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Parallel Dialogue Games for Intermediate Logics

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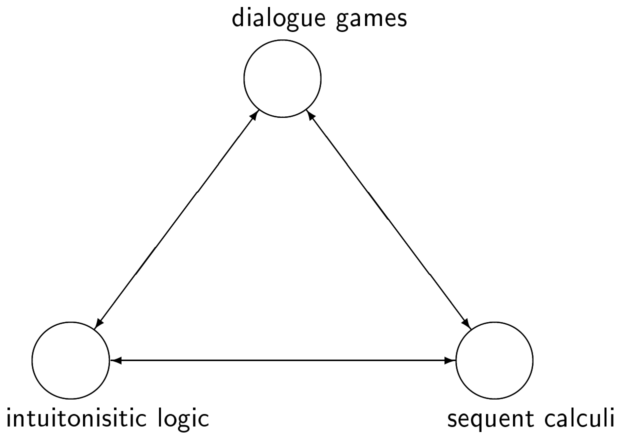
Contents

Motivation

Three ‘entry points’:



parallel

interm. &

hyper

Games in logic

Different types of games are used in modern logic . . .

- **evaluation games:** *Hintikka, Moschovakis*
- **model comparison games:** *Ehrenfeucht-Fraïssé*, bisimulation
- **model building games:** *Beth, Hodges*
- **game logics:** *Parikh*

In particular, there are also different forms of **dialogue games**:

- **Lorenzen-type dialogue games:** 'idealized confrontational dialogues' *(our topic!)*
- **Blass-Abramsky-type dialogue games:** 'connectives as game combinators'

Dialogues as foundations

In an idealized dialogue, a Proponent **P** tries to uphold a statement against (rational) attacks by an Opponent **O**

Idea: logical validity 'means' that **P** can always win the dialogue

Some basic features:

- attacking moves and corresponding defense moves refer to connectives (and quantifiers)
- both, **P** and **O**, may launch attacks and defend against attacks during the course of a dialogue
- the dialogue is asymmetric

Logical Rules:

statement by **X**	attack by **Y**	defense by **X**
$A \wedge B$	l? or r? (**Y** chooses)	A or B, accordingly
$A \vee B$	?	A or B (**X** chooses)
$A \supset B$	A	B

Note: **X**/**Y** stands for **P**/**O** or **O**/**P**

$\neg A$ abbreviates $A \supset \bot$.

Winning conditions for P:

W: **O** has already granted **P**'s current formula.

W$\bot$: **O** has granted $\bot$.

Structural rules:

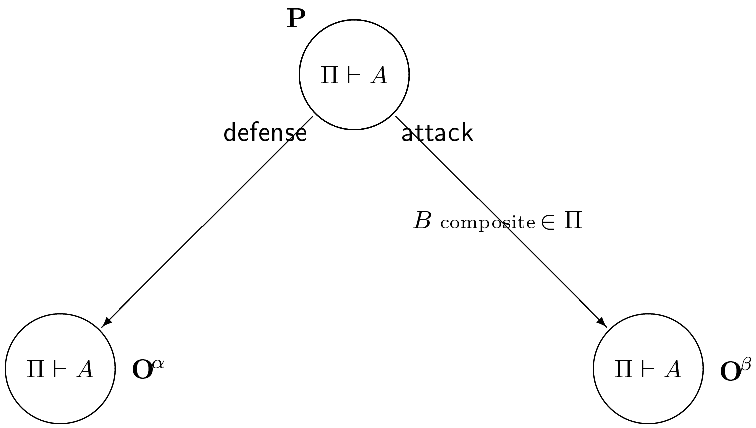
Start: **O** starts (by attacking **P**'s *initial formula*).

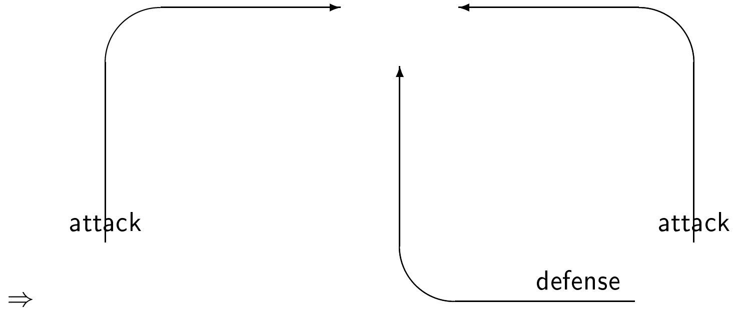
Alternate: Moves strictly alternate between **O** and **P**.

Atom: Atomic formulas (including $\bot$) can neither be attacked nor defended by **P**.

'E-rule': Each move of **O** reacts directly to the immediately preceding move by **P**.

Dialogues as state transitions





Winning strategies

Definition:

A **winning strategy** is a finite tree, whose branches are dialogues that end in winning states for **P**, s.t.

– **P**-nodes have (at most) one successor;

– **O**-nodes have successors for each possible next move by **O**.

(This is equivalent to strategies as functions.)

Note:

Dialogues are traces of the corresponding state transition system.

Winning strategies arise by 'unwinding' the state transition system.

LJ′: (A version of) Gentzen's sequent calculus for intuitionistic logic

Axioms:

$\bot, \Pi \longrightarrow C \quad$ and $\quad A, \Pi \longrightarrow A$

Structural rules:

$$\frac{\Pi \longrightarrow C}{A, \Pi \longrightarrow C}\ (\mathit{weakening}) \qquad \frac{A, A, \Pi \longrightarrow C}{A, \Pi \longrightarrow C}\ (\mathit{contraction})$$

$$\frac{\Pi \longrightarrow A \qquad A, \Pi \longrightarrow C}{\Pi \longrightarrow C}\ (\mathit{cut})$$

Logical rules:

$$\frac{A, A \vee B, \Pi \longrightarrow C \qquad B, A \vee B, \Pi \longrightarrow C}{A \vee B, \Pi \longrightarrow C}\ (\vee, l)$$

$$\frac{\Pi \longrightarrow A}{\Pi \longrightarrow A \vee B}\ (\vee_{\mathsf{l}}, r) \qquad \frac{\Pi \longrightarrow B}{\Pi \longrightarrow A \vee B}\ (\vee_{\mathsf{r}}, r)$$

$$\frac{A, A \wedge B, \Pi \longrightarrow C}{A \wedge B, \Pi \longrightarrow C}\ (\wedge_{\mathsf{l}}, l) \qquad \frac{B, A \wedge B, \Pi \longrightarrow C}{A \wedge B, \Pi \longrightarrow C}\ (\wedge_{\mathsf{r}}, l)$$

$$\frac{\Pi \longrightarrow A \qquad \Pi \longrightarrow B}{\Pi \longrightarrow A \wedge B}\ (\wedge, r)$$

$$\frac{A \supset B, \Pi \longrightarrow A \quad B, A \supset B, \Pi \longrightarrow C}{A \supset B, \Pi \longrightarrow C}\ (\supset, l) \quad \frac{A, \Pi \longrightarrow B}{A, \Pi \longrightarrow A \supset B}\ (\supset, r)$$

Adequateness Theorems

Soundness of dialogues:
Every winning strategy for $\Gamma \vdash C$ can be transformed into an $\mathbf{LJ}'$-proof of $\Gamma \longrightarrow C$.

Completeness of dialogues:
Every $\mathbf{LJ}'$-proof of $\Gamma \longrightarrow C$ can be transformed into a winning strategy for $\Gamma \vdash C$.

Remarks on the soundness proof:

– By induction on the depth of the strategy.

– $\mathbf{P} \longrightarrow \mathbf{O} \longrightarrow \mathbf{P}$-cycles correspond to $\mathbf{LJ}'$-inference steps.

– **O**-choices correspond to branching rules.

Remarks on the completeness proof:

– By cut-elimination and further pruning, $\mathbf{LJ}'$-proofs are first transformed into cut- and weaking-free form.

– $\mathbf{LJ}'$-inference steps then turn into $\mathbf{P} \longrightarrow \mathbf{O} \longrightarrow \mathbf{P}$-cycles.

– Branching corresponds to an **O**-choice.

Parallel dialogue games

Our 'guidelines' for parallelization:

- ◇ Ordinary dialogues (I-dialogues) should appear as subcase of the more general parallel framework.
- ◇ **P** may initiate additional dialogues by 'cloning'.
- ◇ To win a set of parallel dialogues, **P** has to win at least one of the component I-dialogues.
- ◇ Synchronisation between parallel I-dialogues is invoked by **P**'s decision to merge two or more I-dialogues into one.

Notions for parallel dialogue games

A parallel I-dialogue (I-P-**dialogue**) is a sequence of global states connected by internal or external moves.

Global state:
$\{\Pi_1 \vdash_{\iota 1} C_1,\ \ldots,\ \Pi_n \vdash_{\iota n} C_n\}$
(Set of uniquely indexed component I-dialogue sequents.)

Internal move:
Set of I-dialogue moves: at most one for each component.

External move:
May add or remove components, but does not change the status—**P**'s or **O**'s turn to move—of existing components.

Basic external moves:

fork: **P** duplicates a **P**-component of the current global state.

cancel: **P** removes an arbitrary **P**-component
(if the global state contains another **P**-component).

To ensure fairness:
Every I-P-dialogue starts by a sequence of internal moves containing **O**-moves that 'initiate' each component.

Different forms of parallelity

We choose to refer to global and discrete time.

Options for 'implementing' concurrency:

- All component dialogues have to advance at each time step.
- At most k dialogues may advance simultanously:
  P and **O** as teams of k players each.
  (k pairs of dialogue partners.)
- **Sequentialization:** Only one pair of dialogue partners; exactly one component move (or external move) at each step.

Towards proving adequacy:
Sequentialized and normal I-P-dialogues

Sequentiality: internal moves are singletons.

Normality:

- **P**-moves are immediately followed by **O**-moves referring to the same component(s)
- external moves (possibly consisting of a **P**-**O**-round) are followed by **P**-moves

Theorem:
Every finite I-P-dialogue can be translated into an equivalent sequentialized and normal I-P-dialogue.

HLJ′: A hypersequent calculus for intuitionistic logic

Axioms:

$\bot, \Pi \longrightarrow C \mid \mathcal{H}$ and $A, \Pi \longrightarrow A \mid \mathcal{H}$

Logical rules:

As for **LJ′**, except for the presence of side hypersequents:

$$\frac{A \supset B, \Pi \longrightarrow A \mid \mathcal{H} \quad B, A \supset B, \Pi \longrightarrow C \mid \mathcal{H}}{A \supset B, \Pi \longrightarrow C \mid \mathcal{H}} \ (\supset, l)$$

$$\frac{A, \Pi \longrightarrow B \mid \mathcal{H}}{A, \Pi \longrightarrow A \supset B \mid \mathcal{H}} \ (\supset, r)$$

Internal structural rules:

$$\frac{\Pi \longrightarrow C \mid \mathcal{H}}{A, \Pi \longrightarrow C \mid \mathcal{H}} \ (\textit{I-weakening}) \qquad \frac{A, A, \Pi \longrightarrow C \mid \mathcal{H}}{A, \Pi \longrightarrow C \mid \mathcal{H}} \ (\textit{I-contr.})$$

$$\frac{\Pi \longrightarrow A \mid \mathcal{H} \quad A, \Pi \longrightarrow C \mid \mathcal{H}'}{\Pi \longrightarrow C \mid \mathcal{H} \mid \mathcal{H}'} \ (\textit{cut})$$

External structural rules:

$$\frac{\mathcal{H}}{\Pi \longrightarrow C \mid \mathcal{H}} \ (\textit{E-weakening}) \qquad \frac{\Pi \longrightarrow C \mid \Pi \longrightarrow C \mid \mathcal{H}}{\Pi \longrightarrow C \mid \mathcal{H}} \ (\textit{E-contr.})$$

Soundness and completeness of HLJ′:

Theorem: [see, e.g., Avron, Baaz/Ciabattoni/F]
HLJ′ is sound and cut-free complete for intuitionistic logic.

Remark:
Actually, published proofs refer to a slightly different version (and some contain errors ...)

Adequateness of I-P-dialogues:

Soundness of dialogues:
Every (normal sequential) I-P winning strategy for
$\{\Gamma_1 \vdash_{\iota 1} C_1, \ldots, \Gamma_n \vdash_{\iota n} C_n\}$
can be transformed into an **HLJ′**-proof of
$\Gamma_1 \longrightarrow C_1 \mid \ldots \mid \Gamma_n \longrightarrow C_n$.

Completeness of dialogues:
Every **HLJ′**-proof of
$\Gamma_1 \longrightarrow C_1 \mid \ldots \mid \Gamma_n \longrightarrow C_n$
can be transformed into an I-P winning strategy for
$\{\Gamma_1 \vdash_{\iota 1} C_1, \ldots, \Gamma_n \vdash_{\iota n} C_n\}$.

Remarks on proving adequacy:

The correspondence between internal moves and logical rules (in normal sequential dialogues) directly transfer from the non-parallel case.

External rules:

- fork corresponds to external contraction
- cancel corresponds to external weakening

General form of synchronisation rules:

- only **P**-components are involved
- **P** chooses components for merging
- no change of status of exisiting components
- **O** may or may not have a choice as to how to merge the components picked by **P**

An (almost) trivial example: (sub-case of cancel)

join: **P** picks two **P**-components with the same dialogue sequent; one of it is removed from the global state.

Modelling classical logic

cl: **P** removes a **P**-component of form $\Pi \vdash_\iota \bot$ and adds Π to the granted formulas of another **P**-component.

In other words:
P merges $\Pi \vdash_{\iota 1} \bot$ and $\Gamma \vdash_{\iota 2} C$ into $\Pi \cup \Gamma \vdash_{\iota 2} C$.

The winning power for **P** increases:

P has a winning strategy for $\vdash A \vee \neg A$.

Characterizing Jankov logic LQ

A slight modification of rule cl results in winning powers that are weaker than required for classical logic, but allow **P** to force a win for $\vdash \neg A \vee \neg\neg A$:

lq: **P** removes a **P**-component of form $\Pi \vdash_\iota \bot$ and adds Π to the granted formulas of another **P**-component of form $\Gamma \vdash_\iota \bot$.

In other words:
P merges $\Pi \vdash_{\iota 1} \bot$ and $\Gamma \vdash_{\iota 2} \bot$ into $\Pi \cup \Gamma \vdash_{\iota 2} \bot$.

The resulting logic of weak excluded middle is also known as Jankov logic or as LQ.

Characterizing Gödel-Dummett logic

We consider the following '**P**–**O**'-choice rule:

> lc: **P** picks two **P**-components $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$ for merging. In response, **O** chooses either C_1 or C_2 as the current formula of the merged component with granted formulas $\Pi_1 \cup \Pi_2$.

In other words:

– **P** wants to merge $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$

– **O** then chooses either $\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1$ or $\Pi_1 \cup \Pi_2 \vdash_{\iota 2} C_2$ as merged component.

System	rule	external move(s)
Cl-P	cl	**P** merges $\Pi \vdash_{\iota 1} \bot$ and $\Gamma \vdash_{\iota 2} C$ into $\Pi \cup \Gamma \vdash_{\iota 2} C$
LQ-P	lq	**P** merges $\Pi \vdash_{\iota 1} \bot$ and $\Gamma \vdash_{\iota 2} \bot$ into $\Pi \cup \Gamma \vdash_{\iota 2} \bot$
LC-P	lc	**P** picks $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$ **O** chooses $\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1$ or $\Pi_1 \cup \Pi_2 \vdash_{\iota 2} C_2$
LC′-P	lc′	**P** picks* $\Pi_1 \cup \Gamma_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \cup \Gamma_2 \vdash_{\iota 2} C_2$ **O** chooses $\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1$ or $\Gamma_1 \cup \Gamma_2 \vdash_{\iota 2} C_2$
sLC-P	lc0	**P** picks $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$ **O** chooses $\Pi_2 \vdash_{\iota 1} C_1$ or $\Pi_1 \vdash_{\iota 2} C_2$
	sp	**P** merges $\Pi \vdash_{\iota 1} C$ and $\Gamma \vdash_{\iota 2} C$ into $\Pi \cup \Gamma \vdash_{\iota 2} C$
G_n-P	g_n	**P** picks the components $\Pi_1 \vdash_{\iota 1} C_1$, and $\ldots \Pi_{n-1} \vdash_{\iota[n-1]} C_{n-1}$, and $\Pi_n \vdash_{\iota n}$ **O** chooses one of $\Pi_1 \cup \Pi_2 \vdash_{\iota 1} C_1$, $\Pi_2 \cup \Pi_3 \vdash_{\iota 2} C_2$, $\ldots$, or $\Pi_{n-1} \cup \Pi_n \vdash_{\iota[n-1]} C_{n-1}$

Hypersequent calculi for intermediate logics

…arise from **HLJ′** by adding additional external rules.

E.g., a hypersequent calculus **HCl** for classical logic is obtained by adding the following 'splitting rule':

$$\frac{\Pi, \Gamma \longrightarrow C \mid \mathcal{H}}{\Pi \longrightarrow \bot \mid \Gamma \longrightarrow C \mid \mathcal{H}}\ (cl)$$

Theorem: [see, e.g., Avron, Ciabattoni]
HCl is sound and cut-free complete for classical logic.

A hypersequent calculus for Jankov logic LQ

The calculus **HLQ** is obtained by adding the following rule to **HLJ′**:

$$\frac{\Pi, \Gamma \longrightarrow \bot \mid \mathcal{H}}{\Pi \longrightarrow \bot \mid \Gamma \longrightarrow \bot \mid \mathcal{H}}\ (lq)$$

Theorem: [Ciabattoni/Gabbay/Olivetti]
HLQ is sound and cut-free complete for LQ.

A hypersequent calculus for Gödel-Dummet logic LC

The calculus **HLC** is obtained by adding the following rule — 'communication rule' — to **HLJ′**:

$$\frac{\Pi_1, \Pi_2 \longrightarrow C_1 \mid \mathcal{H} \qquad \Pi_1, \Pi_2 \longrightarrow C_2 \mid \mathcal{H}}{\Pi_1 \longrightarrow C_1 \mid \Pi_2 \longrightarrow C_2 \mid \mathcal{H}}\ (com)$$

Theorem: [see Avron, Gabbay/Olivetti]
HLC is sound and cut-free complete for LC.

Remark: The characterization of LC via (analytic) hypersequent calculi is due to Avron.
For the version needed here, we refer to observations of Olivetti/Gabbay.

Hypersequent calculi for finite-valued Gödel logics G_n

The calculus $\mathbf{HG_n}$ ($n \geq 2$) is obtained by adding the following rule to **HLJ′**:

$$\frac{\Pi_1, \Pi_2 \longrightarrow C_1 \mid \mathcal{H} \quad \Pi_2, \Pi_3 \longrightarrow C_2 \mid \mathcal{H} \quad \ldots \quad \Pi_{n-1}, \Pi_n \longrightarrow C_{n-1} \mid \mathcal{H}}{\Pi_1 \longrightarrow C_1 \mid \ldots \mid \Pi_{n-1} \longrightarrow C_{n-1} \mid \Pi_n \longrightarrow \mid \mathcal{H}}\ (g_n)$$

Theorem: [see, e.g., Ciabattoni]
For all $n \geq 2$, $\mathbf{HG_n}$ is sound and cut-free complete for G_n.

Adequateness proofs for the various parallel dialogue systems ...

... procede — once more — by translating (normal sequentialized) winning strategies into cut-free proofs, and vice versa.

Remark:
(Parallel) dialogues are slightly more 'finegrained' than (hypersequent) proofs.
This allows to model some subtle forms of proof search.

Lorenzen's foundational program in a wider perspective

$$\begin{array}{ccc}
\text{dialogue} & \longleftrightarrow & \text{[state of proof search]} \\
\Downarrow & & \Downarrow \\
\text{winning strategy} & \longleftrightarrow & \text{(sequent) proof} \\
\Downarrow & & \Downarrow \\
\text{winning strategy for parallel dialogues} & \longleftrightarrow & \text{hypersequent proof} \\
\downarrow & & \downarrow \\
\text{`naïve' winning strategy} & \longleftrightarrow & \text{proof search strategy}
\end{array}$$

Concluding remarks

Understanding Avron's 'communication rule(s)': The parallel dialogue view allows for a better evaluation of Avron's claim that *LC* relates to parallel programs in the same way as intuitionistic logic relates to sequential programs.

Relations between intermediate logics: *Vide* the close relation between, e.g., rules cl and lq.

Interesting variants: A lot of (interesting?) questions are triggered by the parallel dialogue format.
E.g.: how to add more choices for **O**? What about other (natural) forms of scheduling, merging and exchange of information between different dialogue components?

Beyond intermediate logics: Resource bounded-ness, but also modalities can be handled elegantly in the dialogue format!

A final remark

A (longer) paper corresponding roughly to the contents of this talk is *almost* finished.

I'll put it on my webpage

`http://www.logic.at/staff/chrisf/`

as soon as it is ready.

Alternatively, send me an email:

`chrisf@logic.at`