Sigma) = \Sigma \cup \{U_1^\alpha\}$ and $U(R) = R$ except that rule α is replaced by the rules $f(x) \to U_1^\alpha(x,x)$ and $U_1^\alpha(e,x) \to x$. $\mathcal{R}$ is quasi-reductive, nevertheless $U(\mathcal{R})$ is non-terminating ([8]).

Roughly speaking, the problem in the latter example is that subterms at the second position of U_1^α are reduced, which are actually only supposed to "store" the variable bindings for future rewrite steps. These reductions can be avoided by using context-sensitivity.

Definition 3. *Let $\mathcal{R} = (\Sigma, R)$ be a deterministic conditional term rewriting system. The context-sensitive rewrite system $U_{cs}(\mathcal{R})$ uses the same signature and the same rules as $U(R)$. Additionally, we use a replacement map $\mu_{U_{cs}(\mathcal{R})}$ with $\mu_{U_{cs}(\mathcal{R})}(f) = \{1\}$ if $f \in U_{cs}(\Sigma) \backslash \Sigma$ and $\mu_{U_{cs}(\mathcal{R})}(f) = \{1, ..., ar(f)\}$ if $f \in \Sigma$.*

Proposition 1 (simulation completeness). *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS and $U_{cs}(\mathcal{R})$ its transformed CSRS. For every $s, t \in \mathcal{T}(\Sigma, V)$ we have: If $s \to_{\mathcal{R}} t$, then $s \to^+_{U_{cs}(\mathcal{R})} t$.*

Proposition 2 (simulation soundness). *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS and let $U_{cs}(\mathcal{R}) = (U(\Sigma), U(R))$ be its transformed CSRS. For every $s, t \in \mathcal{T}(\Sigma, V)$ we have: If $s \to^+_{U_{cs}(\mathcal{R})} t$, then $s \to^+_{\mathcal{R}} t$.*

Furthermore, $\mu_{U_{cs}(\mathcal{R})}$-termination of the transformed system $U_{cs}(\mathcal{R})$ implies cs-quasi-reductivity of the original DCTRS $\mathcal{R}$, cf. the more general Theorem 2 below.

Example 2. Consider the DCTRS $\mathcal{R}$ of Example 1. The transformed system $U_{cs}(\mathcal{R})$ (which is identical to $U(\mathcal{R})$, except for the fact that an additional replacement map is used) is $\mu_{U_{cs}(\mathcal{R})}$-terminating. To show this one can use induction on the term depth.

Unfortunately, and interestingly, cs-quasi-reductivity of a DCTRS $\mathcal{R}$ does not imply $\mu_{U_{cs}(\mathcal{R})}$-termination of $U_{cs}(\mathcal{R})$, cf. [8, Ex. 7.2.51]. Yet, we do get $\mu_{U_{cs}(\mathcal{R})}$-termination of $U_{cs}(\mathcal{R})$ on original terms, even for quasi-decreasing systems. Conversely, cs-quasi-reductivity follows from termination of the transformed system on original terms.

Theorem 1. *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS. If $\mathcal{R}$ is quasi-decreasing, then $U_{cs}(\mathcal{R})$ is $\mu_{U_{cs}(\mathcal{R})}$-terminating on $\mathcal{T}(\Sigma, V)$.*

Theorem 2. *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS. If $U_{cs}(\mathcal{R})$ is $\mu_{U_{cs}(\mathcal{R})}$-terminating on $\mathcal{T}(\Sigma, V)$, then $\mathcal{R}$ is cs-quasi-reductive.*

Corollary 1. *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS. The following properties of $\mathcal{R}$ are equivalent: $\mu_{U_{cs}(\mathcal{R})}$-termination of $U_{cs}(\mathcal{R})$ on $\mathcal{T}(\Sigma, V)$, cs-quasi-reductivity, quasi-decreasingness, and operational termination.*

From a practical point of view it remains unclear whether the restricted proof task of showing $\mu_{U_{cs}(\mathcal{R})}$-termination of $U_{cs}(\mathcal{R})$ on $\mathcal{T}(\Sigma, V)$ is really easier than proving $\mu_{U_{cs}(\mathcal{R})}$-termination of $U_{cs}(\mathcal{R})$ on all terms over $\Sigma' = U(\Sigma)$.

Furthermore, it is currently open to what degree termination proofs of DCTRSs based on cs-quasi-reductivity are in practice more successful than those based on the previous notion of quasi-reductivity. This seems to depend strongly on the power of existing termination tools for context-sensitive systems.

4 Related Work and Discussion

A very similar modification of the transformation in [8] was proposed by [7]. However, there, besides context-sensitivity, the authors additionally impose a *membership condition* on the rewrite relation of the transformed CSRS. In [6] a further refinement of the transformation of [7] is presented, which is able to reduce the number of U symbols in the transformed system in some cases.

Our notion of cs-quasi-reductivity provides a new sufficient (in fact, equivalent) criterion for operational termination. Furthermore, cs-quasi-reductivity can be verified by proving termination of the resulting CSRS (on original terms). We have shown that our new transformation yields operational termination of strictly more DCTRSs than Ohlebusch's context-free transformation. However, we could not automatically, i.e., with termination tools, verify operational termination of the DCTRS of Example 1. Thus more powerful termination tools and/or features appear to be necessary.

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