On Operational Termination of Deterministic Conditional Rewrite Systems*

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Abstract. We characterize the practically important notion of operational termination of deterministic conditional term rewriting systems (DCTRSs) by context-sensitive termination of a transformed TRS on original terms. Experimental evaluations show that this new approach yields more power when verifying operational termination than existing ones. Moreover, it allows us to disprove operational termination of DCTRSs.

1 Introduction and Overview

Conditional term rewriting systems (CTRSs) are a natural extension of unconditional such systems (TRSs) allowing rules to be guarded by conditions. Conditional rules tend to be very intuitive and easy to formulate and are therefore used in several rule based programming and specification languages, such as Maude or ELAN.

Here we focus on the particularly interesting class of *deterministic* (oriented) CTRSs (DCTRSs) which allows for extra variables in conditions (corresponding to *let-constructs* or *where-clauses* in other functional-(logic) languages) and has been used for instance in proofs of termination of (well-moded) logic programs [4].

When analyzing the termination behaviour of conditional TRSs, it turns out that the proof-theoretic notion of *operational termination* is more adequate than ordinary termination in the sense that practical evaluations w.r.t. operationally terminating DCTRSs always terminate (which is indeed not true for other similar notions like *effective termination* [5]).

We propose the notion of context-sensitive quasi-reductivity ([6]), that will be proved to be equivalent to operational termination of DCTRSs. Furthermore, we use a simple modification of *unraveling* transformations ([7], [9]) that allows us to completely characterize the new property of context-sensitive quasi-reductivity of a DCTRS by means of termination of a context-sensitive (unconditional) TRS *on original terms*.

In the following, we assume familiarity with the basic concepts and notations of (context-free, context-sensitive and conditional) term rewriting (cf. e.g. [2], [6], [9]). In this work we are exclusively concerned with deterministic conditional term rewrite systems (DCTRSs).

2 Proving operational termination of DCTRSs via context-sensitive quasi-reductivity

The main goal of this work is to provide methods for proving *operational termination* of DCTRSs. For that purpose we will now introduce the concept of *context-sensitive quasi-reductivity* which is equivalent to *operational termination* while being practically easier to verify for given systems.

Definition 1. *A DCTRS $\mathcal{R}$ ($\mathcal{R} = (\Sigma, R)$) is called* context-sensitively quasi-reductive *(*cs-quasi-reductive*) if there is an extension of the signature Σ' ($\Sigma' \supseteq \Sigma$), a replacement*

* Preliminary results of our approach have been presented at WST'07 (Paris, France, 2007).

μ (s.t. $\mu(f) = \{1, ..., ar(f)\}$ for all $f \in \Sigma$) and a μ-monotonic, well-founded partial order $\succ_\mu$ on $\mathcal{T}(\Sigma', V)$ satisfying for every rule $l \to r \Leftarrow s_1 \to^* t_1, ..., s_n \to^* t_n$, every $\sigma: V \to \mathcal{T}(\Sigma, V)$ and every $i \in \{0, ..., n-1\}$:[1]

- *If $\sigma s_j \succeq_\mu \sigma t_j$ for every $1 \leq j \leq i$ then $\sigma s_{i+1} \prec^{st}_\mu \sigma l$.*
- *If $\sigma s_j \succeq_\mu \sigma t_j$ for every $1 \leq j \leq n$ then $\sigma r \prec_\mu \sigma l$.*

The ordering $\prec^{st}_\mu$ is defined as $(\prec_\mu \cup \lhd_\mu)^+$ where $t \lhd_\mu s$ if and only if s is a proper subterm of t at some position $p \in Pos^\mu(t)$.

Now, we define a transformation from DCTRSs into CSRSs, such that μ-termination of the transformed CSRS implies cs-quasi-reductivity of the original DCTRS. The transformation is actually a variant of the one in [9].

Definition 2. **[9]** *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS. For every rule $\alpha : l \to r \Leftarrow s_1 \to^* t_1, ..., s_n \to^* t_n$ we use n new functions symbols U^α_i ($i \in \{1, ..., n\}$). Then α is transformed into a set of unconditional rules in the following way:*

$$\begin{aligned} l &\to U^\alpha_1(s_1, Var(l)) \\ U^\alpha_1(t_1, Var(l)) &\to U^\alpha_2(s_2, Var(l), \mathcal{E}Var(t_1)) \\ &\vdots \\ U^\alpha_n(t_n, Var(l), \mathcal{E}Var(t_1), ..., \mathcal{E}Var(t_{n-1})) &\to r \end{aligned}$$

Here $Var(s)$ denotes the sequence of variables in a term s rather than the set. The set $\mathcal{E}Var(t_i)$ is $Var(t_i) \setminus (Var(l) \cup \bigcup_{j=1}^{i-1} Var(t_j))$. Again, abusing notation, by $\mathcal{E}Var(t_i)$ we mean an arbitrary but fixed sequence of the variables in the set $\mathcal{E}Var(t_i)$. Any unconditional rule of $\mathcal{R}$ is transformed into itself. The transformed system $U_{cs}(\mathcal{R}) = ((U(\Sigma), U(R), \mu)$ is obtained by transforming each rule of $\mathcal{R}$ where $U(\Sigma)$ is Σ extended by all new function symbols. Furthermore, the replacement map μ is given by $\mu(f) = \{1\}$ if $f \in U(\Sigma) \setminus \Sigma$ and $\mu(f) = \{1, \ldots, ar(f)\}$ otherwise.

Apart from analyzing operational termination, this transformation can also be used to exactly simulate conditional derivations.

Theorem 1. *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS and $U_{cs}(\mathcal{R})$ its transformed CSRS. For every $s, t \in \mathcal{T}(\Sigma, V)$ we have $s \to^+_{\mathcal{R}} t$ if and only if $s \to^+_{U_{cs}(\mathcal{R})} t$.*

Unfortunately, and interestingly, cs-quasi-reductivity of a DCTRS $\mathcal{R}$ does not imply $\mu_{U_{cs}(\mathcal{R})}$-termination of $U_{cs}(\mathcal{R})$, cf. [9, Ex. 7.2.51]. However, it implies $\mu_{U_{cs}(\mathcal{R})}$-termination of $U_{cs}(\mathcal{R})$ *on original terms* (i.e., terms over the original signature of $\mathcal{R}$), thus allowing us to characterize cs-quasi-reductivity.

Theorem 2. *Let $\mathcal{R} = (\Sigma, R)$ be a DCTRS. The following properties of $\mathcal{R}$ are equivalent: $\mu_{U_{cs}(\mathcal{R})}$-termination of $U_{cs}(\mathcal{R})$ on $\mathcal{T}(\Sigma, V)$, cs-quasi-reductivity and operational termination.*

From a practical point of view these results yield two contributions for the analysis of operational termination of DCTRSs. First, when reducing the task of proving operational termination of DCTRSs to the task of proving (context-sensitive) termination of TRSs, it is now sufficient to prove termination *on original terms*. In order to exploit this relaxation of the termination property, we developed a dedicated method based on the dependency pair framework of [3] and [1], that allows us to prove termination of CSRSs obtained by the proposed transformation on original terms. First experimental results with a prototype implementation are promising. In particular, we were able to automatically

[1] Note that – in contrast to the definition of quasi-reductivity – the substitution maps variables only into terms over the original signature. This restriction is crucial for some of our main results.

prove termination on original terms of systems that are not terminating in the general sense.

Secondly, our results provide the basis for automatically disproving operational termination of DCTRSs, which was, to the authors' knowledge, impossible with transformational approaches before. On the other hand, as proving non-termination on original terms may be significantly harder than proving ordinary non-termination, the practical benefits of the latter results seem unclear. However, we were able to show that the proposed transformation is sound and complete with respect to *collapse-extended* termination (cf. e.g. [9]), so from (ordinary) non-termination of a transformed system we can at least conclude that the original DCTRS does not enjoy collapse-extended termination.

3 Related Work and Discussion

Our notion of cs-quasi-reductivity provides a new sufficient (in fact, equivalent) criterion for operational termination. Furthermore, cs-quasi-reductivity can be verified by proving termination of the resulting CSRS (on original terms). We have shown that the proposed transformation, which has already been discussed in [8] regarding simulation soundness and completeness and briefly in [10] regarding termination analysis, yields operational termination of strictly more DCTRSs than Ohlebusch's context-free transformation. Furthermore, we developed methods for the termination analysis that are tailored to verify termination on original terms. We implemented a prototype termination prover which, besides other well-known techniques, makes essential use of these methods. Our implementation was able to automatically prove operational termination of several DCTRSs, taken from the standard literature, for which all existing approaches failed. Finally, our work is the first to provide means for disproving operational termination.

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