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Abstract

We study (un)soundness of transformations of conditional term rewriting systems (CTRSs) into unconditional term rewriting systems (TRSs). The focus here is on analyzing (un)soundness of so-called unravelings, the most basic and natural class of such transformations. We extend our previous analysis from normal 1-CTRSs to the more general class of deterministic CTRSs (DCTRSs) where extra variables in right-hand sides of rules are allowed to a certain extent. We prove that the previous soundness results based on weak left-linearity and on right-linearity can be extended from normal 1-CTRSs to DCTRSs. Counterexamples show that such an extension to DCTRSs does not work for the previous criteria which were based on confluence and on non-erasingness, not even for right-stable systems. Yet, we prove weaker versions of soundness criteria based on confluence and on non-erasingness. Finally, we compare our approach and results with other recently established soundness criteria for unraveling DCTRSs.

1 Introduction and Overview

1.1 Background and Motivation

Unconditional term rewriting systems (TRSs) are very well studied and enjoy many nice properties. However, often TRSs are insufficient to appropriately model computations or specifications, since the applicability of rules is inherently conditional. Thus, conditional term rewriting systems (CTRSs) naturally arise in many settings and examples. Since conditional rewriting is known to be much more involved than unconditional rewriting, both in theory and in practice, an attractive approach to analysis and implementation of CTRSs consists in transforming them into unconditional TRSs where ordinary (unconditional) rewriting can simulate the original conditional computations.

There exists abundant literature on conditional rewriting, cf. e.g. [16, 3] and also on transforming CTRSs into TRSs, where computation is sometimes restricted by further mechanism like membership constraints, context-sensitivity or imposed reduction strategies so as to avoid reductions which have no counterpart in the conditional setting (cf. e.g. [6, 8, 14, 13, 17, 19]). Typically, *completeness* of such transformations (w.r.t. reduction) is easily obtained and proved (by construction). However, *soundness* is much more difficult to analyze and to achieve. The reason simply is that in the encoding much more fine-grained rewrite computations are possible and potentially dangerous, since they may lead to reductions between terms in the original signature which have not been possible in the original CTRS.

Concerning soundness of transformations using restricted versions of unconditional rewriting various positive and negative results are known. But concerning soundness of transformation approaches using unrestricted unconditional rewriting little was known until recently, and mostly only for the simplest class of CTRSs, namely oriented normal 1-CTRSs, and only for the simplest class of such transformations, the *unravelings* ([10]).[1]

The first important soundness result for unravelings is due to Marchiori [10] who showed that unraveling left-linear normal 1-CTRSs is sound (cf. also [16]). In [12] and — based on this paper — recently in [15] Nishida et al. presented an analysis of (a slightly optimized form of) unraveling *deterministic* CTRSs (DCTRSs), an interesting subclass of 3-CTRSs, where they showed that soundness is guaranteed if the transformed system is either left-linear or both right-linear and non-erasing. In [8] we have shown that a few other sufficient soundness criteria for the case of normal 1-CTRSs exist, including confluence and non-erasingness. Moreover, we could show there that instead of left-linearity even *weak left-linearity* is sufficient for soundness. In weakly left-linear systems one may have non-left-linear rules like $eq(x, x) \to true$ where the non-linear variables are erased.

Here we extend our analysis for normal 1-CTRSs also to the practically important class of DCTRSs, and finally compare the approach and results with the ones of [15].

1.2 Contributions

First we discuss *simultaneous* versus *sequential* unravelings. Instead of *simultaneously* unraveling normal 1-CTRSs as in [16, 8], a *sequential* unraveling is also perfectly possible thus enforcing a simulated evaluation of the conditions from left to right. In the case of DCTRSs sequentially unraveling still yields an ordinary unconditional TRS, although the original DCTRS may have extra variables in the right-hand sides and the conditions of rules. A careful analysis reveals that for normal 1-CTRSs all soundness results for the simultaneous case from [8] extend to the sequential case.

Our main results for transforming DCTRSs into TRSs via sequential unravelings are the following:

- We show that various tempting extensions of the soundness results for normal 1-CTRSs to DCTRSs do not hold, even for quite restricted sub-classes of DCTRSs, in particular potential criteria based on confluence and on non-erasingness (as in [8]), cf. Example 4).
- Our main positive result is that weak left-linearity of a DCTRS is already sufficient for soundness of its transformed version (Theorem 31).
- For non-erasingness, we show that we obtain soundness only for a restricted class of DCTRSs (Theorem 19).
- Furthermore, right-linearity of the transformed system is also a sufficient condition for soundness (Theorem 14).
- Regarding confluence, we only get a weaker soundness criterion w.r.t. reduction to normal form (Theorem 14).

The rest of the paper is structured as follows. In Section 2 we present the necessary technical and conceptual background. The main Section 3 contains the soundness analysis. In Section 4 we discuss related work, especially [12, 15], and promising directions for future research.

[1] The very idea of *unravelings* is actually much older and appears already e.g. in [4], though in a specialized form (for function definitions).

2 Preliminaries

We assume familiarity with the basic concepts and notations of abstract reductions systems (ARSs) and (conditional) term rewriting systems (CTRSs) (cf. e.g. [2], [16], [3]).

2.1 Basics

For the sake of readability we recall some notions and notations: The set of (non-variable, variable) positions of a term s is denoted as $\mathcal{P}os(s)$ ($\mathcal{FP}os(s)$, $\mathcal{VP}os(s)$). $\mathsf{root}(s)$ denotes the root symbol of the term s. Throughout the paper $\mathcal{V}$ denotes a countably infinite set of variables. x, y, z denote variables from $\mathcal{V}$. By $\mathcal{V}ar(s)$ we denote the set of variables of a term s. If $X = \{x_1, \ldots, x_n\}$ is a set of variables, we denote by $\vec{X}$ the sequence of all variables in X in some arbitrary but fixed order. By $|s_1, \ldots, s_n|_x$ we mean the number of occurrences of the (variable) symbol x in $s_1, \ldots, s_n$.

A term rewriting system $\mathcal{R}$ is a pair $(\mathcal{F}, R)$ of a signature and a set of rewrite rules over this signature. Slightly abusing notation we also write $\mathcal{R}$ instead of R (leaving the signature implicit). A rewrite system $\mathcal{R}$ is called *non-erasing* (NE) if $\mathcal{V}ar(r) = \mathcal{V}ar(l)$ for all $l \to r \in \mathcal{R}$, and *left-linear* (LL) (*right-linear* (RL)) if every $x \in \mathcal{V}ar(l)$ ($x \in \mathcal{V}ar(r)$) occurs exactly once in l (r) for all rules $l \to r \in \mathcal{R}$.

The function $\mathsf{lin}\colon \mathcal{T} \to \mathcal{T}$ renames non-linear variables into fresh new variables while keeping the linear ones.

We denote a rewrite step from a term u to a term v at position p in a rewrite system $\mathcal{R}$ with a rule α from $\mathcal{R}$ as $u \to_{p,\mathcal{R},\alpha} v$. We skip $p, \mathcal{R}$ or α if they are clear from the context or of no relevance. The parallel reduction at positions $P \subseteq \mathcal{P}os(u)$ is denoted as $u \mathrel{+\!\!\!\!\!\!\!\;\to}_{P,\mathcal{R}} v$.

The set of *one-step descendants* of a (subterm) position p of a term u w.r.t. a (one-step) reduction $u = C[s]_p \to_q v$ is the set of positions in v given by $\{p\}$, if $q \geq p$ or $q \parallel p$; $\{q.o'.p' \mid t|_q = l\sigma, l|_o \in \mathcal{V}ar(l), q.o.p' = p, l|_o = r|_{o'}\}$, if $q < p$ and (a superterm of) s is bound to a variable in the matching of $u|_q$ with the lhs of the applied rule; and $\emptyset$, otherwise. Slightly abusing terminology, when $u = C[s]_p \to_q v$ with set $\{p_1, \ldots, p_k\}$ of one-step descendants of p in v, we also say that $u|_p$ has the one-step descendants $v|_{p_i}$ in v. The *descendant relation* (w.r.t. given derivations) is obtained as the (reflexive-)transitive closure of the one-step descendant relation. The relation of (one-step) *ancestors* of a subterm position (w.r.t. a given reduction sequence) is the inverse relation of the (one-step) descendant relation.

A conditional term rewriting system $\mathcal{R}$ (over some signature $\mathcal{F}$) consists of rules $l \to r \Leftarrow c$ where $l \notin \mathcal{V}$ and c is a conjunction of equations $s_i = t_i$. Equality in the conditions may be interpreted (recursively) e.g. as $\leftrightarrow^*$ (semi-equational case), as $\downarrow$ (join case), or as $\to^*$ (oriented case). In the latter case, if all right-hand sides of conditions are ground terms that are irreducible w.r.t. the unconditional version $\mathcal{R}_u = \{l \to r \mid l \to r \Leftarrow c \in \mathcal{R}\}$ of $\mathcal{R}$, the system is said to be a *normal* CTRS. Subsequently, unless otherwise stated, we will always consider oriented CTRSs.

According to the distribution of variables, a conditional rule $l \to r \Leftarrow c$ may satisfy (1) $\mathcal{V}ar(r) \cup \mathcal{V}ar(c) \subseteq \mathcal{V}ar(l)$, (2) $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$, (3) $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l) \cup \mathcal{V}ar(c)$, or (4) no variable constraints at all. If all rules of a CTRS $\mathcal{R}$ are of type (i), $1 \leq i \leq 4$, we say that $\mathcal{R}$ is an i-CTRS. Given a conditional rewrite rule $l \to r \Leftarrow c$ and a variable x such that $x \in \mathcal{V}ar(r) \cup \mathcal{V}ar(c)$ but $x \notin \mathcal{V}ar(l)$, we say that x is an *extra variable.* An oriented 3-CTRS $\mathcal{R}$ is called *deterministic* if for every conditional rule $l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ we have $\mathcal{V}ar(s_i) \subseteq \mathcal{V}ar(l, t_1, \ldots, t_{i-1})$. Note that a normal 1-CTRS is by definition also a DCTRS. A DCTRS $\mathcal{R}$ is *right-stable* (RS, cf. [18]) if for every conditional rule $l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ we have $\mathcal{V}ar(l, s_1, t_1, \ldots, s_{i-1}, t_{i-1}, s_i) \cap \mathcal{V}ar(t_i) = \emptyset$, and t_i is a linear constructor term or a ground $\mathcal{R}_u$-normal form for all $1 \leq i \leq n$. To simplify the presentation of some results we will sometimes denote conditional rules as $t_0 \to s_{n+1} \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$.

The rewrite relation of an oriented CTRS $\mathcal{R}$ is inductively defined as follows: $R_0 = \emptyset$, $R_{j+1} = \{l\sigma \to r\sigma \mid l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \in \mathcal{R} \wedge s_i\sigma \to^*_{R_j} t_i\sigma$ for all $1 \le i \le n\}$, and $\to_{\mathcal{R}} = \bigcup_{j \ge 0} \to_{R_j}$.

2.2 Unravelings

There exists abundant literature on transforming CTRSs into unconditional systems. For a unified parameterized approach to such transformations and the relevant terminology we refer to [7]. Unravelings as introduced and investigated in [10] are the most simple and intuitive ones. We present here a sequential version of unraveling for DCTRS.

Definition 1 (sequential unraveling of DCTRSs, [16]). *Let $\mathcal{R}$ be a DCTRS. For every conditional rule $\alpha\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ we use n new function symbols U_i^α ($i \in \{1, \ldots, n\}$). Then α is transformed into a set of unconditional rules as follows:*[2]

$$\mathbb{U}_{seq}(\alpha) = \{l \to U_1^\alpha(s_1, \vec{X}_1)\ ,\ U_1^\alpha(t_1, \vec{X}_1) \to U_2^\alpha(s_2, \vec{X}_2)\ ,\ \ldots\ ,\ U_n^\alpha(t_n, \vec{X}_n) \to r\}$$

where $X_i = \mathcal{V}ar(l, t_1, \ldots, t_{i-1})$. Any unconditional rule β of $\mathcal{R}$ is transformed into itself: $\mathbb{U}_{seq}(\beta) = \{\beta\}$. The transformed system $\mathbb{U}_{seq}(\mathcal{R}) = \mathcal{R}'_{seq} = (\mathcal{F}', R')$ is obtained by transforming each rule of $\mathcal{R}$ where $\mathcal{F}'$ is $\mathcal{F}$ extended by all new function symbols.

The *simultaneous* unraveling of a rule α from $\mathcal{R}$ as above just yields one introduction rule and one elimination rule: $\mathbb{U}_{sim}(\alpha) = \{l \to U^\alpha(s_1, \ldots, s_n, \vec{X})\ ,\ U^\alpha(t_1, \ldots, t_n, \vec{X}) \to r\}$ where $X = \mathcal{V}ar(l, t_1, \ldots, t_{n-1})$.

Unconditional rules remain invariant. The resulting (*unraveled*) TRS is denoted by $\mathbb{U}_{sim}(\mathcal{R})$ or $\mathcal{R}'_{sim}$.

Remark 2 (simultaneous versus sequential unraveling). *Note that for a normal 1-CTRS $\mathcal{R}$, $\mathbb{U}_{sim}(\mathcal{R})$ is indeed a TRS. However, for a given DCTRS $\mathcal{R}$ its simultaneously unraveled system $\mathbb{U}_{sim}(\mathcal{R})$ does in general not satisfy the variable condition of TRSs $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$ for its rules $l \to r$.*

In [15], an "optimized" version $\mathbb{U}_{opt}$ of $\mathbb{U}_{seq}$ is presented. In $\mathbb{U}_{opt}$ variable bindings are only passed along the computation process if they are eventually used again. Intermediate results in other variable bindings are dropped. The conditional rule α is thereby transformed into $\mathbb{U}_{opt}(\alpha) = \{l \to U_1^\alpha(s_1, \vec{X}_1)\ ,\ U_1^\alpha(t_1, \vec{X}_1) \to U_2^\alpha(s_2, \vec{X}_2)\ ,\ \ldots\ ,\ U_n^\alpha(t_n, \vec{X}_n) \to r\}$ where $X_i = \mathcal{V}ar(l, t_1, \ldots, t_{i-1}) \cap \mathcal{V}ar(t_i, s_{i+1}, t_{i+1}, \ldots, s_n, t_n, r)$.

Symbols from $\mathcal{F}' \setminus \mathcal{F}$ are also called U-symbols. Terms rooted by such symbols are called U-terms or U-rooted terms. Terms containing U-terms are *mixed terms* (as opposed to *original terms*). Every U-symbol corresponds to a particular conditional rewrite rule of the original CTRS according to Definition 1. Hence, we write U_j^α to indicate that U_j^α corresponds to the rewrite rule α. Rewrite rules $l \to r$ of an unraveled DCTRS are called *original unconditional rules* if neither $\mathsf{root}(l)$ nor $\mathsf{root}(r)$ is a U-symbol, *U-introduction rules* if $\mathsf{root}(l)$ is not a U-symbol and $\mathsf{root}(R)$ is a U-symbol, *U-switch rules* (or just *switch rules*) if both $\mathsf{root}(l)$ and $\mathsf{root}(r)$ are U-symbols and *U-elimination rules* if $\mathsf{root}(l)$ is a U-symbol and $\mathsf{root}(r)$ is not a U-symbol.

[2] Using $\overrightarrow{\mathcal{V}ar(s)}$ as sequence of the variables in s goes back to [16], whereas in [10] the sequence is constructed from the *multiset* of variables in s. The former version appears to be generally preferable, because it is more abstract and avoids additional complications due to "non-synchronization effects".

If a property $\mathcal{P}$ is satisfied in the transformed DCTRS $\mathbb{U}_{seq}(\mathcal{R})$ ($\mathbb{U}_{opt}(\mathcal{R})$) then $\mathcal{R}$ satisfies the *ultra-property*, *ultra-*$\mathcal{P}$ w.r.t. $\mathbb{U}_{seq}$ ($\mathbb{U}_{opt}$) (cf. [10]) or short $\mathbb{U}$-$\mathcal{P}$ ($\mathbb{U}_{opt}$-$\mathcal{P}$). Observe, that $\mathbb{U}_{opt}$-$\mathcal{P}$ is in many cases different from $\mathbb{U}$-$\mathcal{P}$. While $\mathbb{U}_{seq}$-LL is equivalent to $\mathbb{U}_{opt}$-LL, $\mathbb{U}_{opt}$-RL and $\mathbb{U}_{opt}$-NE are more general than $\mathbb{U}_{seq}$-RL and $\mathbb{U}_{seq}$-NE, respectively.

From now on, unless stated otherwise, $\mathbb{U}$ is the unraveling $\mathbb{U}_{seq}$, $\mathcal{R} = (\mathcal{F}, R)$ is a DCTRS and $\mathcal{R}' = (\mathcal{F}', R')$ denotes its unraveled TRS (using $\mathbb{U}$). By $\mathcal{T}$ we mean the terms over the original signature $\mathcal{F}$ and by $\mathcal{T}'$ the terms over the extended signature $\mathcal{F}'$.

For proof-technical reasons, in particular in order to show that unraveled systems are not too general and do not enable "too many" reductions, we use a function that maps mixed terms to original terms. The idea of this function is to recursively substitute for each U-term the left-hand side of the corresponding conditional rule instantiated by the substitution which is determined by the variable bindings stored in the U-term.

Definition 3 (translate backwards (tb))**.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a DCTRS. The mapping $\mathsf{tb} : \mathcal{T}' \to \mathcal{T}$ (read "translate back") which is equivalent to Ohlebusch's mapping ∇ ([16, Definition 7.2.53]) is defined as follows:*

$$\mathsf{tb}(t) = \begin{cases} x & \text{if } t = x \in \mathcal{V} \\ f(\mathsf{tb}(t_1), \ldots, \mathsf{tb}(t_{\mathsf{ar}(f)})) & \text{if } t = f(t_1, \ldots, t_{\mathsf{ar}(f)}) \text{ and } f \in \mathcal{F} \\ l\sigma & \text{if } t = U_j^\alpha(u, v_1, \ldots, v_k) \text{ and } x_i\sigma = \mathsf{tb}(v_i) \text{ for } 1 \le i \le k \end{cases}$$

where α is the rule $l \to r \Leftarrow c$ and $\overrightarrow{\mathcal{V}ar(l)} = x_1, \ldots, x_{k'}$ $(1 \le k' \le k)$.

Observe, that tb cannot be sensibly defined for $\mathbb{U}_{opt}$, since in $\mathbb{U}_{opt}$ not all variable bindings of the left-hand side of rules are preserved in U-terms.

In this paper we focus on the property of *soundness* of unravelings which is dual to the (easier to obtain) property of completeness. An unraveling is said to be *complete for reduction* (or *simulation-complete*) for a class of CTRSs if for every CTRS $\mathcal{R}$ of this class, $u \to_{\mathcal{R}}^* v$ for $u, v \in \mathcal{T}$ implies $u \to_{\mathcal{R}'}^* v$. An unraveling is *sound for reduction* (or *simulation-sound*) if $u \to_{\mathcal{R}'}^* v$ implies $u \to_{\mathcal{R}}^* v$. We use the notion of *soundness for reduction to normal form*, which means that $u \to_{\mathcal{R}'}^* v$ with v being a normal form implies $u \to_{\mathcal{R}}^* v$. Given a particular CTRS $\mathcal{R}$, we also say that the unraveling is complete (sound) for $\mathcal{R}$ or, slightly abusing terminology, that $\mathcal{R}'$ is complete (sound) w.r.t. $\mathcal{R}$. For a more thorough discussion of the terminology used for (preservation properties of) transformations we refer to [7].

3 Sufficient Criteria for Soundness of Unraveling DCTRSs

For the case of a normal 1-CTRS $\mathcal{R}$ it was shown in [8] that $\mathcal{R}$ can soundly be (simultaneously) unraveled provided that it is either confluent, or non-erasing, or weakly left-linear or contains only ground conditions. A careful inspection of the proofs in [8] reveals that when using sequential instead of simultaneous unraveling the same results can also be proved in essentially the same way.

The main additional complication here is that when analyzing reduction sequences in the sequential unraveling case one now has not just one introduction and one elimination step for $\alpha\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$, but also $n-1$ intermediate switch rule steps of the shape $U_i^\alpha(t_i, \vec{X}_i) \to U_{i+1}^\alpha(s_{i+1}, \vec{X}_{i+1})$.

When considering general DCTRSs instead of normal 1-CTRSs, there are two major sources of complication. First, right-hand sides of conditions of rules in DCTRSs need not be ground normal forms, and second, these right-hand sides may introduce extra variables that do not occur in the left-hand side of the conditional rule. Both of these properties indeed cause unsoundness in general even for e.g. confluent and $\mathbb{U}$-NE systems (cf. Example 4 below). Thus, it is necessary to restrict our results to certain classes of DCTRSs (cf. e.g. Theorems 9, 14 and 19 below).

3.1 Negative Results

Let us consider potential criteria for soundness of unraveling DCTRSs, namely confluent (CR) DCTRSs and DCTRSs $\mathcal{R}$ where $\mathcal{R}'$ is non-erasing (i.e., $\mathcal{R}$ is $\mathbb{U}$-NE). Unfortunately, both of these criteria do not extend to DCTRSs.

Example 4. *Consider the* $\mathbb{U}$-NE *and* RS DCTRS $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \mathcal{R}_3$ *where*

$$\mathcal{R}_1 = \left\{ \begin{tikzcd}[row sep=small] a \arrow[r] \arrow[dr] & c & s(c) \arrow[r] \arrow[dr] & t(k) \\ b \arrow[r] \arrow[ur] & d & & t(l) \end{tikzcd} \right\} \qquad \begin{array}{l} \mathcal{R}_2 = \{g(x,x) \to h(x,x)\} \\ \\ \mathcal{R}_3 = \{f(x) \to \langle x, y\rangle \Leftarrow s(x) \to^* t(y)\} \end{array}$$

Unraveling of $\mathcal{R}$ *yields* $\mathcal{R}' = \mathcal{R}_1 \cup \mathcal{R}_2 \cup \{f(x) \to U_1^\alpha(s(x), x)\ ,\ U_1^\alpha(t(y), x) \to \langle x, y\rangle\}$. $\mathcal{R}'$ *gives rise to the derivation*

$$\begin{aligned} g(f(a), f(b)) &\mathrel{+\!\!\!\!\!\!\|\,} g(U_1^\alpha(s(a), a), U_1^\alpha(s(b), b)) \to^* g(U_1^\alpha(s(c), d), U_1^\alpha(s(c), d)) \\ &\to h(U_1^\alpha(s(c), d), U_1^\alpha(s(c), d)) \to^* h(U_1^\alpha(t(k), d), U_1^\alpha(t(l), d)) \to^* h(\langle d, k\rangle, \langle d, l\rangle)\,. \end{aligned}$$

However, to get this reduction $g(f(a), f(b)) \to^* h(\langle d, k\rangle, \langle d, l\rangle)$ *in* $\mathcal{R}$*, we need a term* $v \in \mathcal{T}$ *such that (1)* $s(v) \to_{\mathcal{R}}^* t(w)$ *where* $w \in \{k, l\}$*, (2)* $a \to_{\mathcal{R}}^* v$ *and* $b \to_{\mathcal{R}}^* v$*, and (3)* $v \to_{\mathcal{R}}^* d$*. By (1) and (2)* $v = c$*, yet this contradicts (3).*

The DCTRS $\mathcal{R} \cup \{\ c \to e \leftarrow d\ ,\ k \to e \leftarrow l\ ,\ s(e) \to t(e)\ \}$ *is even confluent. Note that this* DCTRS *is even operationally terminating (or, equivalently quasi-decreasing), cf. [6, 17]. Although* $h(\langle d, k\rangle, \langle d, l\rangle)$ *can now be reduced to* $h(\langle e, e\rangle, \langle e, e\rangle)$ *and* $g(f(a), f(b)) \to_{\mathcal{R}}^* h(\langle e, e\rangle, \langle e, e\rangle)$*, still* $g(f(a), f(b)) \not\to_{\mathcal{R}}^* h(\langle d, k\rangle, \langle d, l\rangle)$ *by the same argument as above.*

Example 4 shows that neither confluence nor $\mathbb{U}$-NE are sufficient for soundness of unraveling DCTRSs, even if the system is right-stable and terminating.

For NE, the reason why an extension of the proof approach of [8] to (RS) DCTRSs is not possible in the general case, lies in the fact that for the construction in [8] we used a "translation forward" tf from reductions in the unraveled system to reductions in the original system. From an intermediate stage of evaluating the conditions of the unraveled version of some rule $l \to r \Leftarrow c$ we would need to calculate the final substitution for the right-hand side. However, this is in general impossible due to the incremental left-to-right computation character in DCTRSs. In fact, in Example 4, from the term $U_1^\alpha(s(c), c))$ (with $x = c$) we get two possible instances for y, namely k and l. Therefore tf cannot be sensibly defined here.

3.2 Confluence

Considering again Example 4 (the confluent version), we see that in the unsound $\mathcal{R}'$-reduction $g(f(a), f(b)) \to_{\mathcal{R}'}^* h(\langle d, k\rangle, \langle d, l\rangle)$ the result is not a normal form. If we only consider reduction from original terms to original terms which are normal forms then we can indeed guarantee soundness in the case of confluent right-stable DCTRSs. The proof works similar to the one for confluent normal 1-CTRSs, however, now instead of normality we exploit the fact that the final result of the reduction considered is a normal form.

Due to the specific structure of rules in unraveled DCTRSs, whenever there is an $\mathcal{R}'$ reduction $D\colon s \to^* U_j^\alpha(v, x_1, \ldots, x_n)\sigma$ where $s \in \mathcal{T}$, then intuitively the reduction sequences $s_i\sigma_i \to t_i\sigma_{i+1}$ that explicitly satisfy the conditions $s_i \to^* t_i$ for all $1 \le i < j$ of the conditional rule $\alpha\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ must have occurred as subreductions of D (for some substitutions σ_i where $x\sigma_i \to^*_{\mathcal{R}'} x\sigma_{i+1}$ for all i and all x). This observation is formalized in the following lemma (which is used e.g. in the proof of Lemma 7 below).

Lemma 5 (extraction of condition evaluation)**.** *Let $\mathcal{R}$ be a* DCTRS*. If $u_1 \to_{\mathcal{R}'} u_2 \to_{\mathcal{R}'} \cdots \to_{\mathcal{R}'} u_n \to_{\mathcal{R}'} v$ ($u_1 \in \mathcal{T}$), $v|_p = U^\alpha_j(v_1, \ldots, v_m)$ for some position p and $\alpha = l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$, then there exist substitutions $\sigma_1, \ldots, \sigma_j$ such that*

- *$s_i\sigma_i \to^*_{\mathcal{R}'} t_i\sigma_{i+1}$ for all $1 \le i < j$, $s_j\sigma_j \to^*_{\mathcal{R}'} v_1$; and*
- *$x \in \mathcal{D}om(\sigma_i) \cap \mathcal{D}om(\sigma_{i+1})$ implies $x\sigma_i \to^*_{\mathcal{R}'} x\sigma_{i+1}$ for all $1 \le i < j$; and*
- *if $U^\alpha_j(t_j, x_2, \ldots, x_m)$ is the left-hand side of the unique rule of $\mathcal{R}'$ defining U^α_j, then $x_i\sigma_j \to^*_{\mathcal{R}'} v_i$ for all $2 \le i \le m$.*

Moreover, for each single reduction step $s \to_{\mathcal{R}'} t$ in the reductions $s_i\sigma_i \to^*_{\mathcal{R}'} t_i\sigma_{i+1}$, $s_j\sigma_j \to^*_{\mathcal{R}'} v_1$, $x\sigma_i \to^*_{\mathcal{R}'} x\sigma_{i+1}$ and $x_i\sigma_j \to^*_{\mathcal{R}'} v_i$, there exists an index $k \le n$ and a position q such that $u_k|_q = s$ and $u_{k+1}|_q = t$.

Proof. Proof by induction on the length of the reduction sequence $u_1 \to^*_{\mathcal{R}'} v$. If the length is zero, then $v \in \mathcal{T}$ and thus there is no position p such that $v|_p = U(v_1, \ldots, v_m)$ so in this case the result holds vacuously.

Next, consider an arbitrary position $p \in \mathcal{P}os(v)$ such that $v|_p = U^\alpha_j(v_1, \ldots, v_m)$ where $\alpha = l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to t_n$. Since U-symbols do not occur below the root positions in left-hand sides of rules of $\mathcal{R}'$, $v|_p$ has at least one ancestor in u_n. Let $u_n|_{p'}$ be such an (arbitrarily chosen) ancestor. To prove the result we distinguish several cases depending on the position and type of the reduction step $u_n \to_{\mathcal{R}'} v$.

First, if the reduction step occurs outside or parallel to p', then $u_n|_{p'} = v|_p$ and the induction hypothesis directly yields the result.

Second, if the reduction step occurs below p', then $u_n|_{p'} = U^\alpha_j(v'_1, \ldots, v'_m)$ and the induction hypothesis yields the existence of substitutions $\sigma_i, \ldots, \sigma_j$ such that $s_i\sigma_i \to^*_{\mathcal{R}'} t_i\sigma_{i+1}$ for all $1 \le i < j$ and $s_j\sigma_j \to^*_{\mathcal{R}'} v_1$, $x\sigma_i \to^*_{\mathcal{R}'} x\sigma_{i+1}$ for all $1 \le i < j$ whenever $x \in \mathcal{D}om(\sigma_i) \cap \mathcal{D}om(\sigma_{i+1})$ and $x_i\sigma_j \to^*_{\mathcal{R}'} v_i$ for all $2 \le i \le m$ if $U^\alpha_j(t_j, x_2, \ldots, x_m)$ is the left-hand side of the unique rule of $\mathcal{R}'$ defining U^α_j. We have $v'_i \to^*_{\mathcal{R}'} v_i$ for all $1 \le i \le m$. Hence, $s_j\sigma_j \to^*_{\mathcal{R}'} v'_1 \to^*_{\mathcal{R}'} v_1$ and $x_i\sigma_j \to^*_{\mathcal{R}'} v'_i \to^*_{\mathcal{R}'} v_i$ for all $2 \le i \le m$. Moreover, if $v'_i \to^*_{\mathcal{R}'} v_i$ is non-empty for some $1 \le i \le m$, $v'_i = u_n|_q$ and $v_i = v|_q$ for some position q. Thus, the result holds.

Third, if the reduction step occurs at position p', then $u_n|_{p'}$ may or may not be a U-term. If it is not a U-term, $v|_p = U^\alpha_1(v_1, \ldots, v_m)$ and $v_1 = s_1\sigma$ for some substitution σ. In this case $s_1\sigma \to^*_{\mathcal{R}'} v_1$ is trivial and thus the result holds. Otherwise, $u_n|_{p'} = U^\alpha_{j-1}(v'_1, \ldots, v'_{m'}) = U_{j-1}(t_{j-1}, x_1, \ldots, x_{m'-1})\sigma_j$, and the induction hypothesis yields the existence of substitutions $\sigma_i, \ldots, \sigma_{j-1}$ such that $s_i\sigma_i \to^*_{\mathcal{R}'} t_i\sigma_{i+1}$ for all $1 \le i < j-1$ and $s_{j-1}\sigma_{j-1} \to^*_{\mathcal{R}'} v'_1$, as well as $x\sigma_i \to^*_{\mathcal{R}'} x\sigma_{i+1}$ for all $1 \le i < j-1$ whenever $x \in \mathcal{D}om(\sigma_i) \cap \mathcal{D}om(\sigma_{i+1})$ and $x'_i\sigma_{j-1} \to^*_{\mathcal{R}'} v'_i$ for all $2 \le i \le m'$ if $U^\alpha_{j-1}(t_{j-1}, x'_2, \ldots, x'_{m'})$ is the left-hand side of the unique rule of $\mathcal{R}'$ defining U^α_{j-1}.

Now $s_i\sigma_i \to^*_{\mathcal{R}'} t_i\sigma_{i+1}$ for all $1 \le i < j$ follows directly from the induction hypothesis, because $s_{j-1}\sigma_{j-1} \to^*_{\mathcal{R}'} v'_1 = t_{j-1}\sigma_j$. Moreover, $x_i\sigma_j \to^*_{\mathcal{R}'} v_i$ for all $2 \le i \le m$ if $U^\alpha_j(t_j, x_2, \ldots, x_m)$ is the left-hand side of the unique rule of $\mathcal{R}'$ defining U^α_j is trivial, because $x_i\sigma_j = v_i$ for all $2 \le i \le m$. Finally, it is left to show that $x\sigma_{j-1} \to^*_{\mathcal{R}'} x\sigma_j$ for each $x \in \mathcal{D}om(\sigma_{j-1}) \cap \mathcal{D}om(\sigma_j)$.

The domain of σ_j is restricted to variables occurring in the term $U^\alpha_j(s_j, x_2, \ldots, x_m)$ and variables that occur in s_j also occur in $\{x_2, \ldots, x_m\}$. Now consider some variable x from

$\mathcal{D}om(\sigma_j)$. If x occurs in $\{x'_2, \ldots, x'_{m'}\}$ (say at index k), we have $x_k\sigma_{j-1} \to^*_{\mathcal{R}'} v'_k = x_k\sigma_j$. Otherwise, $x \notin \{x'_2, \ldots, x'_{m'}\}$. This means that x does not occur in s_l or t_l for any $l < j$ and thus we can w.l.o.g. assume that $x \notin \mathcal{D}om(\sigma_{j-1})$. Thus, $x\sigma_{j-1} \to^*_{\mathcal{R}'} x\sigma_j$ for each $x \in \mathcal{D}om(\sigma_{j-1}) \cap \mathcal{D}om(\sigma_j)$ and the involved redexes and reduction steps occur also in $u \to_{\mathcal{R}'} v$. □

For the case of normal 1-CTRSs, confluence of a DCTRS $\mathcal{R}$ is a sufficient criterion for soundness of $\mathcal{R}'$ (cf. [8]). Normality of right hand-sides of conditions is vital for this result, because it means that if a conditional rule $\alpha\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ is applicable to a term $l\sigma \in \mathcal{T}$, then it is also applicable to a term $l\sigma' \in \mathcal{T}$ if $x\sigma \to^*_{\mathcal{R}} x\sigma'$ for all variables x. This is because applicability of α to $l\sigma$ means that $s_i\sigma \to^*_{\mathcal{R}} t_i$ for all $1 \leq i \leq n$. Moreover, we have $s_i\sigma \to^*_{\mathcal{R}} s_i\sigma'$ and thus by confluence and normality of t_i we obtain $s_i\sigma' \to^*_{\mathcal{R}} t_i$ for all $1 \leq i \leq n$.

This observation leads to the conjecture that confluence of a *right-stable* DCTRS is sufficient for soundness of $\mathcal{R}'$, because for right-stable DCTRSs right-hand sides of conditions are either ground normal forms or constructor terms with fresh variables. Hence, we have $s_i\sigma' \to t_i\theta'_i$ for all $1 \leq i \leq n$ and some substitution θ'_i provided that $l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ is a right-stable rewrite rule, $s_i\sigma \to^*_{\mathcal{R}} t_i\theta$ for all $1 \leq i \leq n$ and $x\sigma \to^*_{\mathcal{R}} x\sigma'$ for all variables x. Indeed, as the following lemmas and Theorem 9 below show, $\mathcal{R}'$ is sound w.r.t. reductions to normal forms, provided that $\mathcal{R}$ is confluent and right-stable.

The following lemma states a monotonicity property of tb w.r.t. joining reductions. It is comparable to Lemma 3.8 in [8].

Lemma 6. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a* DCTRS. *If $s \to_{p,\mathcal{R}'} t$ and $\mathsf{tb}(s|_p) \downarrow_{\mathcal{R}} \mathsf{tb}(t|_p)$, then $\mathsf{tb}(s|_q) \downarrow_{\mathcal{R}} \mathsf{tb}(t|_{q'})$ for every descendant $t|_{q'}$ of $s|_q$.*

Proof Sketch. For the interesting case of $p \leq p_i$ we use induction on the length of q where $p.q = p_i$. □

Proof. First, assume $q > p$ or $p \parallel q$, then $s|_q = t|_{q'}$ for every descendant $t|_{q'}$ of $s|_q$ and $\mathsf{tb}(s|_q) \downarrow_{\mathcal{R}} \mathsf{tb}(t|_{q'})$, because $\mathsf{tb}(s|_q) = \mathsf{tb}(t|_{q'})$. Second, if $p = q$, then $q' = q$ and we get $\mathsf{tb}(s|_q) \downarrow_{\mathcal{R}} \mathsf{tb}(t|_{q'})$ by our precondition.

Finally, assume $q < p$. Then the only descendant of $s|_q$ is $t|_q$. By an inductive argument it is sufficient to consider the case where $q.i = p$ for some $i \in \mathbb{N}$. We distinguish several cases depending on whether $\mathsf{root}(s|_q)$ is from $\mathcal{F}$ or not and $i = 1$ or not.

If $\mathsf{root}(s|_q) \in \mathcal{F}$, we have $\mathsf{tb}(s|_q) = f(\mathsf{tb}(s_1), \ldots, \mathsf{tb}(s_i) = \mathsf{tb}(s|_p), \ldots, \mathsf{tb}(s_n))$ if $s|_q = f(s_1, \ldots, s_i, \ldots, s_n)$. Then, $\mathsf{tb}(t|_q) = f(\mathsf{tb}(s_1), \ldots, \mathsf{tb}(t|_p), \ldots, \mathsf{tb}(s_n))$ and by $\mathcal{R}$-joinability of $\mathsf{tb}(s|_p)$ and $\mathsf{tb}(t|_p)$ we obtain $\mathcal{R}$-joinability of $\mathsf{tb}(s|_q)$ and $\mathsf{tb}(t|_q)$.

Now assume $\mathsf{root}(s|_q)$ is a U-symbol. If $i = 1$, then $\mathsf{tb}(s|_q) = \mathsf{tb}(t|_q)$ and thus $\mathsf{tb}(s|_q) \downarrow_{\mathcal{R}} \mathsf{tb}(t|_q)$. Otherwise, assume $i > 1$. Then $s|_q = U^\alpha_j(v, x_1, \ldots, x_n)\sigma$ and $t|_q = U^\alpha_j(v, x_1, \ldots, x_n)\sigma'$ and we have $\mathsf{tb}(x\sigma) \downarrow_{\mathcal{R}} \mathsf{tb}(x\sigma')$ for all variables $x \in \mathcal{D}om(\sigma) = \mathcal{D}om(\sigma')$. Hence, $\mathsf{tb}(s|_q) = l\,\mathsf{tb}(\sigma) \downarrow_{\mathcal{R}} l\,\mathsf{tb}(\sigma') = \mathsf{tb}(t|_q)$ where $\mathsf{tb}(\theta)$ denotes the substitution determined by $x\,\mathsf{tb}(\theta) = \mathsf{tb}(x\theta)$. □

The next lemma is the key lemma for proving soundness of $\mathcal{R}'$ for reduction to normal form of confluent right-stable DCTRSs.

Lemma 7 (technical key lemma)**.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a confluent right-stable* DCTRS *and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \ldots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$ reduction sequence where $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for all $1 \leq i \leq n$. Then $\mathsf{tb}(u_i|_{p_i}) \downarrow_{\mathcal{R}} \mathsf{tb}(u_{i+1}|_{p_i})$ for all $1 \leq i < n$.*

Proof. We prove the result by induction on the length n of D. If $n = 0$, the lemma holds vacuously. Otherwise, the induction hypothesis yields that $\mathsf{tb}(u_i|_{p_i}) \downarrow_{\mathcal{R}} \mathsf{tb}(u_{i+1}|_{p_i})$ for all $1 \leq i < n-1$ and what is left to be shown is that $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) \downarrow_{\mathcal{R}} \mathsf{tb}(u_n|_{p_{n-1}})$. Hence, consider the reduction step $u_{n-1} \rightarrow_{p_{n-1},\mathcal{R}'} u_n$.

We distinguish four cases, depending on the kind of rule of $\mathcal{R}'$ applied in this step. In case the $\mathcal{R}'$ step is due to an original unconditional rule, we have $u_{n-1}|_{p_{n-1}} = l\sigma$ and $u_n|_{p_{n-1}} = r\sigma$ for some unconditional rule $l \to r \in \mathcal{R} \cap \mathcal{R}'$. Thus, we have

$$\mathsf{tb}(u_{n-1}|_{p_{n-1}}) = \mathsf{tb}(l\sigma) = l\,\mathsf{tb}(\sigma) \rightarrow_{\mathcal{R}} r\,\mathsf{tb}(\sigma) = \mathsf{tb}(r\sigma) = \mathsf{tb}(u_n|_{p_{n-1}})$$

and hence $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) \downarrow_{\mathcal{R}} \mathsf{tb}(u_n|_{p_{n-1}})$.

Second, assume the $\mathcal{R}'$ step is due to a U-introduction rule or a U-switch rule, then $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) = \mathsf{tb}(u_n|_{p_{n-1}})$ and thus trivially $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) \downarrow_{\mathcal{R}} \mathsf{tb}(u_n|_{p_{n-1}})$.

Finally, assume the $\mathcal{R}'$ step is due to a U-elimination rule. Then, $u_{n-1}|_{p_{n-1}} = U_j^{\alpha}(t_m, x_1, \ldots, x_k)\sigma$ where α is the conditional rule $l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_m \to^* t_m$. According to Lemma 5, there exist substitutions $\sigma_1, \ldots, \sigma_m$ such that $s_i\sigma_i \rightarrow^*_{\mathcal{R}'} t_i\sigma_{i+1}$ and $x\sigma_i \rightarrow^*_{\mathcal{R}'} x\sigma_{i+1}$ (in case $x \in \mathcal{D}om(\sigma_i) \cap \mathcal{D}om(\sigma_{i+1})$) for all $1 \leq i \leq m$ (where $\sigma_{n+1} = \sigma$) and all x. Moreover, these reduction sequences can be extracted from D. Hence, by Lemma 6 and the induction hypothesis we obtain

$$\mathsf{tb}(s_i\sigma) \downarrow \mathsf{tb}(s_i\sigma_i) \downarrow \mathsf{tb}(t_i\sigma_{i+1}).$$

By confluence of $\mathcal{R}$ we get

$$\mathsf{tb}(s_i\sigma) \rightarrow^*_{\mathcal{R}} v_i \leftarrow^*_{\mathcal{R}} \mathsf{tb}(t_i\sigma_{i+1}).$$

By right-stability of $\mathcal{R}$ we have $v_i = t_i\theta$ and $\mathsf{tb}(x\sigma_{i+1}) \rightarrow^*_{\mathcal{R}} x\theta$ for all $x \in \mathcal{V}ar(t_i)$.

Again by confluence of $\mathcal{R}$, we obtain joinability of $x\theta$ and $\mathsf{tb}(x\sigma)$ for all $x \in \mathcal{D}om(\theta) \cap \mathcal{D}om(\sigma)$. Hence, we obtain $\mathsf{tb}(u_{n-1}|_{p-1}) = l\,\mathsf{tb}(\sigma) \rightarrow_{\mathcal{R}} r\theta$ for some substitution θ and $r\theta \downarrow_{\mathcal{R}} r\,\mathsf{tb}(\sigma) = u_n|_{p_n}$. $\square$

Lemma 8. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a confluent right-stable* DCTRS *and let $D\colon u_1 \rightarrow_{p_1,\mathcal{R}'} u_2 \rightarrow_{p_2,\mathcal{R}'} \ldots \rightarrow_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$ reduction sequence where $u_1, u_n \in \mathcal{T}$, $u_i \in \mathcal{T}'$ for all $1 \leq i \leq n$ and u_n is a normal form. Then $u_1 \rightarrow^*_{\mathcal{R}} \mathsf{tb}(u_n)$.*

Proof. By Lemma 7, $\mathsf{tb}(u_i|_{p_i}) \downarrow_{\mathcal{R}} \mathsf{tb}(u_{i+1}|_{p_i})$ for all $i \in \{1, \ldots, n-1\}$. By Lemma 6, $\mathsf{tb}(u_i|_{\epsilon}) \downarrow_{\mathcal{R}} \mathsf{tb}(u_{i+1}|_{\epsilon})$. Since u_n is a normal form and by confluence of $\mathcal{R}$ this implies $u_1 \rightarrow^*_{\mathcal{R}} \mathsf{tb}(u_n)$. $\square$

Theorem 9. *Let $\mathcal{R}$ be a confluent right-stable* DCTRS*. Then $\mathbb{U}_{seq}$ is sound for reduction to normal form.*

Proof. Straightforward by Lemma 8. $\square$

Note that several results for verifying confluence of DCTRSs exist, cf. e.g. [18, 1].

Example 10. *Consider the* DCTRS $\mathcal{R} = \mathcal{R}_{div} \cup \mathcal{R}_{sub} \cup \mathcal{R}_{leq}$ *where*

$$\mathcal{R}_{div} = \left\{ \begin{array}{c} div(0, s(y)) \to (0, 0) \quad div(s(x), s(y)) \to \langle 0, s(x)\rangle \Leftarrow leq(s(y), s(x)) \to^* false \\ div(s(x), s(y)) \to \langle s(q), r\rangle \Leftarrow leq(s(y), s(x)) \to^* true, div(x - y, s(y)) \to^* \langle q, r\rangle \end{array} \right\}$$

$$\mathcal{R}_{sub} = \{ \quad s(x) - 0 \to s(x) \qquad 0 - s(y) \to 0 \qquad s(x) - s(y) \to x - y \quad \}$$

$$\mathcal{R}_{leq} = \{ \; leq(s(x), 0) \to false \quad leq(0, s(y)) \to true \quad leq(s(x), s(y)) \to leq(x, y) \; \}$$

performing a simple division with remainder. Transforming the conditional rules yields

$$\begin{array}{rl} div(s(x),s(y)) \to U_1^{\alpha_1}(leq(s(y),s(x)),x,y) & div(s(x),s(y)) \to U_1^{\alpha_2}(leq(s(y),s(x)),x,y) \\ U_1^{\alpha_1}(false,x,y) \to \langle 0, s(x)\rangle & U_1^{\alpha_2}(true,x,y) \to U_2^{\alpha_2}(div(x-y,s(y)),x,y) \\ & U_2^{\alpha_2}(\langle q,r\rangle,x,y) \to \langle s(q),r\rangle \end{array}$$

$\mathcal{R}$ is right-stable, left-linear, quasi-reductive (cf. [1, 17]), strongly deterministic ([1]) and has infeasible critical pairs. Thus it is locally confluent and confluent (using either [18] or [1]). Hence, by Theorem 9 we obtain that $\mathcal{R}'$ is sound for reduction to normal form.

3.3 Right-Linearity

In the case of normal 1-CTRSs $\mathcal{R}$ it was shown in [8] that right-linearity of $\mathcal{R}'$ implies that all left-hand sides of conditions are ground terms. Moreover, since right-hand sides of conditions in normal 1-CTRSs are ground terms as well, such systems are of limited practical interest. For the case of DCTRSs, the situation is slightly more interesting, since, while left-hand sides of conditions must still be ground terms in order to guarantee right-linearity of the unraveled unconditional system, right-hand sides of conditions may contain variables. In this section we show that for a DCTRS $\mathcal{R}$, right-linearity of $\mathcal{R}'$ implies soundness of $\mathcal{R}'$.

Lemma 11. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a DCTRS. If $s \to_{p,\mathcal{R}'} t$ and $\mathsf{tb}(s|_p) \to^*_{\mathcal{R}} \mathsf{tb}(t|_p)$, then $\mathsf{tb}(s|_q) \to^*_{\mathcal{R}} \mathsf{tb}(t|_{q'})$ for every descendant $t|_{q'}$ of $s|_q$.*

Proof. Analogous to Lemma 3.8 in [8]. □

Lemma 12. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a DCTRS such that $\mathcal{R}'$ is right-linear and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$ reduction sequence where $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for all $1 \le i \le n$. Then $\mathsf{tb}(u_i|_{p_i}) \to^*_{\mathcal{R}} \mathsf{tb}(u_{i+1}|_{p_i})$ for all $1 \le i < n$.*

Proof. We prove the result by induction on the length n of D. If $n = 0$, the lemma holds vacuously. Otherwise, the induction hypothesis yields that $\mathsf{tb}(u_i|_{p_i}) \to^*_{\mathcal{R}} \mathsf{tb}(u_{i+1}|_{p_i})$ for all $1 \le i < n-1$ and what is left to be shown is that $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) \to^*_{\mathcal{R}} \mathsf{tb}(u_n|_{p_{n-1}})$. Hence, consider the reduction step $u_{n-1} \to_{p_{n-1},\mathcal{R}'} u_n$.

We distinguish four cases, depending on the kind of rule of $\mathcal{R}'$ applied in this step. In case the $\mathcal{R}'$ step is due to an original unconditional rule, we have $u_{n-1}|_{p_{n-1}} = l\sigma$ and $u_n|_{p_{n-1}} = r\sigma$ for some unconditional rule $l \to r \in \mathcal{R} \cap \mathcal{R}'$. Thus, we have

$$\mathsf{tb}(u_{n-1}|_{p_{n-1}}) = \mathsf{tb}(l\sigma) = l\,\mathsf{tb}(\sigma) \to_{\mathcal{R}} r\,\mathsf{tb}(\sigma) = \mathsf{tb}(r\sigma) = \mathsf{tb}(u_n|_{p_{n-1}}).$$

Second, assume the $\mathcal{R}'$ step is due to a U-introduction rule or a U-switch rule, then $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) = \mathsf{tb}(u_n|_{p_{n-1}})$ and thus trivially $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) \to^*_{\mathcal{R}} \mathsf{tb}(u_n|_{p_{n-1}})$.

Finally, assume the $\mathcal{R}'$ step is due to a U-elimination rule. Then, $u_{n-1}|_{p_{n-1}} = U^\alpha(t_m, x_1, \dots, x_k)\sigma$ where α is the conditional rule $l \to r \Leftarrow s_1 \to^* t_1, \dots, s_m \to^* t_m$. According to Lemma 5, there exist substitutions $\sigma_1, \dots, \sigma_m$ such that $s_i\sigma_i \to_{\mathcal{R}'} t_i\sigma_{i+1}$ and $x\sigma_i \to_{\mathcal{R}'} x\sigma_{i+1}$ (in case $x \in \mathcal{D}om(\sigma_i) \cap \mathcal{D}om(\sigma_{i+1})$) for all $1 \le i \le m$ (where $\sigma_{n+1} = \sigma$) and all x. Moreover, all reduction steps occurring in these reduction sequences occur identically in D. Hence, by Lemma 6 and the induction hypothesis we obtain

$$\mathsf{tb}(s_i\sigma) = s_i \to^*_{\mathcal{R}} \mathsf{tb}(t_i\sigma_{i+1}) \to^*_{\mathcal{R}} \mathsf{tb}(t_i\sigma)$$

for all $1 \le i \le m$. Hence, we obtain $\mathsf{tb}(u_{n-1}|_{p-1}) = l\,\mathsf{tb}(\sigma) \to_{\mathcal{R}} r\,\mathsf{tb}(\sigma) = u_n|_{p_n}$. □

Lemma 13. *Let $\mathcal{R} = (\mathcal{F}, R)$ be a* DCTRS *such that $\mathcal{R}'$ is right-linear and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \ldots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$ reduction sequence where $u_1 \in \mathcal{T}$, $u_i \in \mathcal{T}'$ for all $1 \leq i \leq n$. Then $u_1 \to^*_{\mathcal{R}} \mathsf{tb}(u_n)$.*

Proof. By Lemmas 12 and 11 we obtain $u_1 = \mathsf{tb}(u_1) \to^*_{\mathcal{R}} \mathsf{tb}(u_2) \to^*_{\mathcal{R}} \ldots \to^*_{\mathcal{R}} \mathsf{tb}(u_n)$. □

Theorem 14. *Let $\mathcal{R}$ be a $\mathbb{U}_{seq}$-*RL DCTRS*. Then, $\mathbb{U}_{seq}$ is sound for reduction.*

Proof. Straightforward by Lemma 13. □

3.4 Non-Erasingness

In [8] we proved soundness of $\mathbb{U}_{sim}$ for NE normal 1-CTRSs. For our proof it was essential that U-terms are not erased but properly eliminated. In order to ensure this, $\mathbb{U}$-NE is sufficient (which is equivalent to NE for normal 1-CTRSs).

Yet, we cannot extend our result for normal 1-CTRSs to DCTRSs because we used a *translation forward* (tf) that cannot be defined in an appropriate way if the rhs of a rule contains extra variables. In order to prove soundness using tf we therefore restrict ourselves to 2-DCTRSs. Example 4 shows indeed that $\mathbb{U}_{seq}$-NE (and also $\mathbb{U}_{opt}$-NE) is not a sufficient criterion for soundness.

$\mathbb{U}_{opt}$-NE is more general than $\mathbb{U}$-NE and tf can be properly defined for $\mathbb{U}_{opt}$ for 2-DCTRSs. Hence, we show our result for $\mathbb{U}_{opt}$. Since $\mathbb{U}$ is sound for a DCTRS $\mathcal{R}$ if $\mathbb{U}_{opt}$ is (cf. [15, Theorem 4.19]), this also yields soundness of $\mathbb{U}$.

Rules that use the same variable in the lhs and the rhs of conditions can still be a source of unsoundness. A more thorough analysis of CTRSs containing such rules shows that the following property is sufficient to exclude such cases of unsoundness: $\mathcal{V}ar(t_i) \cap \mathcal{V}ar(t_0, \ldots, t_{i-1}) = \emptyset$ for all conditional rules $\alpha\colon t_0 \to s_{n+1} \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$. This property resembles right-stability but is slightly more general since it allows non-linear rhs's in conditions. In the following we will refer to DCTRSs with this property as *right-separated* DCTRSs.

To prove soundness of $\mathbb{U}_{opt}$ for $\mathbb{U}_{opt}$-NE, right-separated 2-DCTRSs we first define the function "translate forward" for $\mathbb{U}_{opt}$:

Definition 15 (translation forward)**.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a 2-*DCTRS*. The mapping $\mathsf{tf} : \mathcal{T}' \to \mathcal{T}$ is defined by*

$$\mathsf{tf}(t) = \begin{cases} x & \text{if } t = x \in \mathcal{V} \\ f(\mathsf{tf}(t_1), \ldots, \mathsf{tf}(t_{\mathsf{ar}(f)})) & \text{if } t = f(t_1, \ldots, t_{\mathsf{ar}(f)}) \text{ and } f \in \mathcal{F} \\ r\sigma & \text{if } t = U^{\alpha}_j(u, v_1, \ldots, v_k) \text{ and } x_i\sigma = \mathsf{tf}(v_i) \text{ for } 1 \leq i \leq k \end{cases}$$

where α is the rule $l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$, $\vec{X}_j = x_1, \ldots, x_k$ and $X_j = \mathcal{V}ar(l, t_1, \ldots, t_{j-1}) \cap \mathcal{V}ar(t_j, s_{j+1}, t_{j+1}, \ldots, s_n, t_n, r)$.

By the definition of $\mathbb{U}_{opt}$, X_j contains all variables occurring in r for 2-DCTRSs so that tf is well-defined. The proof now follows mainly the proof for NE 1-CTRSs in [8]:

Lemma 16 (monotony property of tf)**.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a 2-*DCTRS*. If $s \to_{p,\mathcal{R}} t$ for $s, t \in \mathcal{T}'$ and $\mathsf{tf}(s|_p) \to^*_{\mathcal{R}} \mathsf{tf}(t|_p)$, then $\mathsf{tf}(s|_q) \to^*_{\mathcal{R}} \mathsf{tf}(t|_{q'})$ for every descendants $t|_{q'}$ of $s|_q$.*

Proof Sketch. Analogous to Lemma 3.13 in [8]. □

Lemma 17 (technical key result for $\mathbb{U}_{opt}$-NE right-separated 2-DCTRSs)**.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be an $\mathbb{U}_{opt}$-*NE *and right-separated 2-*DCTRS *and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \cdots u_n$ be a derivation where $u_n \in \mathcal{T}$ and $u_1, \ldots, u_{n-1} \in \mathcal{T}'$, then $\mathsf{tf}(u_i|_{p_i}) \to^*_{\mathcal{R}} \mathsf{tf}(u_{i+1}|_{p_i})$ for all $i \in \{1, \ldots, n-1\}$.*

Proof. We prove the result by induction on the length n of D:

IB If $n = 0$, the lemma holds vacuously.

IH The induction hypothesis yields that $\mathsf{tf}(u_i|_{p_i}) \to^*_{\mathcal{R}} \mathsf{tf}(u_{i+1}|_{p_i})$ for all $i \in \{2, \ldots, n-1\}$.

IS In order to prove that $\mathsf{tf}(u_1|_{p_1}) \to^*_{\mathcal{R}} \mathsf{tf}(u_2|_{p_1})$ we distinguish the following cases based on the rule $l \to r$ of $\mathcal{R}'$ applied in the first rewrite step:

1. If the applied rule is an original unconditional rule, then $u_1|_{p_1} = l\sigma$ and $u_2|_{p_1} = r\sigma$. Since $l \to r \in R$, $\mathsf{tf}(u_1|_{p_1}) = l\ \mathsf{tf}(\sigma) \to_{\mathcal{R}} r\ \mathsf{tf}(\sigma) = \mathsf{tf}(u_2|_{p_1})$.
2. If the applied rule is a U-elimination rule or a U-switch rule, then $u_1|_{p_1} = l\sigma = U^\alpha_j(u, \vec{X}_j)\sigma$ and $u_2|_{p_1} = r\sigma$ so that $\mathsf{tf}(u_1|_{p_1}) = \mathsf{tf}(U^\alpha_j(u, \vec{X}_j)\sigma) = r\ \mathsf{tf}(\sigma) = \mathsf{tf}(u_2|_{p_1})$.
3. If the applied rule is the U-introduction rule of the conditional rule $\alpha\colon t_0 \to s_{n+1} \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$, then $u_1|_{p_1} = t_0\sigma_0$ and $u_2|_{p_1} = U^\alpha_1(s_1\sigma_0, \vec{X}_1\sigma_0)$. Since u_n does not contain any U-terms and $\mathcal{R}'$ is non-erasing, all descendants of $u_2|_{p_1}$ (and therefore $u_1|_{p_1}$) must be eventually eliminated. Therefore, every U-term has at least one one-step descendant and there is (at least) one descendant $u_{m+1}|_{p_m} = s_{n+1}\sigma_n$ of $u_1|_{p_1}$ in D. We therefore can extract the following derivation from D:

$$\begin{aligned} u_1|_{p_1} = t_0\sigma_0 &\to_{\mathcal{R}'} U^\alpha_1(s_1\sigma_0, \vec{X}_1\sigma_0) \to^*_{\mathcal{R}'} U^\alpha_1(t_1\sigma_1, \vec{X}_1\sigma_1) \\ &\to_{\mathcal{R}'} U^\alpha_2(s_2\sigma_1, \vec{X}_2\sigma_1) \to^*_{\mathcal{R}'} U^\alpha_2(t_2\sigma_2, \vec{X}_2\sigma_2) \\ &\to_{\mathcal{R}'} U^\alpha_3(s_2\sigma_2, \vec{X}_3\sigma_2) \to^*_{\mathcal{R}'} \cdots U^\alpha_n(t_n\sigma_n, \vec{X}_n\sigma_n) \to_{\mathcal{R}'} s_{n+1}\sigma_n = u_{m+1}|_{p_m} \end{aligned}$$

   where $\mathcal{D}om(\sigma_i) = X_{i+1}$ for $i \in \{0, \ldots, n-1\}$ and $\mathcal{D}om(\sigma_n) = \mathcal{V}ar(s_{n+1})$.

   By the induction hypothesis we can translate every rewrite step in this derivation into a rewrite step in $\mathcal{R}$ using tf. By repeated application of Lemma 16 we obtain $\mathsf{tf}(x\sigma_{i-1}) \to^*_{\mathcal{R}} \mathsf{tf}(x\sigma_i)$ for all $x \in X_{i-1} \cap X_i$ and $\mathsf{tf}(s_i\sigma_{i-1}) \to^*_{\mathcal{R}} \mathsf{tf}(t_i\sigma_i)$. Hence, $\mathsf{tf}(s_i\sigma_0 \ldots \sigma_{i-1}) \to^*_{\mathcal{R}} \mathsf{tf}(s_i\sigma_{i-1})$.

   Right-separateness implies $t_i\sigma_i = t_i\sigma_0 \ldots \sigma_i$. Henceforth, we obtain $s_i\ \mathsf{tf}(\sigma_0 \ldots \sigma_{i-1}) \to^*_{\mathcal{R}} \mathsf{tf}(s_i\sigma_{i-1}) \to^*_{\mathcal{R}} \mathsf{tf}(t_i\sigma_0 \ldots \sigma_i) = t_i\ \mathsf{tf}(\sigma_0 \ldots \sigma_i)$. Therefore, all conditions of α are satisfied for $\mathsf{tf}(\sigma_0)$ and $\mathsf{tf}(u_1|_{p_1}) = t_0\ \mathsf{tf}(\sigma_0) \to^*_{\mathcal{R}} \mathsf{tf}(u_2|_{p_1}) = s_{n+1}\ \mathsf{tf}(\sigma_0)$.

□

Lemma 18. *Let $\mathcal{R}$ be a $\mathbb{U}_{opt}$-*NE*, right-separated 2-*DCTRS*, and $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \cdots u_n$ be a derivation such that $u_n \in \mathcal{T}$ and $u_1, \ldots, u_{n-1} \in \mathcal{T}'$, then $\mathsf{tf}(u_1) \to^*_{\mathcal{R}} u_n$.*

Proof. By Lemma 17, $\mathsf{tf}(u_i|_{p_i}) \to^*_{\mathcal{R}} \mathsf{tf}(u_{i+1}|_{p_i})$ for all $i \in \{1, \ldots, n-1\}$. By Lemma 16, $\mathsf{tf}(u_i|_\epsilon) \to^*_{\mathcal{R}} \mathsf{tf}(u_{i+1}|_\epsilon)$. Since $\mathsf{tf}(u_n) = u_n$, $\mathsf{tf}(u_1) \to^*_{\mathcal{R}} u_n$. □

Theorem 19 (Soundness for reduction for $\mathbb{U}_{opt}$)**.** *Let $\mathcal{R}$ be an $\mathbb{U}_{opt}$-*NE* and right-separated 2-*DCTRS*, then $\mathbb{U}_{opt}$ is sound for reduction.*

Proof. Straightforward via Lemma 18. □

3.5 Weak Left-Linearity

In [8] the class of weakly left-linear (WLL) normal 1-CTRSs has been introduced. Weakly left-linear normal 1-CTRSs may, in addition to left-linear rules, contain non-left-linear rules provided the variables occurring more than once in the the lhs of such a rule do not occur in its rhs at all. It was shown in [8] that for weakly left-linear normal 1-CTRSs $\mathbb{U}_{sim}$ is sound.

For a DCTRS $\mathcal{R}$ the situation is significantly more involved, since extra variables may occur in rhs's of multiple conditions of one conditional rewrite rule (which indeed can not be the case for normal 1-CTRSs). In this case, $\mathcal{R}'$ can be non-left-linear (indeed non-weakly-left-linear according to Definition [8, 3.22]) even if $\mathcal{R}$ is left-linear and the rhs's of all conditions are linear. Hence, WLL of a DCTRS $\mathcal{R}$ does not necessarily imply WLL of $\mathcal{R}'$ (and this is in sharp contrast to the situation of normal 1-CTRSs).

Hence, the notion of WLL from [8] is inadequate for our treatment of DCTRSs and we instead introduce and use a generalized notion of WLL. The basic idea of our definition of WLL for DCTRSs is to count simultaneous occurrences of variables in the rhs's of conditions and the lhs of a conditional rule, thus anticipating non-left-linear rules in the transformed systems. Variables that occur more than once in the lhs of a conditional rule and the rhs's of conditions should not occur at all in lhs's of conditions or the rhs of the conditional rule:

Definition 20 (Weakly left-linear DCTRSs)**.** *A* DCTRS $\mathcal{R}$ *is* weakly left-linear *(*WLL*) if for every rule* $\alpha\colon t_0 \to s_{n_\alpha+1} \Leftarrow s_1 \to^* t_1, \ldots, s_{n_\alpha} \to^* t_{n_\alpha}$ *in* $\mathcal{R}$ *and all variables* $x \in \mathcal{V}ar(\alpha)$*,* $|t_0, \ldots, t_{n_\alpha}|_x > 1 \implies x \notin \mathcal{V}ar(s_1, \ldots, s_{n_\alpha+1})$*.*

Note that the version of weak left-linearity of [8] and the one from Definition 20 are compatible in the sense that WLL normal 1-CTRSs according to [8] are also WLL according to Definition 20. Hence, whenever we speak of WLL we mean the notion of Definition 20.

Given a WLL DCTRS $\mathcal{R}$, $\mathcal{R}'$ is not necessarily WLL (according to Definition 20). However, WLL of a DCTRS $\mathcal{R}$ does imply that in introduction or switch rules $l \to r$, every non-linear variable in l is linear in r, and elimination rules $l \to r$ erase all such variables.

In [8], we showed soundness of WLL normal 1-CTRSs by using a back translation w.r.t. derivations called tb_D to translate derivations in the transformed CTRS into derivations in the original CTRS. For DCTRSs $\mathcal{R}$, we are using a slightly modified version of tb_D that takes into account potential non-weak left-linearity of $\mathcal{R}'$:

Definition 21 (tb_D)**.** *Let* $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \cdots \to_{p_{n-1},\mathcal{R}'} u_n$ $(u_1 \in \mathcal{T})$*, then* tb_D *is defined as*

$$\mathsf{tb}_D(i,p) = \begin{cases} x & \textit{if } u_i|_p = x \in \mathcal{V} \\ \mathsf{tb}_D(i-1,p') & \textit{if } \mathsf{root}(u_i|_p) = U^\alpha_j \textit{ and } p' \textit{ is the unique one-step-ancestor of } p \\ f(\mathsf{tb}_D(i,p.1), \ldots, \mathsf{tb}_D(i,p.\mathsf{ar}(f))) & \textit{if } \mathsf{root}(u_i|_p) = f, f \in \mathcal{F}, \textit{ and } \\ & \quad \mathsf{tb}_D(i,p.1), \ldots, \mathsf{tb}_D(i,p.\mathsf{ar}(f)) \textit{ are defined} \\ \textit{undefined} & \textit{otherwise} \end{cases}$$

tb_D is undefined for U-terms with multiple one-step ancestors or terms containing such U-terms. This can happen if we apply a non-left-linear introduction or switch rule. After applying an introduction rule $l[x_i, x_i] \to U^\alpha_1(s_1, x_1, \ldots, x_k)$, tb_D would be undefined for U-terms in the $i+1$st argument in U^α_1-rooted terms. In the following, we will refer to such arguments in U-terms as *non-traceable arguments for* U^α_1.

Formally, the ith argument of the symbol U^α_j is a *non-traceable argument for* U^α_j, if in the introduction or switch rule $l \to U^\alpha_j(s_j, x_1, \ldots, x_k)$ in $\mathcal{R}'$, $|l|_{x_{i-1}} > 1$, or the rule is a switch rule, $l = U^\alpha_{j-1}(t_{j-1}, x_1, \ldots, x_{k'})$ and the ith argument of U^α_{j-1} is a *non-traceable argument for* U^α_{j-1}. If the ith argument of U^α_j $(2 \le i \le \mathsf{ar}(U^\alpha_j))$ is not a non-traceable argument, it is a *traceable argument (for U^α_j)*. Analogously, a term $u|_{p.i}$ ($i \in \mathbb{N}^+$) is a *(non-)traceable argument*, if $\mathsf{root}(u|_p) = U^\alpha_j$ and the ith argument of U^α_j is a (non-)traceable argument.

Our goal is to show that we can translate a derivation D of a transformed WLL DCTRS $\mathcal{R}'$ into a derivation of the original DCTRS using tb_D. Yet, tb_D might be undefined for terms occurring in D. To illustrate this problem, consider a (introduction or switch) rule $l[x,x] \to U^\alpha_j(s_j,x)$ (such that either $j=1$, or x is not a non-traceable argument for U^α_{j-1}). Consider the derivation $l[u,u] \to_{\mathcal{R}'} U^\alpha_j(s,u) \to_{\mathcal{R}'} U^\alpha_j(s,v)$. tb_D is not defined for U-terms in u and v. We therefore cannot backtranslate the rewrite step $u \to v$ using tb_D.

Our basic idea to remedy this problem is to show that arbitrary $\mathcal{R}'$ reductions (starting from old terms) can be reconstructed into $\mathcal{R}'$ reductions (having the same starting and end term) such that no rewrite step occurs in a non-traceable argument (of any U-term). Intuitively, a derivation $l[u,u] \to U^\alpha_j(s,u) \to U^\alpha_j(s,v)$ is rebuilt by moving rewrite steps in non-traceable arguments ahead: $l[u,u] \twoheadrightarrow\!\!\!\!\!+ \; l[v,v] \to U^\alpha_j(s,v)$.

However, in general we cannot rebuild all derivations in WLL DCTRSs in this way: If a non-traceable argument has multiple descendants, we cannot move such rewrite steps ahead because they are not unique: Consider the rule $l'[z] \to r'[z,z]$ and the derivation $l'[l[u,u]] \to l'[U^\alpha_j(s,u)] \to r'[U^\alpha_j(s,u), U^\alpha_j(s,u)] \twoheadrightarrow\!\!\!\!\!+ \; r'[U^\alpha_j(s,v_1), U^\alpha_j(s,v_2)]$.

The rewrite steps $u \to v_1$ and $u \to v_2$ occur in non-traceable argument positions of U^α_j. In order to move these rewrite steps ahead, we first have to avoid rewrite steps that duplicate non-traceable arguments of U-terms. To achieve this, we further rearrange reduction sequences by moving applications of duplicating rewrite rules ahead of reductions in non-traceable arguments of involved U-terms. For instance, in a derivation $l'[l[u,u]] \to l'[U^\alpha_j(s,u)] \to r'[U^\alpha_j(s,u), U^\alpha_j(s,u)]$ we shift the rewrite step multiplying the U-terms ahead in the following way: $l'[l[u,u]] \to r'[l[u,u], l[u,u]] \twoheadrightarrow\!\!\!\!\!+ \; r'[U^\alpha_j(s,u), U^\alpha_j(s,u)]$. Then we transform $r'[l[u,u], l[u,u]] \twoheadrightarrow\!\!\!\!\!+ \; r'[U^\alpha_j(s,u), U^\alpha_j(s,u)] \twoheadrightarrow\!\!\!\!\!+ \; r'[U^\alpha_j(s,v_1), U^\alpha_j(s,v_2)]$ into $r'[l[u,u], l[u,u]] \twoheadrightarrow\!\!\!\!\!+ \; r'[l[v_1,v_1], l[v_2,v_2]] \twoheadrightarrow\!\!\!\!\!+ \; r'[U^\alpha_j(s,v_1), U^\alpha_j(s,v_2)]$. Note that in the final reduction sequence no rewrite steps take place in non-traceable arguments of any U-term.

The following Lemma shows that we can transform a derivation $s[u] \to s[v] \to t[v,v]$ that duplicates a U-term containing a non-traceable argument into $s[u] \to t[u,u] \twoheadrightarrow\!\!\!\!\!+ \; t[v,v]$.

Lemma 22 (shifting ahead rewrite steps duplicating U-terms)**.** *Let $\mathcal{R}$ be a WLL DCTRS, $u,v,w \in \mathcal{T}'$ be such that $D\colon u \to_{p,\mathcal{R}'} v \to_{q,\mathcal{R}'} w$ where $v|_{p'.i}$ is a non-traceable argument, $p' \leq p$, either $q \parallel p'$ or $q < p'$, and such that there are no one-step descendants of $v|_{p'}$ inside a non-traceable argument. Then, there is a derivation $u \to_{\mathcal{R}'} w[u|_p]_P \twoheadrightarrow\!\!\!\!\!+_{\mathcal{R}'} w[v|_p]_P = w$ where P is the set containing all one-step descendants of $v|_p$ in D.*

Proof. We prove this result by case distinction on the positions p' and q. If $p' \parallel q$, then we can simply swap the rewrite steps:

$$u = u[u|_p]_p[u|_q]_q \to_{q,\mathcal{R}'} u[u|_p]_p[w|_q]_q \to_{p,\mathcal{R}'} u[v|_p]_p[w|_q]_q = w$$

Otherwise $q < p' \leq p$. Assume the rule applied in the rewrite step $v \to w$ is $l \to r$. There is a $q' \in \mathcal{VP}os(l)$ such that $q.q' < p'$ (since there are no overlaps involving U-terms). Furthermore, l is linear in $x = l|_{q'}$ (otherwise some descendant of $v|_{p'}$ would be inside a non-traceable argument).

Let in the following $Q = \{q \in \mathcal{VP}os(r) \mid r|_q = x\}$ be all one-step descendants of $l|_{q'}$ in D and let q'' be determined by $q.q'.q'' = p$. Then we can rearrange the derivation in the following way: $u = u \to_{q,\mathcal{R}'} w[u|_p]_{q.Q.q''} \twoheadrightarrow\!\!\!\!\!+_{\mathcal{R}',q.Q.q''} w[v|_p]_{q.Q.q''} = w$ where $q.Q.q'' = \{q.q'.q'' \mid q' \in Q\}$. $\square$

In the following lemma, we show that we can transform a derivation $s[u,u] \to t[u] \to t[v]$ into $s[u,u] \twoheadrightarrow\!\!\!\!\!+ \; s[v,v] \to t[v]$, thereby eliminating the rewrite step in the non-traceable argument in t.

Lemma 23 (shifting ahead rewrite steps in non-traceable arguments)**.** *Let $\mathcal{R}$ be a* WLL DCTRS*, $u, v, w \in \mathcal{T}'$ be such that $D\colon u \rightarrow_{p,\mathcal{R}'} v \rightarrow_{q,\mathcal{R}'} w$ where $v|_{q'.i}$ ($q'.i \leq q$) is a non-traceable argument and the only one-step-descendant of all its one-step-ancestors, and either $p < q'.i$ or $p \parallel q'.i$. Then, there is a derivation $u \nparallel\!\!\!\!\rightarrow_{\mathcal{R}'} u[w|_q]_Q \rightarrow_{p,\mathcal{R}'} v[w|_q]_q = w$ where Q is the set containing all one-step ancestors of $v|_q$ in D.*

Proof. We prove this result by case distinction on the positions p and $q'.i$. If $p \parallel q'.i$, then we can simply swap the rewrite steps.

Otherwise $p < q'.i \leq q$. Assume the rule applied in the rewrite step $u \rightarrow v$ is $l \rightarrow r$. There is a $p' \in \mathcal{VP}os(r)$ such that $p.p' < q'.i$. Furthermore, r is linear in $x = r|_{p'}$.

Let in the following $Q = \{q \in \mathcal{VP}os(l) \mid l|_q = x\}$ be all one-step ancestors of $r|_{p'}$ and let p'' be determined by $p.p'.p'' = p$. Then we can rearrange the derivation in the following way: $u = u[v|_q]_Q \nparallel\!\!\!\!\rightarrow_{\mathcal{R}'} u[w|_q]_Q \rightarrow_{p,\mathcal{R}'} v[w|_q]_q = w$. □

By repeatedly applying these two Lemmata, we can eliminate all rewrite steps in non-traceable arguments:

Lemma 24 (elimination of rewrite steps in non-traceable arguments)**.** *Let $D\colon u_1 \rightarrow_{p_1,\mathcal{R}'} u_2 \rightarrow_{p_2,\mathcal{R}'} \cdots u_n$ be a derivation in $\mathcal{R}'$, then there is a derivation $D'\colon u_1 = u'_1 \rightarrow_{p'_1,\mathcal{R}'} u'_2 \rightarrow_{p'_2,\mathcal{R}'} \cdots u'_{n'} = u_n$ such that there are no rewrite steps in non-traceable arguments.*

Proof. We show our result by induction over the number of rewrite steps in non-traceable arguments in D:

IB If there are no such rewrite steps, $D' = D$.

IS Let $u_m \rightarrow_{p_m,\mathcal{R}'} u_{m+1}$ be the first rewrite step in D inside a non-traceable argument. We now repeatedly apply Lemma 22 to the subderivation $D\colon u_1 \rightarrow^*_{\mathcal{R}'} u_{m+1}$. Since there might be nested non-traceable arguments, we use an inductive argument to apply Lemma 22 on the set of all non-traceable arguments above p_m, and another inductive argument on the number of rewrite steps satisfying the precondition of Lemma 22 and thereby obtain a derivation $D'\colon u_1 \rightarrow^*_{\mathcal{R}'} u'_m \nparallel\!\!\!\!\rightarrow_{\mathcal{R}'} u_{m+1}$ where the last parallel rewrite step contains all rewrite steps in non-traceable arguments in D', and there are no rewrite steps in $u_1 \rightarrow^*_{\mathcal{R}'} u'_m$ that duplicate non-traceable arguments (observe that the rewrite step $u_m \rightarrow_{\mathcal{R}'} u_{m+1}$ itself might duplicate a non-traceable argument).

The derivation D' therefore satisfies our preconditions of Lemma 23. Again, by repeated application of this lemma, we can shift the rewrite steps $u'_m \nparallel\!\!\!\!\rightarrow_{\mathcal{R}'} u_{m+1}$ in D' ahead of those rewrite step that introduces all nested non-traceable arguments.

Therefore we obtain a derivation $D''\colon u_1 \rightarrow^*_{\mathcal{R}'} u_{m+1}$ without rewrite steps in non-traceable arguments, so that for the derivation $D'' \rightarrow^* u_n$ our induction hypothesis holds.

□

The following example shows how we can rebuild a complex derivation such that it does not contain rewrite steps in non-traceable arguments:

Example 25. *Consider the* WLL DCTRS *$\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ where*

$$\mathcal{R}_1 = \{between(x, y, z) \rightarrow true \Leftarrow up(x) \rightarrow^* y, down(z) \rightarrow^* y\}$$

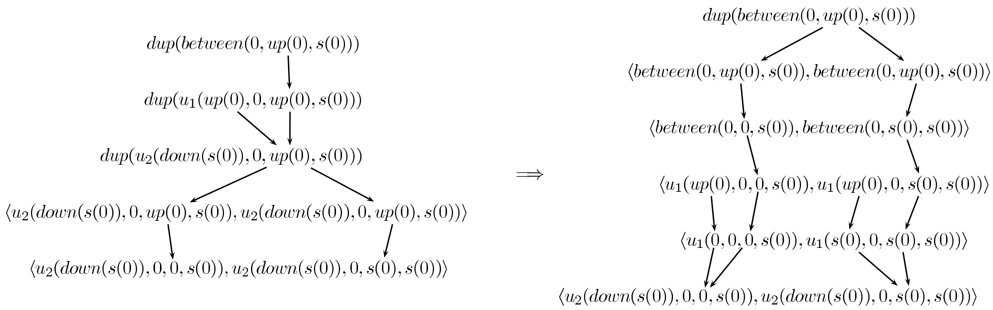
$$\mathcal{R}_2 = \left\{ \begin{array}{lll} up(x) \rightarrow x & down(x) \rightarrow x & dup(x) \rightarrow \langle x, x\rangle \\ up(x) \rightarrow up(s(x)) & down(s(x)) \rightarrow down(x) & \end{array} \right\}$$

It is unraveled into $\mathcal{R}' = \left\{ \begin{array}{l} between(x,y,z) \to U_1^\alpha(up(x),x,y,z) \\ \quad U_1^\alpha(y,x,y,z) \to U_2^\alpha(down(z),x,y,z) \\ \quad U_2^\alpha(y,x,y,z) \to true \end{array} \right\} \cup \mathcal{R}_2$

We obtain the derivation

$$\begin{array}{ll} & dup(between(0,up(0),s(0))) \to dup(u_1(up(0),0,up(0),s(0))) \to \\ \to & dup(u_2(down(s(0)),0,up(0),s(0))) \to \\ \to & \langle u_2(down(s(0)),0,up(0),s(0)), u_2(down(s(0)),0,up(0),s(0))\rangle \to^* \\ \to^* & \langle u_2(down(s(0)),0,0,s(0)), u_2(down(s(0)),0,s(0),s(0))\rangle \end{array}$$

We want to move the rewrite steps in the non-traceable argument ahead of the application of the non-left-linear rule. Yet, the term $up(0)$ *has two different descendants* 0 *and* $s(0)$ *in the last term of the derivation. In order to obtain a derivation without rewrite steps in non-traceable arguments, we move the introduction of the non-traceable argument behind the application of the duplicating rule (repeated application of Lemma 22). Then, we delay the introduction of the non-traceable argument (repeated application of Lemma 23):*



After rebuilding the derivation, we can use the same proof structure we already used in [8], although we have to consider switch rules and take care that tb_D is still undefined for U-terms in non-traceable arguments. Lemma 28 shows that we can use tb for such cases.

First we show monotony of tb_D:

Lemma 26 (monotony of tb_D)**.** *Let* $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \cdots \to_{p_{n-1},\mathcal{R}'} u_n$ *be a derivation without rewrite steps in non-traceable arguments such that* $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i{+}1,p_i)$ *for all* $i \in \{1,\ldots,n-1\}$*. If* $u_i|_p$ *(*$p \in \mathcal{P}os(u_i)$*) has a one-step-descendant* $u_{i+1}|_{p'}$ *and* $\mathsf{tb}_D(i+1,p')$ *is defined, then* $\mathsf{tb}_D(i,p) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p')$*.*

Proof. By case distinction on p and p_i:

1. If $p \parallel p_i$ or $p_i < p$, $u_{i+1}|_{p'}$ is a one-step descendant and $\mathsf{tb}_D(i+1,p'))$ is defined, then $u_i|_p$ is the unique one-step ancestor of $u_{i+1}|_{p'}$. Since tb_D of all maximal U-rooted terms in $u_i|_p$ is equal to tb_D of all maximal U-rooted terms in $u_{i+1}|_{p'}$, $\mathsf{tb}_D(i,p) = \mathsf{tb}_D(i+1,p')$.

2. If $p \le p_i$, then $p = p'$. We use induction on the length of q where $p.q = p_i$:

   IB If $q = \epsilon$, $p = p_i$ and the result holds trivially.

   IS $q = j.q'$: By our inductive hypothesis, $\mathsf{tb}_D(i,p.j) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p.j)$.

   (a) If $u_i|_p = f(t_1,\ldots,t_{\mathsf{ar}(f)})$ is not a U-term, then $\mathsf{tb}_D(i,p) = f(\mathsf{tb}_D(i,p.1),\ldots,\mathsf{tb}_D(i,p.\mathsf{ar}(f)))$ and $\mathsf{tb}_D(i{+}1,p) = f(\mathsf{tb}_D(i,p.1),\ldots,\mathsf{tb}_D(i,p.j-1),\mathsf{tb}_D(i{+}1,p.j),\mathsf{tb}_D(i,p.j+1),\ldots,\mathsf{tb}_D(i,p.\mathsf{ar}(f)))$, hence $\mathsf{tb}_D(i,p) \to^*_{j,\mathcal{R}} \mathsf{tb}_D(i+1,p)$.

(b) If $u_i|_p$ is a U-term, then $u_{i+1}|_p = u_i|_p$.
If $u_i|_p$ is the unique one-step ancestor of $u_{i+1}|_p$, then $u_i|_p$ is defined if $u_{i+1}|_p$ is defined. Since $u_{i+1}|_p$ is a U-term, $\mathsf{tb}_D(i,p) = \mathsf{tb}_D(i+1,p)$ by definition of tb_D. Otherwise $\mathsf{tb}_D(i+1,p)$ is not defined.

□

The next auxiliary lemma helps us to extract derivations inside U-terms:

Lemma 27 (Extraction of U-terms)**.** *Let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \cdots \to_{p_{n-1},\mathcal{R}'} u_n$ be a derivation without rewrite steps in non-traceable arguments such that $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for all $i \in \{1,\ldots,n-1\}$. Let $u_m|_q = U^\alpha_j(v, x_1\tau, \ldots, x_{|X_j|}\tau)$ such that $\mathsf{tb}_D(m,q)$ is defined, then (1) (extraction of [extra] variables) if $\mathsf{tb}_D(m,q.(i+1))$ is defined, then $\mathsf{tb}_D(m-1,q'_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(m,q.(i+1))$ where q'_i is a one-step ancestor of $u_m|_{q.(i+1)} = x_i\tau$ $(1 \leq i \leq |X_j|)$, and (2) (extraction of conditional arguments) if $u_{m-1}|_{q'} = U^\alpha_j(u, x_1\sigma, \ldots, x_k\sigma)$, then $\mathsf{tb}_D(m-1,q'.1) \to^*_{\mathcal{R}} \mathsf{tb}_D(m,q.1)$.*

Proof. Straightforward via Lemma 26. □

Although we can prevent that there are rewrite steps applied in non-traceable arguments we still might obtain terms for which tb_D is undefined, namely terms in non-traceable arguments containing U-terms. For such terms we can use tb instead of tb_D as the following result shows.

Lemma 28 (tb_D to tb)**.** *Let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \cdots \to_{p_{n-1},\mathcal{R}'} u_n$ be a derivation without rewrite steps in non-traceable arguments such that $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for all $i \in \{1,\ldots,n-1\}$. Then, $\mathsf{tb}_D(m,q) \to^*_{\mathcal{R}} \mathsf{tb}(u_m|_q)$ for all $m \in \{1,\ldots,n\}$ and all $q \in \mathcal{P}os(u_m)$ such that $\mathsf{tb}_D(m,q)$ is defined.*

Proof. By induction on the term depth of $u_m|_q$:

IB If $u_n|_p$ is a variable or a constant, $\mathsf{tb}_D(m,q) = \mathsf{tb}(u_m|_q)$.

IS By case distinction on $\mathsf{root}(u_m|_q)$:

1. If $u_m|_q = f(t_1,\ldots,t_{\mathsf{ar}(f)})$ where $f \in \mathcal{F}$, then $\mathsf{tb}_D(m,q) = f(\mathsf{tb}_D(m,q.1),\ldots,\mathsf{tb}_D(m,q.\mathsf{ar}(f)))$ and $\mathsf{tb}(u_m|_q) = f(\mathsf{tb}(t_1),\ldots,\mathsf{tb}(t_{\mathsf{ar}(f)}))$. By the induction hypothesis $\mathsf{tb}_D(m,q.i) \to^*_{\mathcal{R}} \mathsf{tb}(t_i)$ for all $i \in \{1,\ldots,\mathsf{ar}(f)\}$, so that

$$\mathsf{tb}_D(m,q) = f(\mathsf{tb}_D(m,q.1),\ldots,\mathsf{tb}_D(m,q.\mathsf{ar}(f))) \to^*_{\mathcal{R}} f(\mathsf{tb}(t_1),\ldots,\mathsf{tb}(t_{\mathsf{ar}(f)})) = \mathsf{tb}(u_m|_q)$$

2. Otherwise, $u_m|_q = U^\alpha_j(v, x_1\tau, \ldots, x_|X_j|\tau)$ and $\mathsf{tb}_D(m,q) = \mathsf{lin}(l)\sigma$ where l is the lhs of α. If $\mathsf{tb}_D(m,q.i)$ is defined, repeated application of 27 yields $\mathsf{lin}(l)\sigma|_o \to^*_{\mathcal{R}} \mathsf{tb}_D(m,q.i)$ for all $o \in \mathcal{VP}os(l)$ such that $l|_o = x_{i-1}$ $(1 < i \leq |X_j|)$. By the induction hypothesis $\mathsf{tb}_D(m,q.i) \to^*_{\mathcal{R}} \mathsf{tb}(u_m|_{q.i})$.
If $\mathsf{tb}_D(m,q.i)$ is not defined, let k be biggest value such that $u_k|_{q'}$ is an ancestor of $u_m|_q$, $u_k|_{q'.o}$ is an ancestor of $u_m|_{q.i}$ and $\mathsf{tb}_D(k,q'.o)$ is defined. By the definition of tb_D, $\mathsf{tb}_D(m,q) = \mathsf{tb}_D(k,q') = \mathsf{lin}(l)\sigma$. Since there are no rewrite steps inside non-traceable arguments $\mathsf{tb}(u_k|_{q'.o}) = \mathsf{tb}(u_m|_{q.i})$. By multiple application of Lemma 27 $x_{i-1}\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k,q'.o) \to^*_{\mathcal{R}} \mathsf{tb}(m,q.i) = x_{i-1}\ \mathsf{tb}(\tau)$.

□

The following is our key result for WLL DCTRSs:

Lemma 29 (technical key result)**.** *Let $\mathcal{R}$ be a* WLL DCTRS *and* $D\colon u_1 \to_{p_1,\mathcal{R}'} u_1 \to_{p_2,\mathcal{R}'} \cdots \to_{p_{n-1},\mathcal{R}'} u_n$ $(u_1 \in \mathcal{T})$ *be a derivation without rewrite steps in non-traceable arguments, then* $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$.

Proof. By induction over n:

IB For $n=1$ the result holds vacuously.

IS By case distinction on the rule applied in the $n-1$st rewrite step

1. If the rule is an introduction or switch rule, $u_n|_{p_{n-1}}$ is a U-term so that $\mathsf{tb}_D(n,p_{n-1}) = \mathsf{tb}_D(n-1,p_{n-1})$ by the definition of tb_D.
2. Otherwise, it is an elimination rule of the rule $\alpha\colon t_0 \to s_{k+1} \Leftarrow s_1 \to^* t_1, \ldots, s_k \to^* t_k$. Let $u_{m_1}|_{q_{m_1}}, u_{m_1+1}|_{q_{m_1+1}}, \ldots, u_{m_2}|_{q_{m_2}}, \ldots, u_{m_k}|_{q_{m_k}}, \ldots, u_{m_k-1}|_{q_{m_k-1}} = u_{n-1}|_{p_{n-1}}$ be ancestors of $u_n|_{p_{n-1}}$ such that $u_{i-1}|_{q_{i-1}}$ is a one-step ancestor of $u_i|_{q_i}$ $(m_1 < i < n)$, $\mathsf{root}(u_{m_i}|_{q_{m_i}}) = \cdots = \mathsf{root}(u_{m_{i+1}-1}|_{q_{m_{i+1}-1}}) = U_i^\alpha$ $(1 \le i \le k)$ and $q_{m_1} = p_{m_1-1}$ (i.e., the m_1-1sth rewrite step is the introduction step of α). Let furthermore in the following $m_{k+1} = n$.

   Observe, that $\mathsf{tb}_D(i,q_i)$ and the corresponding conditional argument $\mathsf{tb}_D(i,q_i.1)$ is always defined (the latter is due to the definition of WLL) $(m_1 \le i \le n-1)$, and $u_{i-1}|_{q_{i-1}}$ is the unique one-step ancestor of $u_i|_{q_i}$ since there is no non-traceable argument above q_i (otherwise p_{n-1} would be inside a conditional argument which violates our assumption).

   We first define matchers for the lhs of the rule and the (possibly non-linear) rhs's of the conditions: Let $\sigma_1, \sigma_1', \ldots, \sigma_k, \sigma_k'$ be such that $\mathsf{tb}_D(n-1,p_{n-1}) = \mathsf{tb}_D(m_1-1,q_{m_1-1}) = \mathsf{lin}(t_0)\sigma_1\sigma_1'$ and $\mathsf{tb}_D(m_i-1,q_{m_i-1}.1) = \mathsf{lin}(t_i)\sigma_{i+1}\sigma_{i+1}'$ $(1 \le i \le k)$ where $\mathcal{D}om(\sigma_{i+1}) = \{x \in \mathcal{V}ar(t_i) \mid |t_i|_x = 1\}$ is the matcher for all linear variables and $\mathcal{D}om(\sigma_{i+1}') = \mathcal{V}ar(\mathsf{lin}(t_i)) \setminus \mathcal{D}om(\sigma_i)$ is the matcher for all renamed non-linear variables in t_i $(0 \le i \le k)$.

   $\mathbb{U}(\alpha)$ is $t_0 \to U_1^\alpha(s_1, x_1, \ldots, x_{n_1}), U_1^\alpha(t_1, x_{1,1}, \ldots, x_{n_1}) \to U_2^\alpha(s_2, x_1, \ldots, x_{n_1}, \ldots, x_{n_2})$, $\ldots, U_k^\alpha(t_k, x_1, \ldots, x_{n_1}, \ldots, x_{n_k}) \to s_{n+1}$.

   For all variables x_j in traceable arguments for U_i^α, tb_D is defined, and $\mathsf{tb}_D(m_i, q_{m_i}.(j+1)) = x_j\sigma_i$ $(n_i \le j < n_{i+1})$.

   Let τ_i be defined as $x_j\tau_{i'} = \mathsf{tb}_D(m_{i'}-1, q_{m_{i'}-1}.j)$ for all x_j such that $|t_0, \ldots, t_{i'-1}|_{x_j} = 1$ (tb_D is therefore defined and $x_j \in \mathcal{D}om(\sigma_i)$). Observe that $\mathcal{D}om(\tau_i) \cap \mathcal{D}om(\sigma_i) = \emptyset$, since $\mathcal{D}om(\sigma_i)$ only contains variables that are introduced in t_{i-1} (otherwise they would be in $\mathcal{D}om(\sigma_i')$).

   By Lemma 27, $\mathsf{tb}_D(m_i, q_{m_i}.(j+1)) = x_j\sigma_i \to^*_{\mathcal{R}} \mathsf{tb}_D(m_{i'}-1, q_{m_{i'}-1}.(j+1)) = x_j\tau_{i'}$ $(i \le i' \le k)$, $\mathsf{tb}_D(m_{i'}, q_{m_{i'}}.1) = s_{i'}\sigma_{i'}\tau_{i'}$ $(1 \le i' \le n)$ and $\mathsf{tb}_D(n,p_{n-1}) = s_{n+1}\sigma_{k+1}\tau_{k+1}$. Therefore also $s_i\sigma_1 \ldots \sigma_i \to^*_{\mathcal{R}} s_i\sigma_i\tau_i$ for all $1 \le i \le k+1$.

   By repeated application of Lemma 27, also $s_i\tau_i \to^*_{\mathcal{R}} \mathsf{lin}(t_i)\sigma_{i+1}\sigma_{i+1}'$ $(1 \le i \le k)$.

   It remains to show that there is a derivation starting from $\mathsf{lin}(t_i)\sigma_{i+1}\sigma_{i+1}'$ such that we obtain a term matching t_i for all $0 \le i \le k$.

   Let $o \in \mathcal{VP}os(t_i)$ such that $t_i|_o = x$ and $|t_0, \ldots, t_i|_x > 1$. By the definition of tb_D, $\mathsf{lin}(t_i)\sigma_{i+1}'|_o = \mathsf{tb}_D(m_{i+1}-1, q_{m_{i+1}}.1.o)$ for $1 \le i \le k$ and $\mathsf{lin}(t_0)\sigma_1'|_o = \mathsf{tb}_D(m_1-1, q_{m_1}.o)$.

   By Lemma 28, $\mathsf{tb}_D(m_{i+1}-1, q_{m_{i+1}}.1.o) \to^*_{\mathcal{R}} \mathsf{tb}(u_{m_{i+1}-1}|_{q_{m_{i+1}}.1.o}) = x\tau_i'$ for $1 \le i \le k$ and $\mathsf{tb}_D(m_1-1, q_{m_1}.1.o) \to^*_{\mathcal{R}} \mathsf{tb}(u_{m_1-1}|_{q_{m_1}.1.o}) = x\tau_1'$.

Since there are no rewrite steps in non-traceable arguments in D, $x\tau'_i = x\tau'_{i'}$ for all $1 \leq i, i' \leq k+1$. Let therefore $\tau^* = \tau'_1 \dots \tau'_{k+1}$. Then $\mathsf{lin}(t_i)\sigma_{i+1}\sigma'_{i+1} \rightarrow^*_{\mathcal{R}} t_i\sigma_{i+1}\tau^*$.

By the definition of WLL, $s_i\sigma_1 \dots \sigma_i = s_i\sigma_1 \dots \sigma_{k+1}\tau^*$ and $t_i\sigma_{i+1}\tau^* = t_i\sigma_1 \dots \sigma_{k+1}\tau^*$. Therefore $s_i\sigma \rightarrow^*_{\mathcal{R}} s_i\sigma_i\tau_i \rightarrow^*_{\mathcal{R}} \mathsf{lin}(t_i)\sigma_i\sigma'_i \rightarrow^*_{\mathcal{R}} t_i\sigma\tau^*$ where $\sigma = \sigma_1 \dots \sigma_{k+1}\tau^*$. Therefore also $t_0\sigma = \mathsf{tb}_D(n-1, p_{n-1}) \rightarrow^*_{\mathcal{R}} s_{k+1}\sigma$. Since $x\sigma \rightarrow^*_{\mathcal{R}} x\tau_{k+1}$ for all $x \in \mathcal{V}ar(s_{k+1})$, finally $s_{k+1}\sigma \rightarrow^*_{\mathcal{R}} s_{k+1}\sigma_{k+1}\tau_{k+1} = \mathsf{tb}_D(n, p_{n-1})$.

□

Finally, we obtain soundness for WLL DCTRSs:

Lemma 30 (Soundness for weakly left-linear DCTRSs)**.** *Let $\mathcal{R}$ be a* WLL DCTRS *and* $D\colon u_1 \rightarrow_{p_1,\mathcal{R}'} u_1 \rightarrow_{p_2,\mathcal{R}'} \cdots \rightarrow_{p_{n-1},\mathcal{R}'} u_n$ *($u_1 \in \mathcal{T}$) then* $u_1 \rightarrow^*_{\mathcal{R}} \mathsf{tb}(u_n)$.

Proof. By Lemma 24, there is a derivation $D'\colon u_1 \rightarrow^*_{\mathcal{R}'} u_n$ without rewrite steps in non-traceable arguments. In D', $\mathsf{tb}_D(n, \epsilon)$ is defined so that via Lemma 29 and repeated application of Lemma 26 we obtain $u_1 \rightarrow^*_{\mathcal{R}} \mathsf{tb}_D(n, \epsilon)$. By Lemma 28 we finally get $\mathsf{tb}_D(n, \epsilon) \rightarrow^*_{\mathcal{R}} \mathsf{tb}(u_n)$. □

Theorem 31 (Soundness for reduction of weakly left-linear DCTRSs)**.** *The unraveling is sound for reduction w.r.t. weakly left-linear* DCTRSs.

Proof. Straightforward via Lemma 30. □

The definition of WLL DCTRSs allows conditional non-left-linear rules so that this result is more general than our result in [8] even for normal 1-CTRSs.

4 Discussion, Perspectives and Related Work

We have shown that the main ideas and approaches of [8] for soundness of normal 1-CTRSs also extend to DCTRSs, but with a couple of complications and subtleties. We think that especially the results involving weak left-linearity, non-erasingness and confluence as sufficient conditions for soundness (in the case of confluence restricted to soundness for reductions to normal forms) are quite interesting and practically relevant.

We are not aware of many related works on DCTRS or other classes of 3- or 4-CTRSs. An early one is [11] (which contains only a claim but no proofs). Yet, there is a notable exception. In [12] and, building on that paper, in [15] by the same authors closely related questions are investigated and similar soundness results for $\mathbb{U}_{opt}$ are obtained using quite different proof techniques.

More precisely, the main results of [15] are:

(a) $\mathbb{U}_{opt}$ is sound for $\mathbb{U}_{opt}$-LL (and for $\mathbb{U}$-LL) DCTRSs([15, Theorem 4.5]).

(b) $\mathbb{U}_{opt}$ is sound for $\mathbb{U}_{opt}$-RL-NE DCTRSs([15, Theorem 4.12]).

(c) If $\mathbb{U}_{opt}$ is sound w.r.t. $\mathcal{R}$, then $\mathbb{U}$ is so, too ([15, Theorem 4.19]).

Result (a) is of course important and practically relevant, since ($\mathbb{U}_{opt}$- or $\mathbb{U}$-)LL are reasonable requirements for DCTRSs. It is subsumed by our result on weak left-linearity (WLL) as sufficient criterion for soundness of $\mathbb{U}$ (Theorem 31). As Example 32 shows, $\mathbb{U}_{opt}$ is unsound even for WLL normal 1-CTRSs.

Result (b) resembles our Theorem 19, since it also requires $\mathbb{U}_{opt}$-NE. However, our result additionally forbids extra variables on rhs's. On the other hand, being right-separated is a much less restrictive property than $\mathbb{U}_{opt}$-RL. Proof-technically, the approach for (b) in [15] is remarkable and elegant. The proof works by reduction to the proof of (a) using the inverse reduction relation.

Result (c) is of particular interest, since via $\mathbb{U}_{opt}$ certain properties (e.g. soundness criteria) like NE become more likely to be satisfied. However, the reverse of (c) does not hold in general. In particular, we cannot reproduce our soundness results for WLL and CR (in the latter case w.r.t. soundness for reductions to normal forms) for $\mathbb{U}_{opt}$:

Example 32 (unsoundness of $\mathbb{U}_{opt}$)**.** *Consider the* WLL *and confluent normal 1-*CTRS $\mathcal{R}$ *consisting of the rules* $or(x, y) \to true \Leftarrow x \to^* true$ *and* $eq(x, x) \to true$*.* $\mathcal{R}$ *is unraveled using* $\mathbb{U}_{opt}$ *into* $\mathcal{R}'_{opt} = \{or(x, y) \to U_1^\alpha(x)\ ,\ U_1^\alpha(true) \to true\ ,\ eq(x, x) \to true\}$*.*

In $\mathcal{R}'_{opt}$*,* $or(false, true)$ *and* $or(false, false)$ *both rewrite to the irreducible U-term* $U_1^\alpha(false)$*, so that* $eq(or(false, true), or(false, false)) \to^*_{\mathcal{R}'_{opt}} true$*.*

Yet, $or(false, true)$ *and* $or(false, false)$ *are irreducible (and therefore not joinable) in* $\mathcal{R}$ *so that* $eq(or(false, true), or(false, false)) \not\to^*_{\mathcal{R}} true$*. Therefore,* $\mathbb{U}_{opt}$ *is not sound for* $\mathcal{R}$*.*

In future work we will try to improve the (un)soundness analysis for DCTRSs (e.g. by isolating abstract principles or characterization results), apply the sondness criteria for deriving properties of the original systems, and exploit / transfer the analysis and criteria for other well-known transformation approaches from CTRSs to TRSs.

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