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Abstract

We revisit (un)soundness of transformations of conditional into unconditional rewrite systems. The focus here is on so-called unravelings, the most simple and natural kind of such transformations, for the class of normal conditional systems without extra variables. By a systematic and thorough study of existing counterexamples and of the potential sources of unsoundness we obtain several new positive and negative results. In particular, we prove the following new results: Confluence, non-erasingness and weak left-linearity (of a given conditional system) each guarantee soundness of the unraveled version w.r.t. the original one. The latter result substantially extends the only known sufficient criterion for soundness, namely left-linearity. Furthermore, by means of counterexamples we refute various other tempting conjectures about sufficient conditions for soundness.

1 Introduction

1.1 Background and Motivation

Conditional term rewriting systems (CTRSs) are a very natural, though non-trivial and complex extension of unconditional ones (TRSs). This concerns both the theoretical foundations as well as applications and implementations of such systems. A well-studied approach to dealing with conditional rewriting is via transformation to unconditional systems such that the resulting unconditional system can simulate the original conditional one in an appropriate manner. Various transformations have been developed for that purpose. It is well-known that completeness of these transformations is easy to achieve and usually holds, whereas soundness is much harder to obtain and typically does not hold without imposing further conditions, e.g., restrictions on the rewrite relation in the resulting unconditional system. Informally, by *(simulation) soundness* we mean that whenever an original term reduces to another original term in the transformed system, then such a reduction is also possible in the original system. *(Simulation) completeness* is the dual property.

The above unsoundness phenomenon was discovered by Marchiori ([9, 8]) for the case of so-called unravelings,[1] but is also present in virtually all other known transformation

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[1] The very idea of *unravelings* is actually much older and appears already e.g. in [4], though in a specialized form (for function definitions).

approaches. Approaches to more faithfully simulating rewriting in a conditional system via restricted rewriting in a transformed unconditional system include: *conditional eagerness* ([19], [16]), *innermost rewriting* ([15]), *membership conditional* and *context-sensitive* rewriting ([18], [14, 11, 13], [5], [17])). Yet, in all these approaches the imposed restrictions on rewriting in the transformed unconditional rewrite relation are a major source of complications for reasoning over and deriving properties of the respective transformation approaches. Hence, a deeper knowledge about the borderline between unsoundness and soundness would help to identify cases (classes of initial conditional systems) where soundness is guaranteed even for unrestricted rewriting in the transformed unconditional system. In such cases, one can safely use (unrestricted) rewriting in the transformed system, thus facilitating the analysis and implementation of the respective transformation. These are the main goals of the analysis that we are going to present in this paper.

1.2 Overview and Outline

We focus on the most basic class of conditional systems without extra variables, normal 1-CTRSs. This is motivated by the fact that even for these systems the analysis is rather non-trivial and properly understanding this case appears to be indispensable for later extending the results to other and more general classes of CTRSs. Furthermore, the focus is also restricted to unravelings, the most simple and intuitive class of transformations from CTRSs into TRSs. Again, simplicity and the goal of properly understanding the essential source(s) of unsoundness is the main motivation for this restriction. We expect that a substantial part of the analysis can also be reused for other transformation approaches for CTRSs.

The main contributions of the paper are as follows. Starting from an analysis of existing counterexamples to the unsoundness of unravelings we prove that each of the following conditions on a given normal 1-CTRS is sufficient for soundness of its unraveled version:

- confluence (Theorem 3.12)
- non-erasingness (Theorem 3.16)
- groundness of all conditions (Theorem 3.17)
- *weak left-linearity* (Theorem 3.33).

Especially interesting and practically relevant are the first criterion and the last one which substantially extends the only known criterion for soundness, left-linearity (cf. [8, 9]). In essence, *weak left-linearity* (cf. Definition 3.22) weakens left-linearity by allowing unconditional non-left-linear rules provided that variables that appear non-linear in the left-hand side do not appear at all in the right-hand side.

On the negative side, we disprove various other tempting conjectures about the sufficiency of conditions for soundness, regarding e.g. non-overlappingness, non-collapsingness and right-linearity.

The rest of the paper is structured as follows. After the preliminaries in Section 2, where we introduce unravelings and basic projection functions used later on, we develop the analysis in the main Section 3. Before concluding, the results obtained, potential extensions, open problems and related work are finally discussed in Section 4. For the sake of readability and completeness, missing / more detailed proofs have been postponed to the appendix.

2 Preliminaries

We assume familiarity with the basic concepts and notations of abstract reductions systems (ARSs) and (conditional) term rewriting systems (CTRSs) (cf. e.g. [1], [3]). For the sake of readability we recall some notions and notations here. Moreover, we use the typical abbreviations for properties of rewrite systems, such as CR, NF, UN, $\mathsf{UN}^{\rightarrow}$, ...

The set of (non-variable, variable) positions of a term s is denoted as $Pos(s)$ ($FPos(s)$, $VPos(s)$). By $root(s)$ we denote the root symbol of the term s. Throughout the paper $\mathcal{V}$ denotes a countably infinite set of variables and x, y, z denote variables from $\mathcal{V}$. By $Var(s)$ we denote the set of variables of a term s. Moreover $\overrightarrow{Var(s)}$ denotes the sequence of variables obtained by arranging the variables of $Var(s)$ in an arbitrary but fixed order.

A term rewriting system $\mathcal{R}$ is a pair $(\mathcal{F}, R)$ of a signature and a set of rewrite rules over this signature. Slightly abusing notation we also write $\mathcal{R}$ instead of R (leaving the signature implicit).

We denote a rewrite step from a term s to a term t at position p with respect to a rewrite system $\mathcal{R}$ and with a rule δ from $\mathcal{R}$ as $s \rightarrow_{p,\mathcal{R},\delta} t$. We also write $s \rightarrow t$ ($s \rightarrow_p t$ resp. $s \rightarrow_{p,\mathcal{R}} t$) if the position, rewrite system and applied rule (the rewrite system and applied rule resp. the applied rule) are clear from the context or of no relevance. Parallel reduction is denoted by $\mathrel{+\!\!\!\!\!\!\rightarrow}$ and $\rightarrow^{\leq 1}$ means reduction with one or zero steps.

The set of *one-step descendants* of a (subterm) position p of a term t w.r.t. a (one-step) reduction $t = C[s]_p \rightarrow_q t'$ is the set of subterm positions in t' given by

- $\{p\}$, if $q \geq p$ or $q \mid\mid p$,
- $\{q.o'.p' \mid t|_q = l\sigma, l|_o \in Var(l), q.o.p' = p, l|_o = r|_{o'}\}$, if $q < p$ and (a superterm of) s is bound to a variable in the matching of $t|_q$ with the left-hand side of the applied rule, and
- $\emptyset$, otherwise.

Slightly abusing terminology, when $t = C[s]_p \rightarrow_q t'$ with set $\{p_1, \ldots, p_k\}$ of one-step descendants in t', we also say that $t|_p$ has the one-step descendants $t'|_{p_i}$ in t'. The *descendant relation* (w.r.t. given derivations) is obtained as the (reflexive-)transitive closure of the one-step descendant relation. The relation of (one-step) *ancestors* of a subterm position (w.r.t. a given reduction sequence) is the inverse relation of the (one-step) descendant relation.

A conditional term rewriting system $\mathcal{R}$ (over some signature $\mathcal{F}$) consists of rules $l \rightarrow r \Leftarrow c$ where c is a conjunction of equations $s_i = t_i$. Equality in the conditions may be interpreted (recursively) e.g. as $\leftrightarrow^*$ (semi-equational case), as $\downarrow$ (join case), or as $\rightarrow^*$ (oriented case). In the latter case, if all right-hand sides of conditions are ground terms that are irreducible w.r.t. the unconditional version $\mathcal{R}_u = \{l \rightarrow r \mid l \rightarrow r \Leftarrow c \in \mathcal{R}\}$ of R, the system is said to be a *normal* one.

According to the distribution of variables, a conditional rule $l \rightarrow r \Leftarrow c$ may satisfy (1) $Var(r) \cup Var(c) \subseteq Var(l)$, (2) $Var(r) \subseteq Var(l)$, (3) $Var(r) \subseteq Var(l) \cup Var(c)$, or (4) no variable constraints. If all rules of a CTRS $\mathcal{R}$ are of type (i), $1 \leq i \leq 4$, we say that $\mathcal{R}$ is an i-CTRS. Given a conditional rewrite rule $l \rightarrow r \Leftarrow c$ and a variable x such that $x \in Var(r)$ but $x \notin Var(l)$, we say that x is an *extra variable*.

There exists abundant literature on transforming CTRSs into unconditional systems such that the original system can be appropriately simulated via reduction in the unconditional transformed one. For a unified parametrized approach to such transformations and the relevant terminology we refer to [6]. Unravelings as introduced and investigated in [8, 9] are the most simple and intuitive ones.

Definition 2.1 ((simultaneous) unraveling for normal 1-CTRSs ([9, 8], cf. also [15])). *Given a normal 1-CTRS $\mathcal{R} = (\mathcal{F}, R)$, every conditional rule*

$$\delta\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$$

of $\mathcal{R}$ is transformed into[2]

$$\begin{array}{rcll} l & \to & U^\delta(s_1, \ldots, s_n, \overrightarrow{Var(l)}) & \text{(introduction rule)} \\ U^\delta(t_1, \ldots, t_n, \overrightarrow{Var(l)}) & \to & r & \text{(elimination rule)} \end{array}$$

*Unconditional rules remain invariant. The resulting (*unraveled*)* TRS *is denoted as $U(\mathcal{R})$ or $\mathcal{R}'$ (over the signature $\mathcal{F}' = \mathcal{F} \cup \{U^\delta \mid \delta\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \in \mathcal{R}\}$). Instead of the new symbols U^δ (corresponding to rule δ) we sometimes use other ones if appropriate.*

Symbols from $\mathcal{F}' \setminus \mathcal{F}$ are also called U-symbols. Terms rooted by such symbols are called U-terms or U-rooted terms. Every U-symbol corresponds to a particular conditional rewrite rule of the original CTRS according to Definition 2.1. Hence, we write U^δ to indicate that U^δ corresponds to the rewrite rule δ. Moreover, if there is only one conditional rule defining a function symbol f we may also write U^f to identify this rule. Henceforth, $\mathcal{R}$ denotes a normal 1-CTRS unless stated otherwise.

The signature of an unraveled CTRS $\mathcal{R}'$ is a superset of the signature of the CTRS $\mathcal{R}$. Hence, terms in $\mathcal{R}'$-reductions are terms over this extended signature in general (we also call them *mixed* terms). Throughout the paper, when dealing with CTRSs $\mathcal{R} = (\mathcal{F}, R)$ we denote by $\mathcal{R}'$ the corresponding unraveled TRS, by $\mathcal{F}'$ the extended signature of the TRS, by $\mathcal{T}$ the terms over the signature $\mathcal{F}$ and by $\mathcal{T}'$ the terms over the extended signature $\mathcal{F}'$. For proof-theoretical reasons, in particular to show that unraveled systems are not too general and do not enable "too many" reductions, we introduce functions that map mixed terms to terms over the original signature of the CTRS in question.

We define two basic approaches of projecting mixed terms in the transformed system back into corresponding original terms. The crucial idea is that when we consider a U-(sub)term $U^\delta(s_1, \ldots, s_n)$ in a given $\mathcal{R}'$-reduction we know that the root-symbol U^δ indicates that previously the introduction rule for $U^\delta\colon l \to r \Longleftarrow u_1 \to^* v_1, \ldots, u_n \to^* v_n$ must have been applied. Now, in order to get rid of U^δ, there are two natural ways of doing so: We can go back to the corresponding instance of the *lhs* l, or we anticipate the result by taking the corresponding instance of the *rhs* r. In both cases, the projection needs to recursively translate also U-subterms of the given term.

Definition 2.2 (translate backwards (tb)). *Let $\mathcal{R} = (\mathcal{F}, R)$ be a normal 1-*CTRS*. Then the* translate backward *function* $\mathsf{tb}\colon \mathcal{T} \to \mathcal{T}'$ *is defined by*

$$\mathsf{tb}(t) = \begin{cases} x & \text{if } t = x \in \mathcal{V} \\ f(\mathsf{tb}(t_1), \ldots, \mathsf{tb}(t_m)) & \text{if } t = f(t_1, \ldots, t_m) \text{ and } f \in \mathcal{F} \\ l\sigma & \text{if } t = U^\delta(v_1, v_2, \ldots, v_n, w_1, \ldots, w_k) \\ & \quad \text{and } \delta\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \end{cases}$$

where $\overrightarrow{Var(l)} = x_1, \ldots, x_k$ and σ is (recursively) defined as $x_i\sigma = \mathsf{tb}(w_i)$ for $1 \le i \le k$.

[2]Using $\overrightarrow{Var(t)}$ as sequence of the *set* of variables in t goes back to [15], whereas in [9, 8] the sequence is constructed from the *multiset* of variables in t. The former version appears to be generally preferable, because it is more abstract and avoids additional complications due to "non-synchronization effects".

Definition 2.3 (translate forward (tf)). *Let $\mathcal{R} = (\mathcal{F}, R)$ be an normal 1-*CTRS*. Then the* translate forward *function* $\mathsf{tf}\colon \mathcal{T} \to \mathcal{T}'$ *is defined by*

$$\mathsf{tf}(t) = \begin{cases} x & \text{if } t = x \in \mathcal{V} \\ f(\mathsf{tf}(t_1), \ldots, \mathsf{tf}(t_m)) & \text{if } t = f(t_1, \ldots, t_m) \text{ and } f \in \mathcal{F} \\ r\sigma & \text{if } t = U^\delta(v_1, v_2, \ldots, v_n, w_1, \ldots, w_k) \\ & \quad \text{and } \delta\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n \end{cases}$$

where $\overrightarrow{Var(l)} = x_1, \ldots, x_k$ *and* σ *is (recursively) defined as* $x_i\sigma = \mathsf{tf}(w_i)$ *for* $1 \leq i \leq k$.

In this paper we focus on the property of *soundness* of unravelings which is dual to the (easier to obtain) property of completeness. An unraveling is said to be *complete* (for reductions) (or *simulation-complete*) if for all CTRSs $\mathcal{R}$, $s \to^*_{\mathcal{R}} t$ for $s, t \in \mathcal{T}$ implies $s \to^*_{\mathcal{R}'} t$. Furthermore, an unraveling is *sound* for reductions (or *simulation-sound*) if $s \to^*_{\mathcal{R}'} t$ implies $s \to^*_{\mathcal{R}} t$. Subsequently, we sometimes use a slightly more general notion of soundness by demanding that $s \to^*_{\mathcal{R}'} t$ (for $t \in \mathcal{T}'$) implies $s \to^*_{\mathcal{R}} \mathsf{tb}(t)$ resp. $\mathsf{tf}(t)$. This notion is indeed more general since $\mathsf{tb}(t) = \mathsf{tf}(t) = t$ whenever $t \in \mathcal{T}$ (i.e. t is an original term). Given a particular CTRS $\mathcal{R}$, we also say that the unraveling is complete (sound) for $\mathcal{R}$ or, slightly abusing terminology, that $\mathcal{R}'$ is complete (sound) w.r.t. $\mathcal{R}$. For a more thorough discussion of the terminology used for (preservation properties of) transformations we refer to [6].

3 (Un)Soundness for Normal 1-CTRSs

By carefully analyzing known counterexamples to soundness (of unravelings for normal 1-CTRSs) from the literature we first collect a couple of (mainly syntactic) properties whose absence may be viewed as tempting candidates for guaranteeing soundness (Subsection 3.1). We then show that some of them are not really essential for the unsoundness phenomenon.

3.1 Known and New Counterexamples

First of all, as observed in [6], there is a simple source of unsoundness in unravelings (as well as in most other transformations) which is due to an "optimized" version of unraveling as it is used in several papers. The underlying idea for this "optimization" is that when starting a conditional rule application via an introduction step, not all variable bindings of the *lhs* (instance) are stored in the corresponding U-term introduced, but only those that are needed to eventually produce the final *rhs* (instance), provided all conditions are satisfied. This motivates the definition of U_{opt} as follows: Transform

$$\delta\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$$

into

$$\begin{array}{rcll} l & \to & U^\delta(s_1, \ldots, s_n, \overrightarrow{Var(r)}) & \quad (\textit{introduction rule}) \\ U^\delta(t_1, \ldots, t_n, \overrightarrow{Var(r)}) & \to & r & \quad (\textit{elimination rule}) \end{array}$$

Given $\mathcal{R}$, let us denote the resulting system as $\mathcal{R}'_{opt}$. Then it is easy to see that simulating $\mathcal{R}$ (on $\mathcal{T}$) is indeed possible via $\mathcal{R}'_{opt}$, i.e., $\mathcal{R}'_{opt}$ is (simulation) complete (w.r.t. $\mathcal{R}$). However, concerning soundness (and consequently also e.g. completeness w.r.t. termination) there is a problem (due to non-left-linear rules in $\mathcal{R}$).

Example 3.1. *When we unravel*

$$\mathcal{R} = \left\{ \begin{array}{rcl} f(x) \to a & \Longleftarrow & b \to^* c \\ g(x,x) \to d & & \end{array} \right\}$$

with U_{opt} into

$$\mathcal{R}'_{opt} = \left\{ \begin{array}{c} f(x) \to U(b) \\ U(c) \to a \\ g(x,x) \to d \end{array} \right\}$$

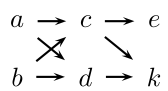
we get $g(f(a), f(b)) \to^*_{\mathcal{R}_{opt}} g(U(b), U(b)) \to_{\mathcal{R}_{opt}} d$, but obviously $g(f(a), f(b)) \not\to^*_{\mathcal{R}} d$, because $f(t)$ is $\mathcal{R}$-irreducible for every $\mathcal{R}$-irreducible $t \in \mathcal{T}$.

If we now add the rule $d \to g(f(a), f(b))$ to $\mathcal{R}$, the resulting system is still terminating, but its unraveled version becomes non-terminating. □

This subtle flaw of "optimized" transformations (caused by omitting certain seemingly unnecessary variable bindings) as for U_{opt} above has been overlooked in various papers on transformations (cf. e.g. [2], [8]).[3] But even if we exclude such "optimizations" and insist on keeping all variable bindings in introduction steps (as in U), unraveled systems are in general not sound, as discovered by Marchiori in his pioneering paper [9].[4] This is a striking fact that — at least at first glance – is rather counterintuitive!

The following is a slightly simplified version of the basic ingenious counterexample of Marchiori [8, Example 4.3], similar to [6, Example 1].

Example 3.2. *Unraveling of $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ with*



$$\underbrace{\qquad h(x,x) \to g(x,x,f(k)) \quad g(d,x,x) \to A}_{\mathcal{R}_1} \qquad \underbrace{f(x) \to x \Leftarrow x \to^* e}_{\mathcal{R}_2}$$

yields $\mathcal{R}' = \mathcal{R}_1 \cup \mathcal{R}'_2$ with

$$\underbrace{f(x) \to U(x,x) \quad U(e,x) \to x}_{\mathcal{R}'_2}$$

In $\mathcal{R}'$ we get

$$\begin{array}{rlllll} & h(f(a), f(b)) & \to^+ & h(U(c,d), U(c,d)) & \to & g(U(c,d), U(c,d), f(k)) \\ \to^+ & g(d, U(c,d), f(k)) & \to^+ & g(d, U(k,k), U(k,k)) & \to & A \end{array}$$

However, in $\mathcal{R}$ we do not have $h(f(a), f(b)) \to^* A$, since otherwise this would imply

$$h(f(a), f(b)) \to^* h(s,s) \to g(s,s,f(k)) \to^* g(d,t,t) \to^* A$$

for some s, t satisfying (1) $f(a) \to^* s$, $f(b) \to^* s$, (2) $s \to^* d$, and (3) $s \to^* t$, $f(k) \to^* t$.

But (1) and (2) imply $s = d$, hence $t = d$ or $t = k$. However, by (3), $f(k) \to^* t$ is neither possible for $t = d$ nor for $t = k$. □

Inspection of Example 3.2 reveals that it has numerous properties that one might be tempted to conjecture to be essential for the counterexample property.

Observation 3.3. *The system $\mathcal{R}$ of Example 3.2 enjoys the following (mostly syntactical) properties: It is non-left-linear ($\neg$LL), non-confluent ($\neg$CR), erasing, i.e. not non-erasing ($\neg$NE), non-right-linear ($\neg$RL), not a constructor system ($\neg$CS), not an overlay system ($\neg$OS), overlapping, i.e. not non-overlapping ($\neg$NO) and collapsing, i.e. not non-collapsing ($\neg$NCOL).*

[3] Also in [12] a similar optimized transformation is used. Although the results presented in [12] do not contradict examples like Example 3.1 above, the general problem with such "optimized" transformations remains hidden, cf. [12, counterex. R_4, p. 9].

[4] More precisely, the details are only included in the extended technical report version [8] of [9].

We will now investigate whether each of these properties is essential for unsoundness or not.

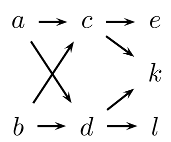
Proposition 3.4. *None of the properties of being*

- *not a constructor system* ($\neg\mathsf{CS}$)
- *not an overlay system* ($\neg\mathsf{OS}$)
- *collapsing* ($\neg\mathsf{NCOL}$)
- *non-right-linear* ($\neg\mathsf{RL}$)

is essential for unsoundness of unravelings.

Proof. Cf. Example 3.5 □

Example 3.5. *Unraveling of* $\mathcal{R} = \mathcal{R}_1 \cup \mathcal{R}_2$ *with*



$$\underbrace{\qquad g(x,x) \to A}_{\mathcal{R}_1} \qquad \underbrace{\begin{array}{ll} f(x) \to m(x) & \Leftarrow x \to^* e \\ h(x,x) \to g(x, f(k)) & \Leftarrow x \to^* m(l) \end{array}}_{\mathcal{R}_2}$$

yields $\mathcal{R}' = \mathcal{R}_1 \cup \mathcal{R}'_2$ *with*

$$\underbrace{\begin{array}{ll} f(x) \to U^f(x,x) & h(x,x) \to U^h(x,x) \\ U^f(e,x) \to m(x) & U^h(m(l),x) \to g(x, f(k))\,. \end{array}}_{\mathcal{R}'_2}$$

In $\mathcal{R}'$ *we have*

$$\begin{array}{rllll} & h(f(a), f(b)) & \to^+ & h(U^f(c,d), U^f(c,d)) & \to & U^h(U^f(c,d), U^f(c,d)) \\ \to^+ & U^h(m(l), U^f(c,d)) & \to & g(U^f(c,d), f(k)) & \to^+ & g(U^f(k,k), U^f(k,k)) & \to A\,. \end{array}$$

However, in $\mathcal{R}$ *we do not have* $h(f(a), f(b)) \to^* A$*, analogously to the reasoning in Example 3.2.*

Proposition 3.6. *The property of being overlapping is not essential for unsoundness of unravelings.*

Proof. The non-confluent overlapping part of the Examples 3.2 and 3.5 can easily be changed into a non-overlapping (but still non-confluent) sub-system such that the counterexample property is preserved by using rules of the shape $a(x,x) \to c(p,p)$ and $a(x, i(x)) \to d(p,p)$ to simulate a divergence $c \leftarrow a \to d$. Additionally, a rule $p \to i(p)$ is added. □

3.2 Sufficient Criteria for Soundness

In this section we will prove that each of the remaining properties of the CTRS of Example 3.2, namely being non-left-linear, non-confluent and erasing (i.e. not non-erasing) is indeed crucial for the counterexample, thus yielding corresponding soundness criteria.

For the case of left-linearity this has already been proved by Marchiori in [8].

Theorem 3.7 (left-linearity is sufficient ([8], cf. also [15])). *Left-linearity of* $\mathcal{R}$ *is sufficient for soundness of* $\mathcal{R}'$.

In the following we establish that confluence and non-erasingness of a CTRS $\mathcal{R}$ are sufficient to deduce that the unraveling of Definition 2.1 is sound w.r.t. $\mathcal{R}$. Moreover, we generalize the soundness result for left-linear systems by demanding only weak left-linearity (see Definition 3.22 below) instead of left-linearity.

3.2.1 Confluence

An important property of unravelings is that variables may be duplicated when U-symbols are introduced. For instance in Example 3.2 such a duplication occurs in the rule $f(x) \to U(x, x)$. Thus, in an $\mathcal{R}'$-reduction after this rule is applied, the instantiated variables could be reduced to different terms. In Example 3.2 this happens when $U(a, a)$ is reduced to $U(c, d)$.

However, when transforming a term like $U(c, d)$ into a term from $\mathcal{T}$ for instance using tb either c or d is selected as instantiation of the single variable of the left-hand side of the corresponding conditional rewrite rule. In case of tb we would get $\mathsf{tb}(U(c, d)) = f(d)$. Regarding soundness this is problematic in general, since $U(c, d) \to^+_{\mathcal{R}'} d$ but $\mathsf{tb}(U(c, d)) = f(d) \not\to^*_{\mathcal{R}} d = \mathsf{tb}(d)$. The particular problem here is that $d \not\to^*_{\mathcal{R}} e$ and thus the conditional rule is not applicable to $f(d)$. Non-confluence, i.e. $d \leftarrow^+_{\mathcal{R}} a \to^+_{\mathcal{R}} e$ but d and e are not joinable, is crucial for this problem.

If $\mathcal{R}$ is confluent and $U(u, v)$ (with $u, v \in \mathcal{T}$) appears as redex w.r.t. a U-elimination rule in a $\mathcal{R}'$-reduction sequence starting from an original term (provided that U has been introduced by a rule $l \to U(x, x)$), we can prove that $v \to^*_{\mathcal{R}} u$ holds. This is achieved by showing that u and v have a common ancestor in $\mathcal{T}$ and since u is a ground normal form, confluence of $\mathcal{R}$ implies $v \to^*_{\mathcal{R}} u$.

First we prove an auxiliary lemma basically stating a kind of monotony under $\mathcal{T}'$-contexts of $\to_{\mathcal{R}}$ when tb is applied.

Lemma 3.8 (monotony property of tb)**.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a 1-CTRS. If $u \to_{p,\mathcal{R}'} v$ for terms $u, v \in \mathcal{T}'$ and $\mathsf{tb}(u|_p) \to^{\leq 1}_{\mathcal{R}} \mathsf{tb}(v|_p)$, then $\mathsf{tb}(u|_q) \mathrel{+\!\!\!\!\!\!\to}_{\mathcal{R}} \mathsf{tb}(v|_{q'})$ for all $q \in Pos(u)$ and all descendants q' of q in v.*

Proof (sketch). For the interesting case where $q \leq p$ we use induction on the size of p' determined by $q.p' = p$. ☐

The next lemma is the technical key result for the proof of Theorem 3.12 below. It states that in an $\mathcal{R}'$-reduction sequence D starting from an original term, for every redex u and its (one-step) reductum v appearing in D we have $\mathsf{tb}(u) \to^{\leq 1}_{\mathcal{R}} \mathsf{tb}(v)$.

Lemma 3.9 (technical key result for confluent systems)**.** *Let $\mathcal{R} = (\mathcal{F}, R)$ be a* confluent *normal 1-CTRS and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \ldots \to_{p_{n-1},\mathcal{R}'} u_n$ be a reduction sequence where $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. Then, $\mathsf{tb}(u_i|_{p_i}) \to^{\leq 1}_{\mathcal{R}} \mathsf{tb}(u_{i+1}|_{p_i})$ for all $1 \leq i < n$.*

Proof (sketch). Proof by induction on the length of D and case distinction on the rule applied in the last step of D. The interesting case is where this last step is a U-elimination step. There, we get for every condition $s_i \to^* t_i$ of the corresponding conditional rule α that $\mathsf{tb}(s_i\sigma) \to^*_{\mathcal{R}} \mathsf{tb}(t_i)$ and $\mathsf{tb}(s_i\sigma) \to^*_{\mathcal{R}} \mathsf{tb}(s_i\tau)$ holds, where τ is the matcher used in the last step of D and σ the matcher used in the corresponding U-introduction step of α, according to the induction hypothesis. Then, confluence of $\mathcal{R}$ yields $\mathsf{tb}(s_i\tau) \to^*_{\mathcal{R}} t_i$ since $\mathsf{tb}(t_i) = t_i$ and t_i is a (ground $\mathcal{R}_u$-)normal form. Hence, α is applicable to $\mathsf{tb}(u_{n-1}|_{p_{n-1}})$ and we get $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) \to_{\mathcal{R}} \mathsf{tb}(u_n|_{p_{n-1}})$. ☐

In Lemma 3.9 the confluence assumption cannot be dropped.

Example 3.10. *Consider the following normal 1-*CTRS $\mathcal{R}$.

$$a \to b \qquad a \to c \qquad f(x) \to x \Leftarrow x \to^* b$$

$\mathcal{R}$ *is not confluent since* b *and* c *are not joinable. Consider the* $\mathcal{R}'$*-reduction sequence* $f(a) \to_{\mathcal{R}'} U(a,a) \to^+_{\mathcal{R}'} U(b,c) \to_{\mathcal{R}'} c$ *and the term* $U(b,c)$. *In the proof of Lemma 3.9 we showed that* b *and* c *must have a common ancestor. However, while in the proof we used this fact to deduce that they also have a common descendant and further that this descendant must be* b, *in the example this conclusion is wrong because of non-confluence of* $\mathcal{R}$. *Indeed, Lemma 3.9 does not hold for this example, since* $\mathsf{tb}(U(b,c)) = f(c) \not\to_{\mathcal{R}} c = \mathsf{tb}(c)$.

Lemma 3.11 (projecting reductions issuing from original terms)**.** *Let* $\mathcal{R}$ *be confluent. Then for every* $\mathcal{R}'$*-reduction* $u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ *with* $u_1 \in \mathcal{T}$ *we have* $u_1 = \mathsf{tb}(u_1) \twoheadrightarrow_{\mathcal{R}} \mathsf{tb}(u_2) \twoheadrightarrow_{\mathcal{R}} \dots \twoheadrightarrow_{\mathcal{R}} \mathsf{tb}(u_n)$.

Proof. For every redex $u_j|_{p_j}$ and corresponding reductum $u_{j+1}|_{p_j}$ $(1 \le j < n)$ we have $\mathsf{tb}(u_j|_{p_j}) \to^{\le 1}_{\mathcal{R}} \mathsf{tb}(u_{j+1}|_{p_j})$ because of Lemma 3.9. This implies $\mathsf{tb}(u_j) \twoheadrightarrow_{\mathcal{R}} \mathsf{tb}(u_{j+1})$ according to Lemma 3.8 (with $q = q' = \epsilon$). ☐

As corollary we obtain the following result.

Theorem 3.12 (confluence is sufficient)**.** *Confluence of* $\mathcal{R}$ *is sufficient for soundness of* $\mathcal{R}'$.

Proof. Straightforward using Lemma 3.11. ☐

3.2.2 Non-Erasingness

In Example 3.2 the $\mathcal{R}'$-reduction that is a witness for unsoundness contains U-(sub)-terms that are not reducible to original terms, since the U-symbol cannot be eliminated (e.g. the term $U(k,k)$). Hence, since the final term A of the reduction is an original term, these terms must be erased.

When considering a non-erasing CTRS $\mathcal{R}$ (and thus a non-erasing $\mathcal{R}'$), every U-symbol in every (finite) $\mathcal{R}'$ reduction sequence D ending in a term from $\mathcal{T}$ must be properly eliminated. This fact motivates and justifies the use of tf when simulating $\mathcal{R}'$-reductions in $\mathcal{R}$, as whenever some U-term is encountered in D it will eventually be eliminated in D and this elimination is anticipated when applying tf.

The following lemma is dual to Lemma 3.8 in that tf instead of tb is used for transforming terms from $\mathcal{T}'$ into terms from $\mathcal{T}$.

Lemma 3.13 (monotony property of tf)**.** *Let* $\mathcal{R} = (\mathcal{F}, R)$ *be a 1-*CTRS. *If* $u \to_{p,\mathcal{R}'} v$ *for* $u, v \in \mathcal{T}'$ *and* $\mathsf{tf}(u|_p) \to^{\le 1}_{\mathcal{R}} \mathsf{tf}(v|_p)$, *then* $\mathsf{tf}(u|_q) \twoheadrightarrow_{\mathcal{R}} \mathsf{tf}(v|_{q'})$ *for all* $q \in Pos(u)$ *and all descendants* q' *of* q *in* v.

Proof (sketch). The proof is analogous to the one of Lemma 3.8. For the interesting case where $q \le p$ we use induction on the size of p' determined by $q.p' = p$. ☐

The next lemma is the technical key result for the proof of Theorem 3.16 below. It is dual to Lemma 3.9 in that tf is used instead of tb.

Lemma 3.14 (technical key result for non-erasing systems)**.** *Let* $\mathcal{R} = (\mathcal{F}, R)$ *be a* non-erasing *normal 1-*CTRS *and let* $D : u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} s_n$ *be a reduction sequence where* $u_n \in \mathcal{T}$ *and* $u_i \in \mathcal{T}'$ *for* $1 \le i < n$. *Then,* $\mathsf{tf}(u_i|_{p_i}) \to^{\le 1}_{\mathcal{R}} \mathsf{tf}(u_{i+1}|_{p_i})$ *for* $1 \le i < n$.

Proof (sketch). Proof by induction on the length of D and case distinction on the rule applied in the first step of D. The interesting case is where this first step is a U-introduction step. Since $\mathcal{R}'$ is non-erasing, the introduced U-symbol is eventually eliminated in D and hence by the induction hypothesis and Lemma 3.13 we get $\mathsf{tf}(s_i\sigma) \to_{\mathcal{R}}^* t_i$ for all conditions of the conditional rule corresponding to the introduced U-symbol. Hence $\mathsf{tf}(u_1|_{p_1}) \to_{\mathcal{R}} \mathsf{tf}(u_2|_{p_1})$. ☐

Finally, we can prove soundness of unravelings for non-erasing normal 1-CTRSs.

Lemma 3.15 (projecting reductions issuing from original term)**.** *Let $\mathcal{R}$ be non-erasing. Then for every $\mathcal{R}'$-reduction $u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ with $u_n \in \mathcal{T}$ we have $\mathsf{tf}(u_1) \mathrel{+\!\!\!\!\!+\!\!\!\!\to}_{\mathcal{R}} \mathsf{tf}(u_2) \mathrel{+\!\!\!\!\!+\!\!\!\!\to}_{\mathcal{R}} \dots \mathrel{+\!\!\!\!\!+\!\!\!\!\to}_{\mathcal{R}} \mathsf{tf}(u_{n-1}) \mathrel{+\!\!\!\!\!+\!\!\!\!\to}_{\mathcal{R}} \mathsf{tf}(u_n) = u_n$.*

Proof. For every redex $u_j|_{p_j}$ and corresponding reductum $u_{j+1}|_{p_j}$ $(1 \leq j < n)$ we have $\mathsf{tf}(u_j|_{p_j}) \to_{\mathcal{R}}^{\leq 1} \mathsf{tf}(u_{j+1}|_{p_j})$ because of Lemma 3.14. This implies $\mathsf{tf}(u_j) \mathrel{+\!\!\!\!\!+\!\!\!\!\to}_{\mathcal{R}} \mathsf{tf}(u_{j+1})$ according to Lemma 3.13 (with $q = q' = \epsilon$). ☐

Theorem 3.16 (non-erasingness is sufficient)**.** *Non-erasingness of $\mathcal{R}$ is sufficient for soundness of $\mathcal{R}'$.*

Proof. Straightforward using Lemma 3.15. ☐

3.2.3 Right-Linearity Revisited

Next we reconsider right-linearity. In Example 3.5 we have shown that non-right-linearity of $\mathcal{R}$ is not essential for unsoundness. However, in this example the unraveled system $\mathcal{R}'$ becomes non-right-linear. This property of $\mathcal{R}'$ is crucial for Example 3.5 (as we will see). Yet, demanding that $\mathcal{R}'$ is right-linear is a severe restriction, since right-linearity of $\mathcal{R}'$ implies that $\mathcal{R}$ contains only ground conditions. To see this consider some conditional rule $l \to r \Leftarrow s \to t$, such that $x \in \mathit{Var}(s)$. Since we consider 1-CTRSs this implies $x \in \mathit{Var}(l)$ and hence the unraveled system contains a non-right-linear rule $l \to U(s, x)$.

It turns out that for CTRSs $\mathcal{R}$ having only ground conditions (GC), $\mathcal{R}'$ is sound even if $\mathcal{R}$ is not right-linear.

Theorem 3.17 (GC is sufficient for soundness)**.** *If $\mathcal{R}$ has only ground conditions, then $\mathcal{R}'$ is sound (w.r.t. $\mathcal{R}$).*

Proof (sketch). The proof is basically analogous to the proof of soundness for confluent CTRSs. There, confluence was (exclusively) needed to show that $t_i \leftarrow_{\mathcal{R}}^* \mathsf{tb}(s_i\sigma) \to_{\mathcal{R}}^* \mathsf{tb}(s_i\tau)$ implies $\mathsf{tb}(s_i\tau) \to_{\mathcal{R}}^* t_i$ for conditions $s_i \to^* t_i$ of some conditional rule and certain substitutions τ and σ (cf. the proof of Lemma 3.9). However, for CTRSs with ground conditions this is trivial since $s_i\sigma = s_i\tau$ for all substitutions σ and τ and thus $\mathsf{tb}(s_i\sigma) = \mathsf{tb}(s_i\tau)$. ☐

Of course, systems with only ground conditions are of limited practical use (and could in principle, though not necessarily effectively, be replaced by equivalent unconditional systems).

3.2.4 Normal Form Property

Reconsidering the sufficiency of confluence of $\mathcal{R}$ for soundness (Theorem 3.12), we can get another slightly more general criterion.

Regarding confluence properties, the following proper implications (for TRSs and also for ARSs) are well-known (cf. e.g. [15]):

$$(*) \quad \mathsf{CR} \Longrightarrow \mathsf{NF} \Longrightarrow \mathsf{UN} \Longrightarrow \mathsf{UN}^{\rightarrow}\,.$$

In the proof of Theorem 3.12, what is actually needed, is not full confluence, but only the property

$$(+) \quad t \;{}^{*}\!\!\leftarrow_{\mathcal{R}} s \rightarrow^{*} u \in \mathsf{NF}(\mathcal{R}) \Longrightarrow t \rightarrow^{*} u\,.$$

Proposition 3.18. *Property (+) is equivalent to* NF.

Proof. Straightforward. □

Consequently we can generalize Theorem 3.12 slightly as follows.

Theorem 3.19 (NF is sufficient). *The normal form property (*NF*) of $\mathcal{R}$ is sufficient for soundness of $\mathcal{R}'$.*

Regarding the above proper implications (*) and Theorem 3.19, an obvious question is whether UN or $\mathsf{UN}^{\rightarrow}$, respectively, is sufficient for soundness.

Proposition 3.20 (UN and $\mathsf{UN}^{\rightarrow}$ are not sufficient for soundness). UN *and* $\mathsf{UN}^{\rightarrow}$ *are not sufficient for soundness.*

Proof. Cf. Example 3.21. □

Example 3.21 (Example 3.2 continued). *Consider the system $\widehat{\mathcal{R}}$ obtained from $\mathcal{R}$ as in Example 3.2 by adding the additional unconditional rule $k \rightarrow k$. Then it is easy to verify that $\widehat{\mathcal{R}}$ is not* NF*, but* UN *and* $\mathsf{UN}^{\rightarrow}$*. Moreover, $\widehat{\mathcal{R}}'$ is still unsound w.r.t. $\widehat{\mathcal{R}}$.*

3.2.5 Left-Linearity Revisited

It is well-known that left-linear join (1-)CTRSs can be simulated by left-linear normal (1-)CTRSs extended by an additional rule like $eq(x,x) \rightarrow tt$ (yielding $\mathcal{R}_{eq}$), via encoding join conditions $u_i \downarrow_{\mathcal{R}} v_i$ as $eq(u_i,v_i) \rightarrow^{*}_{\mathcal{R}_{eq}} tt$. Hence, it would be interesting to know whether – regarding left-linearity of $\mathcal{R}$ as sufficient criterion for soundness (Theorem 3.7) – this class could be extended slightly so as to cover also left-linear systems extended by (non-left-linear) "*eq*-like" rules. This is indeed the case as we will show next.

Definition 3.22 (weak left-linearity). *A normal 1-*CTRS *is said to be* weakly left-linear *if every rule $l \rightarrow r \Leftarrow s_1 \rightarrow^{*} t_1, \ldots, s_n \rightarrow^{*} t_n$ of $\mathcal{R}$ is either left-linear or, if not, is unconditional and every non-linear variable in l does not occur at all in r.*[5]

In particular, extending (not necessarily disjointly concerning the signature) a left-linear normal 1-CTRS by $eq(x,x) \rightarrow tt$ yields a weakly left-linear system.

Before proving that weak-left-linearity of $\mathcal{R}'$ is indeed sufficient for soundness of unravelings, we state two observations regarding the preservation of weak left-linearity under unravelings and the existence and uniqueness of one-step ancestors of U-terms in reductions w.r.t. weakly left-linear systems $\mathcal{R}'$.

Observation 3.23. *$\mathcal{R}$ is weakly left-linear iff $\mathcal{R}'$ is so.*

[5]Note that this definition also covers the case of TRSs.

Observation 3.24. *Let $\mathcal{R}$ be a weakly left-linear normal 1-*CTRS*. If $u \to_{p,\mathcal{R}'} v$, then every U-rooted subterm position of v has exactly one one-step ancestor in u.*

Proof. For all normal 1-CTRSs every U-(sub)-term has at least one one-step ancestor (in an $\mathcal{R}'$-reduction), because U-symbols do not occur strictly below the root of *rhs*'s of rules in $\mathcal{R}'$. Weak left-linearity of $\mathcal{R}$ implies weak left-linearity of $\mathcal{R}'$ and thus in every $\mathcal{R}'$-reduction every term has at most one one-step ancestor. □

Observation 3.24 motivates the definition of a function tb_D w.r.t. to a $\mathcal{R}'$-reduction sequence D, starting from an original term, which basically transforms terms from $\mathcal{T}'$ into terms from $\mathcal{T}$. Since we can trace a U-(sub)term uniquely backwards in D (uniqueness is due to Observation 3.24), the idea is that we can find the first (when traced backwards) non-U-rooted ancestor of the U-term (i.e., the one appearing in the term of D with the highest index) and thus replace the U-(sub)term by this ancestor.

Definition 3.25 (tb_D)**.** *Let $\mathcal{R}$ be a weakly left-linear normal 1-*CTRS *and let $D : u_1 \to_{\mathcal{R}'} u_2 \to_{\mathcal{R}'} \ldots, \to_{\mathcal{R}'} u_n$ be a reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. We define the (partial) function $\mathsf{tb}_D \colon \{1, \ldots, n\} \times \mathbb{N}_+^* \to \mathcal{T}$, i.e. from pairs (i, p), where i is an index and p is a position, as*

$$\mathsf{tb}_D(i,p) = \begin{cases} \textit{undefined} & \textit{if } p \notin Pos(u_i) \\ x & \textit{if } u_i|_p = x \in \mathcal{V} \\ f(\mathsf{tb}_D(i,p.1), \ldots, \mathsf{tb}_D(i,p.l)) & \textit{if } u_i|_p = f(t_1,\ldots,t_l) \textit{ and } f \in \mathcal{F} \\ \mathsf{tb}_D(i-1,p') & \textit{if } root(u_i|_p) \in \mathcal{F}' \setminus \mathcal{F},\ i > 1 \textit{ and } u_{i-1}|_{p'} \textit{ is the} \\ & \quad \textit{unique one-step ancestor of } u_i|_p\,. \end{cases}$$

Note that pairs (i, p) are supposed to determine a subterm occurrence at position p in the i^{th} term of D. Hence, tb_D is undefined if the pair does not determine such a term, i.e. if $p \notin Pos(u_i)$.

Example 3.26. *Let $\mathcal{R}$ be as in Example 3.2 and consider the $\mathcal{R}'$-derivation $D \colon u_1 = f(a) \to_{\mathcal{R}'} U(a,a) \to_{\mathcal{R}'} U(a,d) \to_{\mathcal{R}'} U(c,d) = u_4$. Then we have $\mathsf{tb}(U(c,d)) = f(d)$, but $\mathsf{tb}_D(4,\epsilon) = f(a)$ (here, $u_4 = U(c,d)|_\epsilon = U(c,d)$). Note that the backtranslation tb_D goes back further than tb. For instance, we have $\mathsf{tb}_D(U(c,d)) \to^*_{\mathcal{R}'} d$, but $\mathsf{tb}(U(c,d)) \not\to^*_{\mathcal{R}'} d$.*

The following lemma roughly states that whether the tb_D-version of some (sub)term is reachable in $\mathcal{R}$ by the tb_D-version of its ancestor depends only on whether the tb_D-version of the reductum is reachable by the tb_D-version of the redex in the corresponding step.

Lemma 3.27 (monotony property of tb_D)**.** *Let $\mathcal{R}$ be a weakly left-linear normal 1-*CTRS *and let $D \colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \ldots \to_{p_{n-1},\mathcal{R}'} u_n$ be an $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. If $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for every $1 \leq i \leq n-1$, then $\mathsf{tb}_D(i,p) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p')$ for every $1 \leq i \leq n-1$, every $p \in Pos(u_i)$ and every descendant $u_{i+1}|_{p'}$ of $u_i|_p$.*

Proof (sketch). For the interesting case where $p \leq p_i$ we use induction on the size of $\overline{p}$ determined by $p.\overline{p} = p_i$. □

In Lemma 3.28 below we prove a restricted monotony property of tb_D.

Lemma 3.28 (extraction of tb_D in U-rooted terms)**.** *Let $\mathcal{R}$ be a weakly left-linear normal 1-*CTRS *and* $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ *be an $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. If $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for every $1 \leq i \leq n-1$, $u_k|_p = U^\alpha(v_1,\dots,v_{m_1},x_1,\dots,x_{m_2})\tau$, $\alpha = l \to r \Leftarrow s_1 \to^* t_1,\dots,s_{m_1} \to t_{m_1}$ and $\mathsf{tb}_D(k,p) = l\sigma$, then $s_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k,p.i)$ for all $1 \leq i \leq m_1$ and $x_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k,p.(m_1+i))$ for all $1 \leq i \leq m_2$.*

Proof (sketch). Proof by induction on k and using Lemma 3.27. □

The next lemma shows that the backtranslation of tb is intuitively not "as far back" as the one of tb_D by stating that $\mathsf{tb}_D(i,p) \to^*_{\mathcal{R}} \mathsf{tb}(u_i|_p)$ for certain $\mathcal{R}'$-reductions D.

Lemma 3.29 (tb_D to tb)**.** *Let $\mathcal{R}$ be a weakly left-linear normal 1-*CTRS *and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. If $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for every $1 \leq i \leq n-1$, then $\mathsf{tb}_D(j,p) \to^*_{\mathcal{R}} \mathsf{tb}(u_j|_p)$ for all $1 \leq j \leq n$ and all $p \in Pos(u_j)$.*

Proof (sketch). Proof by induction on the term depth of $u_j|_p$ and using Lemma 3.28. □

The following lemma is the technical key result for soundness in the weakly left-linear case. It states that in every $\mathcal{R}'$-reduction D we have $\mathsf{tb}_D(i,p) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p)$, if $u_i|_p$ is the redex contracted in D.

Lemma 3.30 (technical key result for weakly left-linear systems)**.** *Let $\mathcal{R}$ be a weakly left-linear normal 1-*CTRS *and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. Then $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for all $1 \leq i < n$.*

Proof (sketch). Proof by induction on the length of D and case distinction over the applied rule in the last reduction step of D. There are two interesting cases. First, if the rule is an unconditional non-left-linear rule $l \to r$, this rule might not be applicable to $\mathsf{tb}_D(n-1,p_{n-1})$ since $u_{n-1}|_q = u_{n-1}|_{q'} \not\Rightarrow \mathsf{tb}_D(n-1,q) = \mathsf{tb}_D(n-1,q')$. However, by Lemma 3.29 we get $\mathsf{tb}_D(n-1,q) \to^*_{\mathcal{R}} \mathsf{tb}(u_{n-1}|_q)$ and $\mathsf{tb}_D(n-1,q') \to^*_{\mathcal{R}} \mathsf{tb}(u_{n-1}|_{q'})$. Hence, $\mathsf{tb}_D(n-1,p_{n-1}.q) \downarrow_{\mathcal{R}} \mathsf{tb}_D(n-1,p_{n-1}.q')$ for all positions q,q' where $l|_q = l|_{q'} = x \in \mathcal{V}$. Moreover, these reductions do not effect the reductum after the rule is applied, since all non-linear variables are erased due to weak left-linearity of $\mathcal{R}$.

For the second interesting case where the last applied rule is a U-elimination rule we get $s_i\sigma \to^*_{\mathcal{R}} t_i$ according to Lemma 3.28 for every condition $s_i \to^* t_i$ of the conditional rewrite rule corresponding to the eliminated U-symbol, where σ is given by $\mathsf{tb}_D(n-1,p_{n-1}) = l\sigma$. Hence, this implies $\mathsf{tb}_D(n-1,p_{n-1}) \to^*_{\mathcal{R}} \mathsf{tb}_D(n,p_{n-1})$ by again applying Lemma 3.28. □

Weak left-linearity is crucial in Lemma 3.30 to ensure that non-left-linear rules are applicable in the tb_D-versions of redexes.

Example 3.31. *Consider the weakly left-linear normal 1-*CTRS *$\mathcal{R}$ given by*

$$eq(x,x) \to tt \qquad f(x) \to b \Leftarrow x \to^* b$$
$$a \to b$$

and the $\mathcal{R}'$-derivation

$$D\colon eq(f(a),f(b)) \to^+_{\mathcal{R}'} eq(U(a,a),U(b,b)) \to^+_{\mathcal{R}'} eq(b,b) \to_{\mathcal{R}'} tt\,.$$

Let $u_{n-1} = eq(b,b)$, then $\mathsf{tb}_D(n-1,\epsilon) = u_1 = eq(f(a),f(b))$ and $eq(f(a),f(b)) \not\to_{\mathcal{R}} tt$ (i.e., with one single $\mathcal{R}$-step). However, $f(a)$ and $f(b)$ are joinable (in general this is justified by

Lemma 3.29) and reducing them is not problematic as the non-linear variable x is erased whenever the eq-rule is applied (this must in general be the case because of weak left-linearity of $\mathcal{R}$). Hence, we have $\mathsf{tb}_D(n-1,\epsilon) \to^*_{\mathcal{R}} tt = \mathsf{tb}_D(n,\epsilon)$.

The following lemma and theorem state the main soundness result for weakly left-linear normal 1-CTRSs.

Lemma 3.32. *Let $\mathcal{R}$ be a weakly left-linear normal 1-CTRS and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \ldots \to_{p_{n-1},\mathcal{R}'} u_n$ be an $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 \le i \le n$. Then, $u_1 = \mathsf{tb}_D(1,\epsilon) \to^*_{\mathcal{R}} \mathsf{tb}(u_n)$.*

Proof. Lemma 3.30 yields that $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for all $1 \le i < n$. Hence, Lemma 3.27 is applicable and its repeated application yields $\mathsf{tb}_D(1,\epsilon) \to^*_{\mathcal{R}} \mathsf{tb}_D(n,\epsilon)$. Finally, Lemma 3.29 yields $\mathsf{tb}_D(n,\epsilon) \to^*_{\mathcal{R}} \mathsf{tb}(u_n)$. ☐

Theorem 3.33. *Weak left-linearity of $\mathcal{R}$ is sufficient for soundness of $\mathcal{R}'$.*

Proof. Straightforward using Lemma 3.32. ☐

Obviously, Theorem 3.33 properly generalizes Theorem 3.7. Intuitively, the former result and its proof show that non-left-linearity due to "*eq*-like" rules is not problematic, since the effects of applying such a non-left-linear rule are only local (and do not cause complex sharing of equal subterms along longer derivations).

A nice consequence of Theorem 3.33 is that left-linear join 1-CTRSs $\mathcal{R}_j$ can be soundly unraveled (via the unraveling U for the case of normal 1-CTRSs) by first encoding $\mathcal{R}_j$ into a normal 1-CTRS $\mathcal{R}_n$ (in a many-sorted setting, by adding the rule $eq(x,x) \to tt$ to $\mathcal{R}_j$ where $eq\colon s \times s \to bool$ is a fresh binary function symbol of sort *bool* and *tt* a fresh constant of sort *bool*, and all terms $s \in \mathcal{T}$ are considered as s-sorted, with $s \neq bool$, and by representing conditions $u_i \downarrow v_i$ as $eq(u_i,v_i) \to^* tt$) and a subsequent unraveling of $\mathcal{R}_n$ into $\mathcal{R}'_n$.

4 Discussion, Perspectives and Related Work

First let us summarize the results obtained. The table in Figure 1 lists the properties (of $\mathcal{R}$) investigated in the first row, indicates whether they are sufficient for soundness (of $\mathcal{R}'$) in the second row (+ means "Yes", − "No"), and gives references for the results in the last row.

LL	CS	OS	RL	NO	CR	NE	NF	GC	UN	$\mathsf{UN}^{\to}$	WLL
+	−	−	−	−	+	+	+	+	−	−	+
3.7 ([8, 6.12])	3.4	3.4	3.4	3.6	3.12	3.16	3.16	3.17	3.20	3.20	3.33

Figure 1: Sufficiency of conditions for soundness of unravelings (of normal 1-CTRSs)

Due to the carefully designed modular proof structure of the obtained positive results and to the conceptually clear underlying ideas and the corresponding projection approaches (via tb, tf and tb_D) we expect that at least some of the results can be extended to other classes of CTRSs and to other transformations from CTRSs to TRSs. One case, for which this is indeed possible, concerns an alternative *sequential* version of unraveling normal 1-CTRSs. Here, the idea is that the conditions of a conditional rule are not processed simultaneously (by the unraveling), but sequentially, one at a time. This means, given the rule $\delta\colon l \to r \Longleftarrow s_1 \to^* t_1, \ldots, s_n \to t_n$, instead of one introduction rule $l \to U^\delta(s_1,\ldots,s_n,\overrightarrow{Var(l)})$

and one elimination rule $U^\delta(t_1, \ldots, t_n, \overrightarrow{Var(l)}) \to r$ we have one first introduction rule $l \to U_1^\delta(s_1, \overrightarrow{Var(l)})$, $n-2$ further intermediate "switch"-rules $U_i^\delta(t_i, \overrightarrow{Var(l)}) \to U_{i+1}^\delta(s_{i+1}, \overrightarrow{Var(l)})$, $1 \leq i \leq n-1$ (which act as elimination rules for U_i^δ and as introduction rules for U_{i+1}^δ) and a final elimination rule $U_{n-1}^\delta(t_n, \overrightarrow{Var(l)}) \to r$. All results (for U) presented in the paper actually also hold for this *sequential unraveling* U_{seq} as can be shown by a careful inspection and adaptation of the proofs.

The corresponding analysis of U_{seq} for normal 1-CTRSs provides the appropriate basis for dealing with the more general class of *deterministic* (oriented) 3-CTRSs where bindings for extra variables in the conditions and in right-hand side r of $l \to r \Longleftarrow s_1 \to^* t_1, \ldots, s_n \to^* t_n$ are "determined" by sequentially processing the conditions, i.e., $Var(s_i) \subseteq Var(l) \cup \bigcup_{1 \leq j \leq i-1} Var(t_j)$. But the details of this extension still need to be carefully worked out.

There are various open questions in the area. For instance, it remains unclear whether an even better (more precise) characterization of unsoundness exists, in the form of a general characterization result for unsoundness, similar to the one for non-modularity of termination (cf. e.g. [7, Theorem 7]), from which (most) known sufficient criteria for soundness follow.

Regarding related work, as far as we know left-linearity (of $\mathcal{R}$) was the only established sufficient criterion for soundness (of $\mathcal{R}'$), cf. [8, 9], [15, Chapter 7]. Compared to the proofs in these papers, we think that our proof of the more general Theorem 3.33 is in a sense more modular and less operational than these previous ones, and is also better suited for potential extensions.

Regarding more general classes of CTRSs (as compared to normal 1-CTRSs), the only works that we aware of, are [10] and [12]. However, in [10] there is only a claim ([10, Theorem 5.2], without any proof or proof sketch) stating soundness of (sequential) unravelings for *semilinear* DCTRSs, and in [12] the basic unraveling transformation used is a kind of optimized version analogous to U_{opt}, cf. Section 3.1 and Example 3.1, for which we have argued that such an optimization is generally problematic from the point of view of soundness.

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A Missing and Completed Proofs

Lemma 3.8 (monotony property of tb). Let $\mathcal{R} = (\mathcal{F}, R)$ be a 1-CTRS. If $u \rightarrow_{p,\mathcal{R}'} v$ for terms $u, v \in \mathcal{T}'$ and $\mathsf{tb}(u|_p) \rightarrow_{\mathcal{R}}^{\leq 1} \mathsf{tb}(v|_p)$, then $\mathsf{tb}(u|_q) \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \mathsf{tb}(v|_{q'})$ for all $q \in Pos(u)$ and all descendants q' of q in v.

Proof. First assume that $q > p$. For every descendant q' of q we have $u|_q = v|_{q'}$ and thus $\mathsf{tb}(u|_q) = \mathsf{tb}(v|_{q'})$ and also $\mathsf{tb}(u|_q) \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \mathsf{tb}(v|_{q'})$ with an empty parallel reduction step.

Second, if $p \parallel q$, $q = q'$ and $\mathsf{tb}(u|_q) = \mathsf{tb}(v|_q)$ and also $\mathsf{tb}(u|_q) \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \mathsf{tb}(v|_{q'})$ again with an empty parallel reduction step.

Third, assume $q \leq p$. Then $q = q'$ and we prove $\mathsf{tb}(u|_q) \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \mathsf{tb}(v|_q)$ by induction on the length of the position p' determined by $q.p' = p$. If $p' = \epsilon$ we trivially have $\mathsf{tb}(u|_q) \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \mathsf{tb}(v|_q)$ since $q = p$.

Otherwise $p' = i.p''$ ($i \in \mathbb{N}_+$) and

$$\begin{array}{rcl} u|_q & = & f(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n) \\ v|_q & = & f(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_n). \end{array}$$

Moreover, $u_i \rightarrow_{p'',\mathcal{R}'} v_i$ and $\mathsf{tb}(u|_{q.p'}) = \mathsf{tb}(u_i|_{p''}) = \mathsf{tb}(u|_p) \rightarrow_{\mathcal{R}}^{\leq 1} \mathsf{tb}(v|_p) = \mathsf{tb}(v_i|_{p''}) \rightarrow_{\mathcal{R}} \mathsf{tb}(v_i|_{p''}) = \mathsf{tb}(v|_{q.p'})$. Hence, the induction hypothesis yields

$$\mathsf{tb}(u|_{q.i}) = \mathsf{tb}(u_i) \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \mathsf{tb}(v_i) = \mathsf{tb}(v|_{q.i}) \tag{1}$$

It remains to show that $\mathsf{tb}(u|_q) \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} \mathsf{tb}(v|_q)$.

We distinguish two cases depending on whether $f \in \mathcal{F}$ or not. First, if $f \in \mathcal{F}$ we have

$$\begin{array}{rcl} \mathsf{tb}(u|_q) & = & f(\mathsf{tb}(u_1), \ldots, \mathsf{tb}(u_{i-1}), \mathsf{tb}(u_i), \mathsf{tb}(u_{i+1}), \ldots, \mathsf{tb}(u_n)) \\ & \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} & f(\mathsf{tb}(u_1), \ldots, \mathsf{tb}(u_{i-1}), \mathsf{tb}(v_i), \mathsf{tb}(u_{i+1}), \ldots, \mathsf{tb}(u_n)) \;=\; \mathsf{tb}(v|_q) \end{array}$$

by (1) and thus the result holds.

Otherwise, if $f \in \mathcal{F}' \setminus \mathcal{F}$ (say $f = U^\alpha$ where $\alpha = l \rightarrow r \Leftarrow s_1 \rightarrow^* t_1, \ldots s_m \rightarrow^* t_m$) we have $\mathsf{tb}(u|_q) = l\sigma$ where σ is given by $x_j\sigma = \mathsf{tb}(u_{m+j})$ ($j \geq 1$). Analogously, $\mathsf{tb}(v|_q) = l\sigma'$ where σ' is given by $x_j\sigma' = \mathsf{tb}(v_i)$ with $m + j = i$, and by $x_j\sigma' = \mathsf{tb}(u_{m+j})$ otherwise.

If $i \leq m$, then $m + j$ cannot be equal to i and thus, $\sigma = \sigma'$ implying $\mathsf{tb}(u|_q) = l\sigma \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} l\sigma' = \mathsf{tb}(v|_q)$ with an empty parallel step. Otherwise, if $i = m + j$, we get $x_j\sigma \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} x_j\sigma'$ because of (1) and $x_k\sigma = x_k\sigma'$ for $k \neq j$. Hence, $\mathsf{tb}(u|_q) = l\sigma \mathrel{+\!\!\!\!\twoheadrightarrow}_{\mathcal{R}} l\sigma' = \mathsf{tb}(v|_q)$. ☐

Lemma 3.9 (technical key result for confluent systems). Let $\mathcal{R} = (\mathcal{F}, R)$ be a *confluent* normal 1-CTRS and let $D\colon u_1 \rightarrow_{p_1,\mathcal{R}'} u_2 \rightarrow_{p_2,\mathcal{R}'} \ldots \rightarrow_{p_{n-1},\mathcal{R}'} u_n$ be a reduction sequence where $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. Then, $\mathsf{tb}(u_i|_{p_i}) \rightarrow_{\mathcal{R}}^{\leq 1} \mathsf{tb}(u_{i+1}|_{p_i})$ for all $1 \leq i < n$.

Proof. We prove the result by induction on the length of D. If the length is 0 the result holds vacuously. Otherwise, let $n + 1$ be the length of D and $D = D' \rightarrow_{p_{n-1},\mathcal{R}'} u_n$. The induction hypothesis yields $\mathsf{tb}(u_j|_{p_j}) \rightarrow_{\mathcal{R}}^{\leq 1} \mathsf{tb}(u_{j+1}|_{p_j})$ for $1 \leq j < n - 1$. Hence, what is left to prove is that $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) \rightarrow_{\mathcal{R}}^{\leq 1} \mathsf{tb}(u_n|_{p_{n-1}})$.

To this end we make a case distinction depending on which type of rewrite rule of $\mathcal{R}'$ is applied in the last step of D, i.e., $u_{n-1} \rightarrow_{p_{n-1},\mathcal{R}'} u_n$. The three possible types of rules $l \rightarrow r \in \mathcal{R}'$ are

$$U\text{-introduction rules, i.e., } root(l) \in \mathcal{F}, root(r) \in \mathcal{F}' \setminus \mathcal{F} \tag{2}$$

$$U\text{-elimination rules, i.e., } root(l) \in \mathcal{F}' \setminus \mathcal{F}, root(r) \in \mathcal{F} \cup \mathcal{V} \tag{3}$$

$$\text{original unconditional rules, i.e., } root(l) \in \mathcal{F}, root(r) \in \mathcal{F} \cup \mathcal{V} \tag{4}$$

First, if the last step is due to the application of a rewrite rule of type (2), we have $\mathsf{tb}(u_{n-1}|_{p_{n-1}}) = \mathsf{tb}(u_n|_{p_{n-1}})$ by the definition of tb and thus the statement holds.

Second, if the last step is due to the application of a rewrite rule $l \to r \in \mathcal{R}'$ of type (4) (i.e., a rule that occurs as an unconditional rule in $\mathcal{R}$), we have

$$u_{n-1}|_{p_{n-1}} = l\sigma \Rightarrow \mathsf{tb}(u_{n-1}|_{p_{n-1}}) = l\ \mathsf{tb}(\sigma) \to_{\mathcal{R}} r\ \mathsf{tb}(\sigma) = \mathsf{tb}(u_n|_{p_{n-1}})$$

due to $l, r \in \mathcal{T}$ (and the definition of tb). Here, $\mathsf{tb}(\sigma)$ is defined by $x\,(\mathsf{tb}(\sigma)) = \mathsf{tb}(x\sigma)$.

Finally, assume the last step on D is due to the application of a U-elimination rule, i.e. a rule of type (3). Then, $u_{n-1}|_{p_n} = U^\alpha(t_1, \ldots, t_{m_1}, x_1, \ldots, x_{m_2})\sigma$ and $\mathsf{tb}(u_{n-1}|_{p_n}) = l\mathsf{tb}(\sigma)$, where $\alpha = l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_{m_1} \to t_{m_1}$ is the conditional rule corresponding to the symbol U^α and m_2 is the number of distinct variables in the left-hand side of this rule.

Note that in every reduction $u \to^*_{\mathcal{R}'} v$ with $u \in \mathcal{T}$, every U-rooted subterm $v|_p$ of v has at least one ancestor, since no U-symbol occurs strictly below the root in the right-hand side of any reduction rule in $\mathcal{R}'$. Hence, as $u_1 \in \mathcal{T}$, there is an index $j > 1$ and a position p, such that $u_j|_p$ is a U-rooted ancestor of $u_{n-1}|_{p_{n-1}}$, some one-step ancestor of $u_j|_p$ is not U-rooted and every descendant of $u_j|_p$ that is also an ancestor of $u_{n-1}|_{p_{n-1}}$, is U-rooted. This means that the U-symbol that is eliminated in the last step of D is introduced in the step $u_{j-1} \to_{p_{j-1},\mathcal{R}'} u_j$ of D and that $p = p_{j-1}$.

Let $u_j|_p = U^\alpha(s_1, \ldots, s_{m_1}, x_1, \ldots, x_{m_2})\tau$. Since every descendant of $u_j|_p$ that is also an ancestor of $u_{n-1}|_{p_{n-1}}$ is U-rooted, every immediate subterm $u_{n-1}|_{p_{n-1}.i}$ of $u_{n-1}|_{p_{n-1}}$ is a descendant of $u_j|_{p.i}$ ($1 \leq i \leq m_1 + m_2$). Hence, the induction hypothesis and repeated application of Lemma 3.8 yield

$$\begin{aligned} \mathsf{tb}(s_i\tau) = s_i\ \mathsf{tb}(\tau) &\to^*_{\mathcal{R}} t_i\ \mathsf{tb}(\sigma) = \mathsf{tb}(t_i\sigma) \text{ for } 1 \leq i \leq m_1 \\ \mathsf{tb}(x_i\tau) &\to^*_{\mathcal{R}} \mathsf{tb}(x_i\sigma) \text{ for } 1 \leq i \leq m_2 \end{aligned}$$

Since t_i is a ground normal form for all $1 \leq i \leq m_1$ (because $\mathcal{R}$ is a normal 1-CTRS) we thus get a divergence in $\mathcal{R}$

$$t_i \leftarrow^*_{\mathcal{R}} \mathsf{tb}(s_i\tau) \to^*_{\mathcal{R}} tb(s_i\sigma) \text{ for } 1 \leq i \leq m_1.$$

Hence, confluence of $\mathcal{R}$ and the fact that t_i is a normal form for all $1 \leq i \leq m_1$ yield

$$\mathsf{tb}(s_i\sigma) = s_i\ \mathsf{tb}(\sigma) \to^*_{\mathcal{R}} t_i \text{ for } 1 \leq i \leq m_1. \tag{5}$$

Now we have

$$\mathsf{tb}(u_{n-1}|_{p_{n-1}}) = l\ \mathsf{tb}(\sigma) \to_{\mathcal{R}} r\ \mathsf{tb}(\sigma) = \mathsf{tb}(u_n|_{p_{n-1}})$$

with rule α which is applicable because of (5). □

Lemma 3.13 (monotony property of tf). Let $\mathcal{R} = (\mathcal{F}, R)$ be a 1-CTRS. If $u \to_{p,\mathcal{R}'} v$ for $u, v \in \mathcal{T}'$ and $\mathsf{tf}(u|_p) \to^{\leq 1}_{\mathcal{R}} \mathsf{tf}(v|_p)$, then $\mathsf{tf}(u|_q) \mathrel{+\!\!\!\!\!\!\Rightarrow}_{\mathcal{R}} \mathsf{tf}(v|_{q'})$ for all $q \in Pos(u)$ and all descendants q' of q in v.

Proof. First assume that $q > p$. For every descendant q' of q we have $u|_q = v|_{q'}$ and thus $\mathsf{tf}(u|_q) = \mathsf{tf}(v|_{q'})$ and also $\mathsf{tf}(u|_q) \mathrel{+\!\!\!\!\!\!\Rightarrow}_{\mathcal{R}} \mathsf{tf}(v|_{q'})$ with an empty parallel reduction step.

Second, if $p \parallel q$, $q = q'$ and $\mathsf{tf}(u|_q) = \mathsf{tf}(v|_q)$ and also $\mathsf{tf}(u|_q) \mathrel{+\!\!\!\!\!\!\Rightarrow}_{\mathcal{R}} \mathsf{tf}(v|_{q'})$ again with an empty parallel reduction step.

Third, assume $q \leq p$. Then $q = q'$ and we prove $\mathsf{tf}(u|_q) \mathrel{+\!\!\!\!\!\!\Rightarrow}_{\mathcal{R}} \mathsf{tf}(v|_q)$ by induction on the length of the position p' determined by $q.p' = p$. If $p' = \epsilon$ we trivially have $\mathsf{tf}(u|_q) \mathrel{+\!\!\!\!\!\!\Rightarrow}_{\mathcal{R}} \mathsf{tf}(v|_q)$ since $q = p$.

Otherwise $p' = i.p''$ ($i \in \mathbb{N}_+$) and

$$\begin{array}{rcl} u|_q & = & f(u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_n) \\ v|_q & = & f(u_1, \dots, u_{i-1}, v_i, u_{i+1}, \dots, u_n). \end{array}$$

Moreover, $u_i \to_{p'',\mathcal{R}'} v_i$ and $\mathsf{tf}(u|_{q.p'}) = \mathsf{tf}(u_i|_{p''}) \to_{\mathcal{R}} \mathsf{tf}(v_i|_{p''}) = \mathsf{tf}(v|_{q.p'})$. Hence, the induction hypothesis yields

$$\mathsf{tf}(u|_{q.i}) = \mathsf{tf}(u_i) \mathrel{+\!\!\!\!\!\!\Vdash}_{\mathcal{R}} \mathsf{tf}(v_i) = \mathsf{tf}(v|_{q.i}) \tag{6}$$

It remains to show that $\mathsf{tf}(u|_q) \mathrel{+\!\!\!\!\!\!\Vdash}_{\mathcal{R}} \mathsf{tf}(v|_q)$.

We distinguish two cases depending on whether $f \in \mathcal{F}$ or not. First, if $f \in \mathcal{F}$ we have

$$\begin{array}{rcl} \mathsf{tf}(u|_q) & = & f(\mathsf{tf}(u_1), \dots, \mathsf{tf}(u_{i-1}), \mathsf{tf}(u_i), \mathsf{tf}(u_{i+1}), \dots, \mathsf{tf}(u_n)) \\ & \mathrel{+\!\!\!\!\!\!\Vdash}_{\mathcal{R}} & f(\mathsf{tf}(u_1), \dots, \mathsf{tf}(u_{i-1}), \mathsf{tf}(v_i), \mathsf{tf}(u_{i+1}), \dots, \mathsf{tf}(u_n)) \quad = \quad \mathsf{tf}(v|_q) \end{array}$$

by (6) and thus the result holds.

Otherwise, if $f \in \mathcal{F}' \setminus \mathcal{F}$ (say $f = U^\alpha$ where $\alpha = l \to r \Leftarrow s_1 \to^* t_1, \dots s_m \to^* t_m$) we have $\mathsf{tf}(u|_q) = r\sigma$ where σ is given by $x_j\sigma = \mathsf{tb}(u_{m+j})$ where m is the number of conditions of the conditional rule α ($j \geq 1$). Analogously, $\mathsf{tb}(v|_q) = r\sigma'$ where σ' is given by $x_j\sigma' = \mathsf{tf}(v_i)$ with $m + j = i$, and by $x_j\sigma' = \mathsf{tf}(u_{m+j})$ otherwise.

If $i \leq m$, then $m + j$ cannot be equal to i and thus, $\sigma = \sigma'$ implying $\mathsf{tf}(u|_q) = r\sigma \mathrel{+\!\!\!\!\!\!\Vdash}_{\mathcal{R}} r\sigma' = \mathsf{tf}(v|_q)$ with an empty parallel step. Otherwise, if $i = m + j$, we get $x_j\sigma \mathrel{+\!\!\!\!\!\!\Vdash}_{\mathcal{R}} x_j\sigma'$ because of (6) and $x_k\sigma = x_k\sigma'$ for $k \neq j$. Hence, $\mathsf{tf}(u|_q) = r\sigma \mathrel{+\!\!\!\!\!\!\Vdash}_{\mathcal{R}} r\sigma' = \mathsf{tf}(v|_q)$. □

Lemma 3.14 (technical key result for non-erasing systems). Let $\mathcal{R} = (\mathcal{F}, R)$ be a *non-erasing* normal 1-CTRS and let $D : u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ be a reduction sequence where $u_n \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 \leq i < n$. Then, $\mathsf{tf}(u_i|_{p_i}) \to_{\mathcal{R}}^{\leq 1} \mathsf{tf}(u_{i+1}|_{p_i})$ for $1 \leq i < n$.

Proof. We prove the result by induction on the length of D. If the length is 0 the result holds vacuously. Otherwise, let $n + 1$ be the length of D and $D = u_1 \to_{p_1,\mathcal{R}'} D'$. The induction hypothesis yields $\mathsf{tf}(u_j|_{p_j}) \to_{\mathcal{R}}^* \mathsf{tf}(u_{j+1}|_{p_j})$ for $2 \leq j < n$. Hence, what is left to prove is that $\mathsf{tf}(u_1|_{p_1}) \to_{\mathcal{R}}^* \mathsf{tf}(u_2|_{p_1})$.

To this end we make a case distinction depending on which type of rewrite rule of $\mathcal{R}'$ has been applied in the first step of D, i.e., $u_1 \to_{p_1,\mathcal{R}'} u_2$. First, if the first step is due to the application of a rewrite rule of type (3) (i.e., a U-elimination rule), we have $\mathsf{tf}(u_1|_{p_1}) = \mathsf{tf}(u_2|_{p_1})$ and thus the statement holds.

Second, if the first step is due to the application of a rewrite rule $l \to r \in \mathcal{R}'$ of type (4) (i.e., a rule that occurs as an unconditional rule in $\mathcal{R}$), we have

$$u_1|_{p_1} = l\sigma \Rightarrow \mathsf{tf}(u_1|_{p_1}) = l\ \mathsf{tf}(\sigma) \to_{\mathcal{R}} r\ \mathsf{tf}(\sigma) = \mathsf{tf}(u_2|_{p_1})$$

due to $l, r \in \mathcal{T}$.

Finally, assume the first step on D is due to the application of a U-introduction rule, i.e. a rule of type (2). Then, $u_2|_{p_1} = U^\alpha(s_1, \dots, s_{m1}, x_1, \dots, x_{m_2})\sigma$ and $\mathsf{tf}(u_2|_{p_1}) = r\ \mathsf{tf}(\sigma)$, where $\alpha = l \to r \Leftarrow s_1 \to^* t_1, \dots, s_{m_1} \to^* t_{m_1}$ is the conditional rule corresponding to the symbol U^α and m_2 is the number of distinct variables in the left-hand side of this rule.

Since $\mathcal{R}$ and thus also $\mathcal{R}'$ are non-erasing, on every $\mathcal{R}'$-reduction $u \to_{\mathcal{R}'} v$ every U-rooted subterm $u|_p$ of u has at least one one-step descendant. Moreover, since $u_n \in \mathcal{T}$, there exist an index j and a position p such that $u_j|_p$ is a U-rooted descendant of $u_2|_{p_1}$, $u_j|_p$ has a one-step descendant (in D) which is not U-rooted and every descendant of $u_2|_{p_1}$ that is also an ancestor of $u_j|_p$ is U-rooted as well. This means that the U-symbol introduced in u_2 is eliminated in the step $u_j \to_{p_j,\mathcal{R}'} u_{j+1}$ of D and that $p = p_j$ (note that $j \neq n$, since $u_j \notin \mathcal{T}$).

Let $u_j|_p = U^\alpha(t_1, \ldots, t_{m_1}, x_1, \ldots, x_{m_2})\tau$. Since every descendant of $u_2|_{p_1}$ that is also an ancestor of $u_j|_p$ is U-rooted, every immediate subterm $u_j|_{p.i}$ of $u_j|_p$ is a descendant of $u_2|_{p_1.i}$ ($1 \leq i \leq m_1 + m_2$). Hence, the induction hypothesis and repeated application of Lemma 3.13 yield

$$\mathsf{tf}(s_i\sigma) \nrightarrow\!\!\!\!\!\!\nrightarrow_{\mathcal{R}} \mathsf{tf}(t_i\tau) \text{ for } 1 \leq i \leq m_1$$
$$\mathsf{tf}(x_i\sigma) \nrightarrow\!\!\!\!\!\!\nrightarrow_{\mathcal{R}} \mathsf{tf}(x_i\tau) \text{ for } 1 \leq i \leq m_2$$

Since $\mathcal{R}$ is a normal 1-CTRS we have $\mathsf{tf}(t_i\tau) = t_i$ for all $1 \leq i \leq m_1$. Hence,

$$\mathsf{tf}(s_i\sigma) = s_i\ \mathsf{tf}(\sigma) \nrightarrow\!\!\!\!\!\!\nrightarrow_{\mathcal{R}} t_i \text{ for } 1 \leq i \leq m_1\,.$$

Thus the conditional rule α is applicable to $u_1|_{p_1}$ yielding

$$u_1|_{p_1} = l\sigma \Rightarrow \mathsf{tf}(u_1|_{p_1}) = l\ \mathsf{tf}(\sigma) \to_{\mathcal{R}} r\ \mathsf{tf}(\sigma) = \mathsf{tf}(u_2|_{p_1})$$

□

Lemma 3.27 (monotony property of tb_D). Let $\mathcal{R}$ be a weakly left-linear normal 1-CTRS and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \ldots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. If $\mathsf{tb}_D(i, p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1, p_i)$ for every $1 \leq i \leq n-1$, then $\mathsf{tb}_D(i, p) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1, p')$ for every $1 \leq i \leq n-1$, every $p \in Pos(u_i)$ and every descendant $u_{i+1}|_{p'}$ of $u_i|_p$.

Proof. Consider some pair (k, p) such that $p \in Pos(u_k)$. First, if $p > p_k$ or $p \parallel p_k$ and $u_{k+1}|_{p'}$ is a descendant of $u_k|_p$, then $u_k|_p = u_{k+1}|_{p'}$ and $\mathsf{tb}_D(k, p) = u_k|_p[\mathsf{tb}_D(k, p.q_1), \ldots, \mathsf{tb}_D(k, p.q_m)]_{q_1,\ldots,q_m}$ where $q_1, \ldots, q_m$ are the positions of the maximal U-rooted subterms of $u|_k$. Analogously, $\mathsf{tb}_D(k+1, p') = u_{k+1}|_{p'}[\mathsf{tb}_D(k+1, p'.q_1), \ldots, \mathsf{tb}_D(k+1, p'.q_m)]_{q_1,\ldots,q_m}$. The unique ancestor of every U-rooted subterm $u_{k+1}|_{p'.q_j}$ ($1 \leq j \leq m$) is $u_k|_{p.q_j}$ hence $\mathsf{tb}_D(k+1, p'.q_j) = \mathsf{tb}_D(k, p.q_j)$ according to Definition 3.25 and since $u_k|_p = u_{k+1}|_{p'}$ we get $\mathsf{tb}_D(k, p) = u_k|_p[\mathsf{tb}_D(k, p.q_1), \ldots, \mathsf{tb}_D(k, p.q_m)]_{q_1,\ldots,q_m} = u_{k+1}|_{p'}[\mathsf{tb}_D(k{+}1, p'.q_1), \ldots, \mathsf{tb}_D(k{+}1, p'.q_m)]_{q_1,\ldots,q_m} = \mathsf{tb}_D(k+1, p')$.

Second, assume $p \leq p_k$. Note that in this case $p = p'$. We prove the result by induction on the length of the position $\overline{p}$ determined by $p.\overline{p} = p_k$. If $\overline{p} = \epsilon$, we have $p = p_k$ and the result holds trivially since $\mathsf{tb}_D(k, p_k) \to^*_{\mathcal{R}} \mathsf{tb}_D(k+1, p_k)$ is a precondition.

Otherwise $\overline{p} = i.\tilde{p}$ ($i \in \mathbb{N}_+$) and

$$\begin{array}{rcl} u_k|_p &=& f(u_k^1, \ldots, u_k^{i-1}, u_k^i, u_k^{i+1}, \ldots, u_k^{ar(f)}) \\ u_{k+1}|_p &=& f(u_k^1, \ldots, u_k^{i-1}, u_{k+1}^i, u_k^{i+1}, \ldots, u_k^{ar(f)}). \end{array}$$

The induction hypothesis yields

$$\mathsf{tb}(k, p.i) \to^*_{\mathcal{R}} \mathsf{tb}(k+1, p.i) \tag{7}$$

because $p.i.\tilde{p} = p_k$ and $|\tilde{p}| < |\overline{p}|$.

It is left to show that $\mathsf{tb}(k, p) \to^*_{\mathcal{R}} \mathsf{tb}(k+1, p)$. We distinguish two cases depending on whether $f \in \mathcal{F}$ or not. First, if $f \in \mathcal{F}$ we have

$$\begin{array}{rcl} \mathsf{tb}_D(k,p) &=& f(\mathsf{tb}_D(k,p.1), \ldots, \mathsf{tb}_D(k,p.(i-1)), \mathsf{tb}_D(k,p.i), \mathsf{tb}_D(k,p.(i+1)), \ldots, \mathsf{tb}_D(k,p.ar(f))) \\ &\to^*_{\mathcal{R}}& f(\mathsf{tb}_D(k,p.1), \ldots, \mathsf{tb}_D(k,p.(i-1)), \mathsf{tb}_D(k+1,p.i), \mathsf{tb}(k,p.(i+1)), \ldots, \mathsf{tb}(k,p.ar(f))) \\ &=& \mathsf{tb}(k+1,p) \end{array}$$

by (7) and thus the result holds.

Otherwise if $f \in \mathcal{F}' \setminus \mathcal{F}$ we have that $\mathsf{tb}_D(k+1, p) = \mathsf{tb}_D(k, p)$ by Definition 3.25. □

Lemma 3.28 (extraction of tb_D in U-rooted terms). Let $\mathcal{R}$ be a weakly left-linear normal 1-CTRS and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. If $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for every $1 \leq i \leq n-1$, $u_k|_p = U^\alpha(v_1,\dots,v_{m_1},x_1,\dots,x_{m2})\tau$, $\alpha = l \to r \Leftarrow s_1 \to^* t_1,\dots,s_{m_1} \to t_{m_1}$ and $\mathsf{tb}_D(k,p) = l\sigma$, then $s_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k,p.i)$ for all $1 \leq i \leq m_1$ and $x_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k,p.(m_1+i))$ for all $1 \leq i \leq m_2$.

Proof. We prove the result by induction on k. If k is 1 the result holds vacuously, since $u_1 \in \mathcal{T}$. For the case that $k \neq 1$ we distinguish two cases depending on the reduction step $u_{k-1} \to_{p_{k-1},\mathcal{R}'} u_k$. If the (unique, cf. Observation 3.24) ancestor $u_{k-1}|_{p'}$ of $u_k|_p$ is U-rooted, then $\mathsf{tb}_D(k,p) = \mathsf{tb}_D(k-1,p') = l\sigma$ according to Definition 3.25. The induction hypothesis yields $s_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k-1,p.i)$ for all $1 \leq i \leq m_1$ and $x_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k-1,m_1+i)$ for all $1 \leq i \leq m_2$. Moreover, $u_k|_{p.i}$ is a one-step descendant of $u_{k-1}|_{p'.i}$ for all $1 \leq i \leq m_1+m_2$ because $root(u_k|_p) = root(u_{k-1}|_{p'}) \in \mathcal{F}' \setminus \mathcal{F}$. Hence, Lemma 3.27 yields $\mathsf{tb}_D(k-1,p'.i) \to^*_{\mathcal{R}} \mathsf{tb}_D(k,p.i)$ for all $1 \leq i \leq m_1+m_2$. Thus, we get

$$s_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k-1,p'.i) \to^*_{\mathcal{R}} \mathsf{tb}_D(k,p.i) \text{ for } 1 \leq i \leq m_1$$
$$x_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(k-1,p'.(m_1+i)) \to^*_{\mathcal{R}} \mathsf{tb}_D(k,p.(m_1+i)) \text{ for } 1 \leq i \leq m_2$$

On the other hand if the unique ancestor $u_{k-1}|_{p'}$ of $u_k|_p$ is not U-rooted, then $p' = p = p_{k-1}$, $\mathsf{tb}_D(k,p.(m_1+i)) = x_i\sigma$ for all $1 \leq i \leq m_2$ and $s_i\sigma = \mathsf{tb}_D(k,p.i)$ for all $1 \leq i \leq m_1$. ☐

Lemma 3.29 (tb_D to tb). Let $\mathcal{R}$ be a weakly left-linear normal 1-CTRS and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$. If $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for every $1 \leq i \leq n-1$, then $\mathsf{tb}_D(j,p) \to^*_{\mathcal{R}} \mathsf{tb}(u_j|_p)$ for all $1 \leq j \leq n$ and all $p \in Pos(u_j)$.

Proof. We prove the result by induction on the term depth of $u_j|_p$. If $u_j|_p$ is a constant (i.e. from $\mathcal{F}$ since there are no U-constants) or a variable, then $\mathsf{tb}_D(j,p) = \mathsf{tb}(u_j|_p) = u_j|_p$ and thus the statement holds.

Otherwise, $u_j|_p = f(u_j|_{p.1},\dots,u_j|_{p.ar(f)})$ and we distinguish two cases depending on whether $f \in \mathcal{F}$ or not. First, if $f \in \mathcal{F}$, we get $\mathsf{tb}_D(j,p) = f(\mathsf{tb}_D(j,p.1),\dots,\mathsf{tb}_D(j,p.ar(f)))$ and the induction hypothesis yields $\mathsf{tb}_D(j,p.i) \to^*_{\mathcal{R}} \mathsf{tb}(u_j|_{p.i})$ for all $1 \leq i \leq ar(f)$. Hence, $\mathsf{tb}_D(j,p) = f(\mathsf{tb}_D(j,p.1),\dots,\mathsf{tb}_D(j,p.k)) \to^*_{\mathcal{R}} f(\mathsf{tb}(u_j|_{p.1}),\dots,\mathsf{tb}(u_j|_{p.k})) = \mathsf{tb}(u_j|_p)$ where $k = ar(f)$.

Second assume $f \in \mathcal{F}' \setminus \mathcal{F}$. Let $u_j|_p = U(v_1,\dots,v_{m_1},x_1,\dots,x_{m_2})\sigma$, then $\mathsf{tb}_D(j,p) = l\sigma$ and we have $x_i\sigma \to^*_{\mathcal{R}} \mathsf{tb}_D(j,p.(m_1+i))$ for all $1 \leq i \leq m_2$ because of Lemma 3.28. Our induction hypothesis yields $\mathsf{tb}_D(j,p.(m_1+i)) \to^*_{\mathcal{R}} \mathsf{tb}(u_j|_{p.(m_1+i)})$ for all $1 \leq i \leq m_2$. Hence we have

$$\mathsf{tb}_D(j,p) = l\sigma \to^*_{\mathcal{R}} l\,\mathsf{tb}(\sigma) = \mathsf{tb}(l\sigma) = \mathsf{tb}(u_j|_p)$$

and thus the statement holds. ☐

Lemma 3.30 (technical key result for weakly left-linear systems). Let $\mathcal{R}$ be a weakly left-linear normal 1-CTRS and let $D\colon u_1 \to_{p_1,\mathcal{R}'} u_2 \to_{p_2,\mathcal{R}'} \dots \to_{p_{n-1},\mathcal{R}'} u_n$ be a $\mathcal{R}'$-reduction sequence with $u_1 \in \mathcal{T}$ and $u_i \in \mathcal{T}'$ for $1 < i \leq n$, then $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for all $1 \leq i < n$.

Proof. We prove the result by induction on n. If $n = 1$ the result holds vacuously. Otherwise, we write D as $D' \to_{p_{n-1},\mathcal{R}'} u_n$ and the induction hypothesis yields $\mathsf{tb}_D(i,p_i) \to^*_{\mathcal{R}} \mathsf{tb}_D(i+1,p_i)$ for all $1 \leq i < n-1$. We distinguish three cases depending on the rule $l \to r$, that has been applied in the last reduction step of D, namely $u_{n-1} \to_{p_{n-1},\mathcal{R}'} u_n$.

First, if this rule was a U-introduction rule, then we have $\mathsf{tb}_D(n-1, p_{n-1}) = \mathsf{tb}_D(n, p_{n-1})$ according to Definition 3.25.

Second, assume the applied rule occurs as unconditional rule in $\mathcal{R}$. If l is linear we have

$$\mathsf{tb}_D(n-1, p_{n-1}) = l\sigma \to_{\mathcal{R}'} r\sigma = \mathsf{tb}_D(n, p_{n-1})$$

Here, $r\sigma = \mathsf{tb}_D(n, p_{n-1})$ is due to the fact that all maximal U-rooted subterms in σ have their one-step ancestors in $l\sigma$.

Otherwise, if l is not linear $\mathsf{tb}_D(n-1, p_{n-1})$ is not necessarily $l\sigma$ since $u_{n-1}|_q = u_{n-1}|_{q'}$ does not imply $\mathsf{tb}_D(n-1, q) = \mathsf{tb}_D(n-1, q')$ in general for positions $q, q' \in Pos(u_{n-1})$. However, since the induction hypothesis yields $\mathsf{tb}_D(j, p_j) \to^*_{\mathcal{R}'} \mathsf{tb}_D(j+1, p_j)$ for all $1 \leq j < n-1$, Lemma 3.29 is applicable yielding $\mathsf{tb}_D(n-1, q) \to^*_{\mathcal{R}} \mathsf{tb}(u_{n-1}|_q)$ for all $q \in Pos(u_{n-1})$. Now let $ren(l)$ be the linearized version of l, then

$$\mathsf{tb}_D(n-1, p_{n-1}) = ren(l)\sigma.$$

Let $\{q^x_1, ..., q^x_{k_x}\}$ be the set of all positions in $VPos(l)$ such that $l|_{q^x_1} = \cdots l|_{q^x_{k_x}} = x$ and $k_x > 1$. Then we have $ren(l)|_{q^x_i}\sigma = \mathsf{tb}_D(n-1, p_{n-1}.q^x_i) \to^*_{\mathcal{R}} \mathsf{tb}(u_{n-1}|_{p_{n-1}.q^x_i})$ for all $1 \leq i \leq k_x$ according to Lemma 3.29 and also $\mathsf{tb}(u_{n-1}|_{p_{n-1}.q^x_i}) = \mathsf{tb}(u_{n-1}|_{p_{n-1}.q^x_j}) = x\sigma'$ for all $1 \leq i, j \leq k_x$ since tb is a function on terms. Thus, we have

$$\mathsf{tb}_D(n-1, p_{n-1}) = ren(l)\sigma \to^*_{\mathcal{R}} l\sigma'$$

where $y\sigma' = y\sigma$ for all variables y that occur only once in l because of weak left-linearity of $\mathcal{R}'$ (cf. Observation 3.23). Hence, we finally get

$$\mathsf{tb}_D(n-1, p_{n-1}) \to^*_{\mathcal{R}} l\sigma' \to^*_{\mathcal{R}} r\sigma' = \mathsf{tb}_D(n, p_{n-1})$$

Here, $r\sigma' = \mathsf{tb}_D(n, p_{n-1})$ is due to the fact that all maximal U-rooted subterms in σ' have their one-step ancestors in $l\sigma'$ (observe that $r\sigma' = r\sigma$).

Third, assume the rule applied in the last step of D is a U-elimination rule $\alpha\colon l \to r \Leftarrow s_1 \to^* t_1, \ldots, s_{m_1}, \ldots, t_{m_1}$, i.e., $u_{n-1}|_{p_{n-1}} = U(t_1, \ldots, t_{m_1}, x_1, \ldots, x_{m_2})\sigma$ and $\mathsf{tb}_D(n-1, p_{n-1}) = l\tau$. According to Lemma 3.28 we have $s_i\tau \to^*_{\mathcal{R}} \mathsf{tb}_D(n-1, p_{n-1}.i)$ for all $1 \leq i \leq m_1$ and $\mathsf{tb}_D(n-1, p_{n-1}.i) = t_i$, since $u_{n-1}|_{p_{n-1}.i} = t_i$ and $\mathcal{R}$ is a normal 1-CTRS $t_i \in \mathcal{T}$.

Hence the conditional rule α is applicable to $l\tau$ and we get

$$\mathsf{tb}_D(n-1, p_{n-1}) = l\tau \to_{\mathcal{R}} r\tau \to^*_{\mathcal{R}} \mathsf{tb}_D(n, p_{n-1})$$

Note that we have $r\tau \to^*_{\mathcal{R}} \mathsf{tb}_D(n, p_{n-1})$ since $x_i\tau \to_{\mathcal{R}} \mathsf{tb}_D(n-1, p_{n-1}.(m_1 + i))$ for all $1 \leq i \leq m_2$ according to Lemma 3.28. $\square$