REVISITING GILES'S GAME

*Reconciling Fuzzy Logic and Supervaluation**

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Abstract We explain Giles's characterization of Łukasiewicz logic via a dialogue game combined with bets on results of experiments that may show dispersion. The game is generalized to other fuzzy logics and linked to recent results in proof theory. We argue that these results allow one to place t-norm based fuzzy logics in a common framework with supervaluation as a theory of vagueness.

Keywords: Dialogue games, fuzzy logic, vagueness, supervaluation, betting

1. Introduction

In [12, 13] Robin Giles presents a strategic two-person game as a formal model of reasoning in physical theories, in particular quantum theory. Giles strictly separates the treatment of logical connectives from the problem of assigning meaning to atomic propositions in the presence of uncertainty. For the systematic stepwise reduction of arguments about compound statements to arguments about their atomic subformulas he refers to Paul Lorenzen's *dialogue game rules* (see, e.g., [20]). Atomic formulas are interpreted as assertions about (yes/no-)results of elementary experiments with dispersion. (I.e., the experiments may yield different results when repeated; only the *probability* of a particular answer is known.) Finally it is stipulated that each player has to pay a fixed amount of money to the other player for every false atomic assertion. Giles discovered that the propositions that a player can assert initially in the sketched game without having to expect a loss of money on average

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coincide with those that are valid in Łukasiewicz logic Ł, a logic that had been introduced for different purposes already in the 1920s [21].

Giles's remarkable result dates back to 1974; in more recent years Ł has emerged as one of several fundamental *fuzzy logics*. (See, e.g., [14, 24].) With hindsight, Giles has addressed an important philosophical challenge concerning vagueness: how to derive a 'fuzzy logic' from first principles of approximate reasoning? (For alternative approaches to this foundational problem see, e.g., [24–26].)

We aim at two different tasks. First, we want to place Giles's theorem in the context of recent results in the proof theory of fuzzy logics. In particular, we indicate how Giles's game can be generalized to other important fuzzy logics and point out that strategies in the corresponding games are related to analytic proofs in so-called r-hypersequent calculi [3]. A second task arises from a seeming paradox: the game theoretic characterization of fuzzy logics eliminates all reference to fuzziness. More exactly, talk of degrees of truth is replaced by talk of success probabilities of elementary experiments. So how does the game based analysis of fuzzy logics relate to their degree theoretic semantics? This question is of particular significance, since experts insist on the fundamental difference between probabilities (degrees of belief), on the one hand, and degrees of truth (reflecting vagueness), on the other hand. (See, e.g., [5, 14, 15] for a clear and concise explication of this difference.)

More generally, one may ask whether the game based analysis can shed light on the relation between truth functional fuzzy logics and competing models of approximate reasoning. Considering the highly contentious discourse on vagueness in analytic philosophy, our aim, although limited, is rather ambitious. We claim that the relevant games provide a way to reconcile the intuitions behind two prominent, but seemingly contradicting theories of vagueness: namely the degree theoretic approach and *supervaluation* with respect to admissible precisifications. We will interpret both approaches to vagueness as combining a *classical* analysis of logical connectives with a *non-classical* interpretation of the semantic status of atomic propositions. To this aim, we show that not only supervaluation, but also degree based fuzzy logics can be analyzed in terms of admissible precisifications of vague propositions. The dramatic difference in the respective judgements on logical validity does not disappear, but will be seen to result from the different syntactic levels at which supervaluation and fuzzy logics, respectively, refer to precisifications.

This paper is organized as follows. We begin with a short review of t-norm based fuzzy logics, in particular Ł, P, and G (in Section 2). This is followed by a presentation of Giles's game for Ł (in Section 3). We then connect the game with recent results in the proof theory of fuzzy logics

(Section 4) and generalize these results to include the logics P, CHL, and G (Section 5)[1]. This will leave us with the challenge of interpreting the game based characterization of fuzzy logics in terms of conceptions of vagueness (as explained in Section 6). To address this challenge, we connect (in Section 7) the semantic machinery of 'supervaluation' with that of t-norm based fuzzy logics. In the conclusion (Section 8) we hint at further topics for research.

We point out that, mainly due to lack of space, we confine our investigations to propositional logic, here.

2. t-norm based fuzzy logics

Fuzzy logics arise by stipulating that, in the presence of vague notions and propositions, truth comes in degrees. This view is very controversial among philosophers of vagueness. (See, e.g., [16, 29, 17] for an overview of the vagueness discourse in analytic philosophy.) Although we think that serious reflections on the philosophical foundation of logical formalisms are unavoidable in judging their adequateness, one may profit from recognizing at the outset that the 'degrees of truth' approach leads to a mathematically sound, robust and non-trivial formalism. It is not our intention to enter the debate on the significance of mathematical models in philosophical logic here, but we subscribe explicitly to the view that *as broad as possible* a collection of mathematical structures and tools should be in view of every expert—whether philosopher, logician, computer scientist, or technician—in the search for an adequate model of reasoning in a given context.

The degree theoretic approach to approximate reasoning has motivated dozens of different formalisms. Following Petr Hajek [14, 15], we cite some 'design decisions' that lead to the definition of a family of logics worth exploring in this context:

1 The set of truth degrees (truth values) is represented by the real unit interval $[0, 1]$. The usual order relation $\leq$ serves as comparison of truth degrees; 0 represents absolute falsity, and 1 absolute truth.

2 The truth value of a compound statement shall only depend on the truth values of its subformulas. In other words: the logics are truth functional.

3 The truth function for conjunction (&) should be a continuous, commutative, associative, and (in both arguments) monotonically increasing function $* : [0, 1]^2 \to [0, 1]$, where $0 * x = 0$ and $1 * x = x$. In other words: $*$ is a continuous t-norm.

4 The residuum $\Rightarrow_*$ of the t-norm $*$—i.e., the unique function $\Rightarrow_*$: $[0,1]^2 \rightarrow [0,1]$ satisfying $x \Rightarrow_* y = max\{z \mid x * z \leq y\}$—serves as the truth function for implication. The truth function for negation is defined as $\lambda x[x \Rightarrow_* 0]$. (Observe that this is analogous to the relation between conjunction, implication and negation in classical logic.)

The three most fundamental continuous t-norms and their residua are:

	t-norm	associated residuum
Łukasiewicz	$x *_{\mathsf{Ł}} y = max(0, x+y-1)$	$x \Rightarrow_{\mathsf{Ł}} y = min(1, 1-x+y)$
Gödel	$x *_{\mathsf{G}} y = min(x,y)$	$x \Rightarrow_{\mathsf{G}} y = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}$
Product	$x *_{\mathsf{P}} y = x \cdot y$	$x \Rightarrow_{\mathsf{P}} y = \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}$

Any continuous t-norm is an ordinal sum construction of these three (see, e.g., [14]). Note that the minimum and maximum of two values, that serve as alternative truth functions for conjunction ($\wedge$) and disjunction ($\vee$), respectively, can be expressed in terms of $*$ and $\Rightarrow_*$: $min(x,y) = x * (x \Rightarrow_* y)$ and $max(x,y) = min((x \Rightarrow_* y) \Rightarrow_* y, (y \Rightarrow_* x) \Rightarrow_* x)$.

We arrive at the following definition of propositional logics associated with a continuous t-norm:

Definition 1 *For a continuous t-norm $*$ with residuum $\Rightarrow_*$, we define a logic $\mathbf{L}_*$ based on a language with binary connectives $\rightarrow$, $\&$, constant $\bot$, and defined connectives $\neg A =_{def} A \rightarrow \bot$, $A \wedge B =_{def} A \,\&\, (A \rightarrow B)$, $A \vee B =_{def} ((A \rightarrow B) \rightarrow B) \wedge ((B \rightarrow A) \rightarrow A)$. A* valuation *for $\mathbf{L}_*$ is a function v assigning to each propositional variable a truth value from the real unit interval $[0,1]$, uniquely extended to v^* for formulas by:*

$$v^*(A \,\&\, B) = v^*(A) * v^*(B), \quad v^*(A \rightarrow B) = v(A) \Rightarrow_* v^*(B), \quad v^*(\bot) = 0$$

A formula A is valid in $\mathbf{L}_*$ iff $v^*(A) = 1$ for all valuations v^* pertaining to the t-norm $*$.

The logics $\mathbf{L}_{*_{\mathsf{Ł}}}$, $\mathbf{L}_{*_{\mathsf{G}}}$, and $\mathbf{L}_{*_{\mathsf{P}}}$, are called Łukasiewicz logic Ł, Gödel logic G, and Product logic P, respectively. Computational properties as well as semantic aspects of these logics, including their relation to other important logics, are well studied. (Again, [14] is the standard reference.) Various corresponding proof systems have been presented. Below, we will refer to the very recent systems **HŁ** of Metcalfe, Olivetti, and Gabbay [23] for Ł, and **rH** of Ciabattoni, Fermüller, and Metcalfe [3] that provides a uniform treatment of Ł, G, and P.

3. Giles's game for Ł

As already mentioned, Giles [12, 13] arrived at his analysis of Łukasiewicz logic irrespective of any reflections on vagueness or t-norms. His corresponding game consists of two largely independent building blocks:

(1) Betting for positive results of experiments. Two players—let's say me and you—agree to pay 1€ to the opponent player for every false statement they assert. By $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ we denote an *elementary state* in the game, where I assert each of the q_i in the multiset $\{q_1, \ldots, q_n\}$ of atomic statements (i.e., propositional variables), and you, likewise, assert each atomic statement $p_i \in \{p_1, \ldots, p_m\}$.

Each propositional variable q refers to an experiment E_q with binary (yes/no) result. The statement q can be read as 'E_q yields a positive result'. Things get interesting as the experiments may show dispersion; i.e., the same experiment may yield different results when repeated. However, the results are not completely arbitrary: for every run of the game, a fixed *risk value* $\langle q \rangle^r \in [0, 1]$ is associated with q, denoting the probability that E_q yields a negative result.[2] For the special atomic formula $\bot$ (*falsum*) we define $\langle \bot \rangle^r = 1$. The risk associated with a multiset $\{p_1, \ldots, p_m\}$ of atomic formulas is defined as $\langle p_1, \ldots, p_m \rangle^r = \sum_{i=1}^{m} \langle p_i \rangle^r$. The risk $\langle \rangle^r$ associated with the empty multiset is defined as 0. The risk associated with an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ is calculated from my point of view. Therefore the condition $\langle p_1, \ldots, p_m \rangle^r \geq \langle q_1, \ldots, q_n \rangle^r$ expresses that I do not expect any loss (but possibly some gain) when betting on the truth of atomic statements, as explained above.

(2) A dialogue game for the reduction of compound formulas. Giles follows Paul Lorenzen (see, e.g., [20]) in constraining the meaning of connectives by reference to rules of a dialogue game that proceeds by systematically reducing arguments about compound formulas to arguments about their subformulas.

For brevity, we will assume that formulas are built up from propositional variables, the falsity constant $\bot$, and the connective $\to$ only[3]. The central dialogue rule can be stated as follows:

(R) If I assert $A \to B$ then, whenever you choose to attack this statement by asserting A, I have to assert also B. (And *vice versa*, i.e., for the roles of me and you switched.)

This rule reflects the idea that the meaning of implication is specified by the principle that an assertion of 'if A, then B' ($A \to B$) obliges one to assert B, if A is granted.

In contrast to dialogue games for intuitionistic logic [20, 8, 18, 10], no special regulations on the succession of moves in a dialogue are required here. However, we assume that each assertion is attacked at most once: this is reflected by the removal of $A \to B$ from the multiset of all formulas asserted by a player during a run of the game, as soon as the other player has either attacked by asserting A, or has indicated that she will not attack $A \to B$ at all. Note that every run of the dialogue game ends in an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$. Given an assignment $\langle\cdot\rangle^r$ of risk values to all p_i and q_i we say that I *win* the game if I do not expect any loss, i.e., if $\langle p_1, \ldots, p_m\rangle^r \geq \langle q_1, \ldots, q_n\rangle^r$.

As an almost trivial example consider the game where I initially assert $p \to q$ for some atomic formulas p and q; i.e., the initial state is $[\|p \to q]$. In response, you can either assert p in order to force me to assert q, or explicitly refuse to attack $p \to q$. In the first case, the game ends in the elementary state $[p\|q]$; in the second case it ends in state $[\|]$. If an assignment $\langle\cdot\rangle^r$ of risk values gives $\langle p\rangle^r \geq \langle q\rangle^r$, then I win the game, whatever move you choose to make. In other words: I have a winning strategy for $p \to q$ in all assignments of risk values where $\langle p\rangle^r \geq \langle q\rangle^r$.

THEOREM 2 (R. GILES [12, 13]) *Every assignment $\langle\cdot\rangle^r$ of risk values to atomic formulas occurring in a formula F induces a valuation $v_{\langle\cdot\rangle^r}$ for Łukasiewicz logic Ł such that $v_{\langle\cdot\rangle^r}(F) = 1$ iff I have a winning strategy for F in the game presented above.*

COROLLARY 3 *F is valid in Ł iff, for all assignments of risk values to atomic formulas occurring in F, I have a winning strategy for F.*

4. Connecting strategies and proofs

There is a well-known correspondence between winning strategies in dialogue games and cut-free proofs in adequate versions of Gentzen's sequent calculus. For the case of Lorenzen's original dialogue game and (a variant of) Gentzen's **LJ** for intuitionistic logic this has been demonstrated, e.g., in [8]. A similar, even more straightforward relation holds between Gentzen's **LK** and Lorenzen style dialogue games for classical logic. Game based characterizations have been presented for many other logics, including modal logics, paraconsistent logics and substructural logics. To name just one result of relevance to our context, a correspondence between *parallel* versions of Lorenzen's game and so-called hypersequent calculi for intermediary logics, including the fuzzy logic G, has been established in [10, 4].

Returning to the game presented in Section 3, we note that Giles proved Theorem 2 without formalizing the concept of strategies. However, to reveal the close relation to analytic proof systems we need to define structures that allow us to formally register possible choices for both players. These structures, called *disjunctive strategies* or, for short, *d-strategies* appear at a different level of abstraction to strategies. The latter are only defined with respect to given assignments of risk values (and may be different for different assignments), whereas d-strategies abstract away from particular assignments.

DEFINITION 4 *A* d-strategy *(for me) is a tree whose nodes are disjunctions of states:*

$$[A_1^1, \ldots, A_{m_1}^1 \| B_1^1, \ldots, B_{n_1}^1] \bigvee \ldots \bigvee [A_1^k, \ldots, A_{m_k}^k \| B_1^k, \ldots, B_{n_k}^k]$$

which fulfill the following conditions:

1 *All leaf nodes denote disjunctions of elementary states.*

2 *Internal nodes are partitioned into I-nodes and you-nodes.*

3 *Any I-node is of the form* $[A \to B, \Gamma \| \Delta] \bigvee \mathcal{G}$ *and has exactly one successor node of the form* $[B, \Gamma \| \Delta, A] \bigvee [\Gamma \| \Delta] \bigvee \mathcal{G}$*, where* $\mathcal{G}$ *denotes a (possibly empty) disjunction of states, and* Γ*,* Δ *denote (possibly empty) multisets of formulas.*

4 *For every state* $[\Gamma \| \Delta]$ *of a you-node and every occurrence of* $A \to B$ *in* Δ*, the you-node has a successor of the form* $[A, \Gamma \| B, \Delta'] \bigvee \mathcal{G}$ *as well as a successor of the form* $[\Gamma \| \Delta'] \bigvee \mathcal{G}$*, where* Δ' *is* Δ *after removal of one occurrence of* $A \to B$*. (The multiset of occurrences of implications at the right hand sides is non-empty in you-nodes.)*[4]

We call a d-strategy winning *(for me) if, for all leaf nodes* ν *and for all possible assignments of risk values to atomic formulas, there is a disjunct* $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ *in* ν*, such that* $\langle p_1, \ldots, p_m \rangle^r \geq \langle q_1, \ldots, q_n \rangle^r$*.*

In game theory a winning strategy (for me) is usually defined as a function from all possible states, where I have a choice, into the set of my possible moves. Note that winning strategies in the latter sense exist for all assignments of risk values if and only if a winning d-strategy exists.

Strictly speaking we have only defined d-strategies (and therefore, implicitly, also strategies) with respect to some given regulation that, for each possible state, determines who is to move next. Each consistent partition of internal nodes into I-nodes and you-nodes corresponds to such a regulation. However, it has been demonstrated by Giles [12, 13] that the order of moves is irrelevant for determining my expected gain. Therefore no loss of generality is involved here.

The defining conditions for I-nodes and you-nodes clearly correspond to possible moves for me and you, respectively, in the dialogue game.

Thus Giles's theorem can be reformulated in terms of d-strategies. More interestingly, conditions 3 and 4 also correspond to the introduction rules for implication in the hypersequent calculus **HŁ** for Ł, defined in [23].

Hypersequents are a natural and useful generalization of Gentzen's sequents due to Pottinger and Avron [1]. A hypersequent is just a multiset of sequents written as

$$\Gamma_1 \vdash \Delta_1 \mid \cdots \mid \Gamma_n \vdash \Delta_n$$

The interpretation of component sequents $\Gamma_i \vdash \Delta_i$ varies from logic to logic. But the |-sign separating the individual components is always interpreted as classical disjunction (at the meta-level). The logical rules for introducing connectives refer to single components of a hypersequent. The only difference to sequent rules is that the relevant sequents live in a (possibly empty) context $\mathcal{H}$ of other sequents, called side-hypersequent. The rules of **HŁ** for introducing implication are:

$$\frac{B, \Gamma \vdash \Delta, A \mid \mathcal{H}}{A \to B, \Gamma \vdash \Delta \mid \mathcal{H}} \; (\to, l) \qquad \frac{A, \Gamma \vdash \Delta, B \mid \mathcal{H} \qquad \Gamma \vdash \Delta \mid \mathcal{H}}{\Gamma \vdash \Delta, A \to B \mid \mathcal{H}} \; (\to, r)$$

Observe that rules $(\to, l)$ and $(\to, r)$ are just syntactic variants of the defining conditions 3 and 4 for d-strategies. To sum up: the logical rules of **HŁ** can be read as rules for constructing generic winning strategies in Giles's game.

5. Other fuzzy logics: variants of the game

We have shown that a formalization of generic strategies for Giles's game (d-strategies) reveals a direct correspondence with the hypersequent system **HŁ** for Ł. What about other fuzzy logics? Can one generalize the discovered correspondence to include P, G, and related logics?

Giles's characterization of Ł combines Lorenzen style dialogue rules for the analysis of connectives with bets on positive results of elementary experiments. But note that the phrase 'betting for positive results of (a multiset of) experiments' is ambiguous. As we have seen, Giles identifies the combined risk for such a bet with the *sum* of risks associated with the single experiments. However, other ways of interpreting the combined risk are worth exploring. In particular, we are interested in a second version of the game, where an elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ corresponds to my single bet that *all* experiments associated with the q_i, where $1 \leq i \leq n$, show a positive result, against your single bet that *all* experiments associated with the p_i $(1 \leq i \leq m)$ show a positive result. A third form of the game arises if one decides to perform only *one* experiment for each of the two players, where the relevant experiment is chosen by the opponent.

To achieve a direct correspondence between the three mentioned betting schemes and the t-norm based semantics of the connectives in Ł, P, and G, respectively, we invert risk values into probabilities of *positive* results (yes-answers) of the associated experiments. More formally, the *value* of an atomic formula q is defined as $\langle q \rangle = 1 - \langle q \rangle^r$; in particular, $\langle \bot \rangle = 0$.

My expected gain in the elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ in Giles's game for Ł is the sum of money that I expect you to have to pay me minus the sum that I expect to have to pay you. This amounts to $\sum_{i=1}^{m}(1 - \langle p_i \rangle) - \sum_{i=1}^{n}(1 - \langle q_i \rangle)$ €. Therefore my expected gain is greater or equal to zero iff $1 + \sum_{i=1}^{m}(\langle p_i \rangle - 1) \leq 1 + \sum_{i=1}^{n}(\langle q_i \rangle - 1)$ holds. The latter condition is called winning condition $W_{\sum}$.[5]

In the second version of the game, you have to pay me 1€ unless all experiments associated with the p_i test positively, and I have to pay you 1€ unless all experiments associated with the q_i test positively. My expected gain is therefore $1 - \prod_{i=1}^{m} \langle p_i \rangle - (1 - \prod_{i=1}^{n} \langle q_i \rangle)$ €; the corresponding winning condition $W_{\prod}$ is $\prod_{i=1}^{m} \langle p_i \rangle \leq \prod_{i=1}^{n} \langle q_i \rangle$.

To maximize the expected gain in the third version of the game I will choose a $p_i \in \{p_1, \ldots, p_m\}$ where the probability of a positive result of the associated experiment is least; and you will do the same for the q_is that I have asserted. Therefore my expected gain is $(1 - \min_{1 \leq i \leq m} \langle p_i \rangle) - (1 - \min_{1 \leq i \leq n} \langle q_i \rangle)$ €. Hence the corresponding winning condition $W_{\min}$ is $\min_{1 \leq i \leq m} \langle p_i \rangle \leq \min_{1 \leq i \leq n} \langle q_i \rangle$.

We thus arrive at the following definitions for the value of a multiset $\{p_1, \ldots, p_n\}$ of atomic formulas, according to the three versions of the game:

$$\begin{aligned}
\langle p_1, \ldots, p_n \rangle_{\text{Ł}} &= 1 + \textstyle\sum_{i=1}^{n}(\langle p_i \rangle - 1) = \left(\sum_{i=1}^{n} \langle p_i \rangle\right) - (n-1) \\
\langle p_1, \ldots, p_n \rangle_{\mathsf{P}} &= \textstyle\prod_{i=1}^{n} \langle p_i \rangle \\
\langle p_1, \ldots, p_n \rangle_{\mathsf{G}} &= \min_{1 \leq i \leq n} \langle p_i \rangle .
\end{aligned}$$

For the empty multiset we define $\langle \rangle_{\text{Ł}} = \langle \rangle_{\mathsf{P}} = \langle \rangle_{\mathsf{G}} = 1$.

In contrast to Ł, the dialogue game rule (R) does not suffice to characterize P and G. To see this, consider the state $[p \to \bot \| q]$. According to rule (R) I may assert p in order to force you to assert $\bot$. Since $\langle \bot \rangle = 0$, the resulting elementary state $[\bot \| p, q]$ fulfills the winning conditions $\langle \bot \rangle \leq \langle p \rangle \cdot \langle q \rangle$ and $\langle \bot \rangle \leq \min\{\langle p \rangle, \langle q \rangle\}$, that correspond to P and G, respectively. However, this is at variance with the fact that for assignments where $\langle p \rangle = 0$ and $\langle q \rangle < 1$ you have asserted a statement $(p \to \bot)$ that is definitely true ($v(p \to \bot) = 1$), whereas my statement q is not definitely true ($v(q) < 1$).[6]

It is no accident that the above example involves the truth constant $\bot$ as well as a value $\langle p \rangle = 0$. If we remove $\bot$ from the language and evaluate formulas as in P—using multiplication for conjunction and its residuum for implication—but over the left-open interval $(0,1]$ instead of $[0,1]$, then we arrive at a well investigated logic, known as *cancellative hoop logic* CHL (see, e.g., [7, 22]).[7] It is easy to check[8] that the logical rules of system **HŁ** are sound and invertible not only for Ł, but also for CHL. Therefore we can directly transfer the connection, described in Section 4, between d-strategies for Giles's game and **HŁ**-rules to obtain the following.

COROLLARY 5 *F is valid in* CHL *iff for all assignments of values from* $(0,1]$ *to the atomic formulas occurring in F, I have a winning strategy for F in the variant of Giles's game with the winning condition* $W_{\prod}$.

We have thus arrived at a game based characterization of CHL, which uses dialogue rules identical to those for Ł, but differs in the betting schemes determining the winning conditions.

We may justify the elimination of $\bot$ and 0 by the observation that the presence of elementary experiments, which *invariably* yield a negative result, spoils the whole idea of combining bets on positive results according to the schemes for P or G. On the other hand, however, the expressiveness of the language is considerably reduced by removing $\bot$, since negation is defined in terms of $\bot$. One may ask, whether there is a characterization of P and G by a Giles/Lorenzen-style game. To address this problem we analyse the rules of the uniform calculus **rH** [3], mentioned at the end of Section 2. In contrast to **HŁ**, the component sequents of hypersequents in **rH** come in two versions: there are two different sequent signs '$\leq$' and '$<$', instead of the one '$\vdash$' used in **HŁ**. More formally, an r-hypersequent is a finite multiset

$$\Gamma_1 \lhd_1 \Delta_1 \mid \ldots \mid \Gamma_n \lhd_n \Delta_n$$

where $\lhd_i \in \{<, \leq\}$ and Γ_i and Δ_i are finite multisets of formulas for $i = 1, \ldots, n$. The relational symbols indicate the intended semantics: the above r-hypersequent is called *valid* for logic $\mathsf{X} \in \{$Ł$,\mathsf{G},\mathsf{P}\}$ if for all valuations v, that refer to the corresponding t-norm $*_{\mathsf{X}}$, there is some i, $1 \leq i \leq n$, such that $\#^v_{\mathsf{X}}\Gamma_i \lhd_i \#^v_{\mathsf{X}}\Delta_i$, where $\#^v_{\mathsf{X}}\emptyset = 1$ and where

$$\#^v_{\text{Ł}}(\Gamma) = 1 + \sum_{A\in\Gamma}\{v(A) - 1\}, \;\; \#^v_{\mathsf{G}}(\Gamma) = \min_{A\in\Gamma}\{v(A)\}, \;\; \#^v_{\mathsf{P}}(\Gamma) = \prod_{A\in\Gamma}\{v(A)\}.$$

This allows us to check that the following **rH**-rules for introducing implication are sound and invertible for all three logics:

$$\frac{A, \Gamma \lhd \Delta, B \mid A \leq B \mid \mathcal{H} \qquad \Gamma \lhd \Delta \mid \mathcal{H}}{\Gamma \lhd \Delta, A \to B \mid \mathcal{H}} \; (\to, r)^*$$

$$\frac{B, \Gamma \triangleleft \Delta, A \mid \Gamma \triangleleft \Delta \mid \mathcal{H} \qquad \Gamma \triangleleft \Delta \mid B < A \mid \mathcal{H}}{A \to B, \Gamma \triangleleft \Delta \mid \mathcal{H}} \ (\to, l)^*$$

where $\triangleleft$ is either $<$ or $\leq$, uniformly in each rule. Together with (also uniform, even simpler) rules for the other connectives and appropriate initial atomic r-hypersequents (that, of course, are different for each of the three logics) one obtains a sound and complete analytic system for Ł, G, and P, respectively (see [3]).

There are at least two different ways to translate these rules into rules for the construction of winning strategies in versions of our game. A rather direct interpretation of r-hypersequents in terms of disjunctions of states in a dialogue game is obtained by distinguishing two different types of states: One, corresponding to the sequent sign $\leq$, which is exactly as in the original game and one corresponding to the sequent sign $<$, in which an additional flag ¶ is raised to announce that I will be declared winner of the current run of the game only if the evaluation of the final elementary state yields a *strictly positive* (and not just non-negative) expected gain for me.

Dialogue rules, replacing (R) in Giles's game, but directly corresponding to $(\to, r)^*$ and $(\to, l)^*$ can be formulated as follows:

(R_r^*) If I assert $A \to B$ then, whenever you choose to attack this statement by asserting A, I have the following choice: either I assert B in reply or I challenge your attack on $A \to B$ by replacing the current game with a new one in which you assert A and I assert B.

Note that the right hand side premise of rule $(\to, l)^*$ corresponds to the case were you choose not to attack the exhibited occurrence of $A \to B$. As can be seen, the newly introduced flag plays no direct role. It is only needed in the rule corresponding to $(\to, l)^*$:

(R_l^*) If you assert $A \to B$ then, whenever I choose to attack this statement by asserting A, you have the following choice: either you assert B in reply or you challenge my attack on $A \to B$ by replacing the current game with a new one in which the flag ¶ is raised and I assert A while you assert B.

Note that I can also choose not to attack $A \to B$. This corresponds to the component sequents $\Gamma \triangleleft \Delta$ in the two premise r-hypersequents of rule $(\to, r)^*$. The flag ¶ is needed because the winning conditions are not fully complementary for me and you: we may both have a non-negative expected gain. Your 'attack-challenging' claim that I *cannot win* when starting in state $[A \| B]$ is equivalent to the claim that I *can win* when starting in state $[B \| A]$ only if the flag ¶, signalling a strictly positive expected gain as winning condition, is raised in the latter game.

The translation of the r-hypersequent rules for conjunction and disjunction in [3] into dialogue game rules is also straightforward. Admittedly these new versions of Lorenzen style dialogue rules amount to *ad hoc* regulations to circumvent the problematic effects of bets on elementary results that always yield negative results. A different (but still *ad hoc*) way to deal with this problem has been described in [3]. Instead of using the additional flag, one imposes the following constraint on attacking implicative formulas:

(Q) If I have a strategy for winning the game starting in the state $[A \| B]$, then I am not allowed to attack your assertion of $A \to B$. (And *vice versa*, i.e., for the roles of you and me switched.)[9]

Imposing (Q) also results in a game that characterizes $\mathsf{Ł}$, P, and G, if the corresponding versions of the winning conditions are applied (cf. [3]). Here we only point out that applying rule (Q) involves the systematic development of full strategies for subformulas, before it can be judged whether an attack to a formula according to rule (R) is permitted. Whether more satisfying Giles/Lorenzen style characterizations of P and G in the presence of $\perp$ and 0 can be achieved remains an open problem.

6. Where is vagueness?

What has been achieved by the analysis of fuzzy logics in terms of dialogue games? Since the rules for the stepwise reduction of arguments about compound formulas to arguments about their atomic subformulas are identical for $\mathsf{Ł}$, CHL, but also (in a more problematic way) for $\mathsf{Ł}$, P, and G, we have opened a *unified view* of reasoning in t-norm based fuzzy logics. Moreover, the relation to classical logic is clarified: the dialogue part of the game coincides with a version of Lorenzen's original dialogue game adapted to classical logic. If we trivialize the betting schemes by stipulating that all assigned probabilities are either 0 or 1—i.e., if each elementary experiment consistently shows the same result when repeated—then Giles's game, as well as the alternative games for P and G, discussed in Section 5, characterize *classical validity.* To see this, it suffices to check that for every elementary state $[p_1, \ldots, p_m \| q_1, \ldots, q_n]$ and $\mathsf{X} \in \{\mathsf{Ł}, \mathsf{P}, \mathsf{G}\}$ we have:

$$\langle p_1, \ldots, p_m \rangle_{\mathsf{X}} \leq \langle q_1, \ldots, q_n \rangle_{\mathsf{X}} \quad \text{iff} \quad \{p_1, \ldots, p_m\} \cap \{q_1, \ldots, q_n\} \neq \emptyset,$$

for all assignments $\langle \cdot \rangle$ of values where $\langle p_i \rangle, \langle q_j \rangle \in \{0, 1\}$. If we denote elementary states in sequent notation

$$p_1, \ldots, p_m \vdash q_1, \ldots, q_n$$

it gets clear that the latter condition corresponds to classical axiom sequents $p \vdash p$, up to (irrelevant) weakening. Moreover, it corresponds to the standard winning condition for Lorenzen style dialogues: I win the dialogue if you attack a statement that you have already asserted yourself ('*ipse dixisti* rule'). Indeed, it is straightforward to show that winning d-strategies for all versions of the game, described above, correspond to cut-free proofs in versions of hypersequent calculi that are sound and complete for classical logic if valuations are restricted to range over $\{0, 1\}$.

What is the significance of the betting schemes for the evaluation of atomic formulas? Obviously, the betting schemes allow us to characterize the *differences* between Ł, P, and G: different underlying t-norms correspond to different ways of combining bets on results of elementary experiments into a single bet. However, a closer look at this setting reveals a serious foundational problem. One would like to present the game based characterizations of Ł, CHL, P, and G as a derivation of fuzzy logics from first principles about reasoning with *vague propositions*, but all reference to vagueness and degrees of truth seems to have disappeared; more exactly: it has been replaced by references to classical reasoning combined with a *probabilistic* semantics for atomic statements. However, fuzziness should never be confused with probability (as has been emphasized in the literature, e.g., in [15, 14, 5]). Whereas fuzzy logic takes vague propositions to refer to *degrees of truth*, probability theory formalizes *degrees of rational belief*. Even without engaging in discussions on adequate interpretations of vagueness and probability, it should be clear that

(1) 'The next throw of the dice will result in 5 or 6'

is true only with some probability (1/3, if the dice is fair), but does not involve vagueness; whereas,

(2) 'Logicians are weird people'

is vague, but does not refer to probability. (2) may meaningfully be said to be true only to some degree (even in a fixed context); whereas (1), in the intended context, is either definitely true or definitely false, even if it is (not yet) known which of the two holds. Since Giles, in evaluating atomic statements, refers to elementary experiments that are of the same (probabilistic, but non-fuzzy) type as in statement (1), it might seem inadequate to interpret Giles's game as a model for proper reasoning with vague notions.

7. Connecting supervaluation, degrees of truth, and bets on positive results of experiments

There is a prolific discourse in analytic philosophy about the nature of reasoning in the presence of vagueness. This is not the place to comment on these debates;[10] however, in order to connect the game based analysis of Ł, P, and G with degrees of truth and disentangle it from probabilistic logic, we refer to a particular approach to understanding vagueness, called *supervaluation*—currently most popular among philosophers of vagueness (see, e.g., [16, 27, 28]).

Supervaluation, as a theory of vagueness, is canonically developed by Kit Fine in [11]. Since we are only interested in propositional logic without additonal modal operators, only a simplified version of supervaluation will be needed here. The central idea is to formalize reasoning in vague contexts by reference to all *admissible precisifications* of vague expressions. More exactly, formulas are evaluated in reference to a *specification space* $\mathcal{S}$, which is simply a collection (multiset[11]) of partial models. A partial model is a possibly partial assignment of classical truth values, 0 or 1, to propositional variables. An element $w \in \mathcal{S}$ is called a *complete precisification* of $v \in \mathcal{S}$ if w is total and if $v(p) = w(p)$ for all propositional variables p, for which v is defined. A complete precisification of v is a classical model compatible with v. We are only interested in those elements of $\mathcal{S}$ that are complete precisifications of a fixed element ('actual world') $a \in \mathcal{S}$. This sub-multiset of $\mathcal{S}$ is denoted by $\mathcal{C}_a$ and is assumed to be non-empty. Three possibilities for the semantic status of a formula F arise:

- $v(F) = 1$ for all $v \in \mathcal{C}_a$: in this case F is called *supertrue* in $\mathcal{C}_a$;
- $v(F) = 0$ for all $v \in \mathcal{C}_a$: in this case F is called *superfalse* in $\mathcal{C}_a$;
- $\exists v, w \in \mathcal{C}_a$ such that $v(F) = 0$ and $w(F) = 1$: in this case the semantic status of F remains undefined.

Proponents of supervaluation often contend that, in contrast to claims made by degree theoretists, no revision of classical logic is necessary in face of vagueness. (However see [19] for a criticism of the claim that supervaluation does not deviate from classical logic.) Whereas, e.g., the formula $A \vee \neg A$ is not valid in Ł, P, G, or any related logic, it is evaluated true in all classical interpretations, and therefore is supertrue in all precisification spaces $\mathcal{S}$, even if A is evaluated true in some precisifications and false in other precisifications of the actual world of $\mathcal{S}$.

Given the coincidence of supertruth in all specification spaces and classical validity, it is understandable that supervaluation is usually seen as incompatible with fuzzy logic. In contrast, we claim that the game based interpretation reveals much common ground among these competing conceptions of reasoning under vagueness. Both, supervaluationists and defendants of Ł, P, and G as logics of vagueness, can agree on three principles:

1 An atomic statement is *definitely true* only if there is no admissible precisification of it that renders it false.

2 Arguments about compound statements F can be reduced to arguments involving only subformulas of F.

3 The rules used for 2 should only depend on the outmost connective of F and should be sound and complete for classical logic.

That the reduction rules should refer to classical logic, seems, at a first glance, to be at variance with the standard t-norm based interpretation of our fuzzy logics. However, the coincidence of the logical dialogue rules in Giles's game with those in versions of the game for classical logic makes shared intuitions about the meaning of connectives explicit.

Obviously, essential differences between supervaluation and t-norm based valuations remain. To facilitate a more detailed comparison, we interpret the truth value $\in [0, 1]$, that is assigned to a propositional variable p in fuzzy valuation, in terms of the *proportion* of those complete precisifications that make p true. The simplest way to formalize this idea is to assume that the cardinality of $\mathcal{C}_a \in \mathcal{S}$ is finite. We may then define the 'fuzzy valuation' $v_{\mathcal{S}}$ induced by a precisification space $\mathcal{S}$ via $\mathcal{C}_a$ as

$$v_{\mathcal{S}}(p) = \frac{|\{v \in \mathcal{C}_a : v(p) = 1\}|}{|\mathcal{C}_a|}$$

for all propositional variables p. In other words: with respect to a given precisification space, fuzzy valuations and supervaluation *agree* on the assignment of classical truth values 1 and 0 to atomic formulas; but in the remaining cases, where supervaluation assigns no overall truth value, fuzzy logics assign a value that 'measures' the fraction of verifying precisifications.[12] For compound formulas, the difference between supervaluation and fuzzy valuation may be described in terms of the *syntactic level* at which a formula is tied to individual precisifications. For *supervaluation* the whole formula F is evaluated in each complete precisification to determine F's semantic status. Following the game based characterization of Ł, P, and G, *fuzzy valuation* of F may be described as consisting of three stages:

1 an analysis—following classical principles—of F into arguments about its atomic components;

2 a valuation of each of the resulting relevant occurrences of atomic formulas in F in reference to a specification space;

3 a synthesis of the resulting values of the atomic subformulas of F into an overall value for F.

The following table confronts the valuation function $v^{sv}_{\mathcal{S}}$, resulting from supervaluation, with the valuation function $v^{\mathsf{X}}_{\mathcal{S}}$ of a t-norm based fuzzy logic X, where all valuations refer to the specification space $\mathcal{S}$ via the multiset $\mathcal{C}_a$ of its complete precisifications.

Supervaluation	**Valuation in logic** X
$v^{sv}_{\mathcal{S}}(p) = 1\,(0) \Longleftrightarrow \forall v \in \mathcal{C}_a \colon v(A) = 1\,(0)$	$v^{\mathsf{X}}_{\mathcal{S}}(p) = \frac{\lvert\{v \in \mathcal{C}_a \colon v(p)=1\}\rvert}{\lvert\mathcal{C}_a\rvert}$
$v^{sv}_{\mathcal{S}}(\bot) = 0$	$v^{\mathsf{X}}_{\mathcal{S}}(\bot) = 0$
$v^{sv}_{\mathcal{S}}(F \to G) = 1\,(0) \Longleftrightarrow$ $\forall v \in \mathcal{C}_a \colon \big(v(F) \Rightarrow_c v(G)\big) = 1\,(0)$	$v^{\mathsf{X}}_{\mathcal{S}}(F \to G) = \big(v^{\mathsf{X}}_{\mathcal{S}}(F) \Rightarrow_* v^{\mathsf{X}}_{\mathcal{S}}(G)\big)$, where $\mathsf{X} = \mathbf{L}_*$

Remember, that $\Rightarrow_*$ is the residuum of the t-norm $*$ that defines the logic $\mathbf{L}_*$. We have used $\Rightarrow_c$ to denote the classical truth function for implication (which, by the way, can be presented as the residuum of an arbitrary t-norm, restricted to $\{0, 1\}$). Also remember that all other logical connectives can be defined in terms of $\to$ and $\bot$, not only for Ł, but also for classical logic. Of course one can easily extend the above list by the corresponding definitions for conjunction and disjunction (thus including also full P and G).

We think that supervaluation and fuzzy valuation capture contrasting, but individually coherent intuitions about the role of logical connectives in vague statements. Consider a sentence like

(3) "The sky is blue and is not blue".

When formalized as $b \wedge \neg b$, (3) is *superfalse* in all specification spaces. This fits Fine's motivation in [11] to capture 'penumbral connections' that prevent any mono-colored object from having two colors at the same time. According to his intuition the statement "The sky is blue" absolutely contradicts the statement "The sky is not blue", even if neither statement is definitely true or definitely false. Therefore (3) is judged as definitely false, even if admittedly vague. On the other hand, by asserting (3) one may intend to convey the information that both component assertions are true only to some degree. Under this reading (and a certain interpretation of 'and') $b \wedge \neg b$ is *not* definitely false, unless b is supertrue or superfalse. The latter intuition is directly captured in Łukasiewicz logic since $b \wedge \neg b$ may receive a value $\in [0, 0.5]$, where $\wedge$ denotes the 'weak conjunction', i.e., the minimum operator.[13]

As already indicated, the difference between the two interpretations of (3) can be described as a difference of the syntactic level at which the sentence is projected to admissible precisifications. In supervaluation it is checked whether the *whole sentence* $b \wedge \neg b$ is true in every complete precisification; whereas in fuzzy valuation each of the two occurrences of the subformula b is valuated separately with respect to the proportion of complete precisifications that make b true.

We claim that both kinds of intuitions should be accommodated in a full account of approximate reasoning[14]. Technically, supervaluation and various forms of fuzzy valuation can easily be embedded in a common semantic framework, as indicated above. For evaluating a formula F correspondingly, it suffices to mark syntactically—e.g., by using two different types of implication, conjunction, negation, etc.—whether an occurrence of a subformula of F should be supervaluated or valuated according to a certain t-norm based scheme. In both cases, the valuation may refer to the same specification space.

8. Conclusion

Our presentation of Giles's game and its variants is meant to demonstrate that t-norm based fuzzy logics can be derived from first principles about approximate reasoning. As we have seen in Sections 4 and 5, rules for the systematic construction of winning strategies in the games for Ł, CHL, P, and G correspond to the logical rules of analytic calculi for these logics. This partly also clarifies the relation to classical logic: for all investigated logics the (dialogue based) meaning of connectives adheres to constraints pertaining to classical logic. Moreover, the game based analysis allows us to relate supervaluation to the seemingly opposite concept of 'degrees of truth': both models of approximate reasoning can be seen as referring to admissible precisifications in a given specification space.

Many interesting topics for further investigation arise; we conclude by explicitly posing a few relevant questions. Is there a similar analysis of other logics that have been suggested for approximate reasoning? In particular, can Hajek's 'basic logic' [14]—the logic of *all* continuous t-norms—be characterized by an adequate game? What about quantifiers? How does the incompleteness of first-order Ł and P, that contrasts with the existence of complete calculi for G (and classical logic), bear on game based semantics for these logics? How can we account for higher-order vagueness in dialogue games? Can one extend the analysis to logics equipped with a definiteness operator and other relevant modal operators? Can the game based characterization of fuzzy logics shed light on the relation to further conceptions of vagueness, like gap-theoretic, epistemic, pragmatic and information-based approaches?

Notes

1. Sections 3, 4, and 5 extend the brief remarks in the final section of [3].

2. Giles [13] attempts to provide a tangible meaning to basic notions that arise in formalizing physical theories. Our use of Giles's game is independent from this original motivation.

3. Note that in Ł all other connectives can be defined from $\rightarrow$ and $\perp$ alone, since we may define $A \,\&\, B$ as $(A \rightarrow (B \rightarrow \perp)) \rightarrow \perp$. The other connectives are defined as indicated in Definition 1.

4. For a total of n occurrences of compound formulas on the right hand sides of states in a you-node, there are $2n$ successor nodes, corresponding to $2n$ possible moves for you.

5. The term 'winning condition' is slightly misleading here, since I can loose money in a particular run of the game even if this condition holds; only a non-negative *expected* gain is guaranteed by what we choose to call 'winning condition'.

6. The problem does not arise in logic Ł, since there the expected gain for state $[\perp \| p, q]$ is $\langle p, q\rangle_{\text{Ł}} - \langle\perp\rangle_{\text{Ł}} = 1 - (\langle p\rangle - 1) - (\langle q\rangle - 1) - (1 - 1) = \langle p\rangle + \langle q\rangle - 1$ and therefore indeed negative, as expected, if $\langle p\rangle = 0$ and $\langle q\rangle < 1$.

7. Note that CHL is different from $\perp$-free P: e.g., $(A \rightarrow (A \,\&\, B)) \rightarrow B$ is valid in CHL, but not in P.

8. For rule $(\rightarrow, l)$ it suffices to observe that for all $a, b, c_i, d_j \in (0, 1]$: $(a \Rightarrow_{\mathsf{P}} b) \cdot \prod_i g_i \leq \prod_j d_j$ iff $b \cdot \prod_i g_i \leq \prod_j d_j \cdot a$. For rule $(\rightarrow, r)$ the relevant fact is that $\prod_i g_i \leq \prod_i d_i \cdot (a \Rightarrow_{\mathsf{P}} b)$ iff both $a \cdot \prod_i g_i \leq \prod_j d_j \cdot b$ and $\prod_i g_i \leq \prod_j d_j$.

9. Recall that the strategies mentioned in (Q) refer to a given assignment $\langle\cdot\rangle$ of values and thus appear at a more concrete level than d-strategies.

10. For an overview of theories of vagueness and their problematic relation to fuzzy logic we refer to [16], [29], [2] and [9].

11. As long as one is not interested in measuring the cardinality of precisifications that satisfy certain properties, the difference between precisifications spaces as sets and as multisets, respectively, disappears.

12. At the propositional level, on which we focus here, it is not unreasonable to assume that only a finite number of different plausible precisifications is relevant when evaluating a given statement in a fixed context. Anyway, it is not difficult to extend the concept to more general situations. E.g., one may wish to weight precisifications according to some measure of their individual plausibility. One may also take into account non-complete precisifications in different ways. In any case, an assignment of a truth value $\in [0, 1]$ to a propositional variable p in fuzzy logic can be interpreted as a way to *quantify the information* pertaining to p that is contained in a given specification space.

13. Note that $b \,\&\, \neg b$ is always evaluated to 0, where $\&$ is the 'strong conjunction' (t-norm) of Ł. Thus one may argue that Ł is capable of representing both interpretations of a sentence like (3). Also remember that in P and G the value of $\neg b$ is 0 if the value of b is not equal to 1. Therefore $b \wedge \neg b$ is always evalutated to 0 in P and G.

14. This is of particular significance for a successful analysis of *Sorites paradoxa* and phenomena of *higher order vagueness*, as we shall argue elsewhere.

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