From Intuitionistic Logic to Gödel-Dummett Logic via Parallel Dialogue Games

Christian G. Fermüller   Agata Ciabattoni
Technische Universität Wien, Austria

Abstract— **Building on a version of Lorenzen's dialogue foundation for intuitionistic logic, we show that a suitable game of communicating parallel dialogues is sound and complete for Gödel-Dummett logic G. Among other things, this provides a computational interpretation of Avron's hypersequent calculus for G.**

I. Introduction

Gödel-Dummett logic (called **G** here, from now on) arguably is one of the most interesting many-valued logics. It naturally turns up in different fields in logic and computer science. Already in the 1930's Gödel [9] used it (implicitly) to shed light on aspects of intuitionistic logic; later Dunn and Meyer [6] pointed out its relevance for relevance logic; Visser [16] employed it in investigations of the provability logic of Heyting arithmetic; and eventually it was recognized as one of the most useful species of 'fuzzy logic' (see [10], [15]).

Considered as a fuzzy logic, propositional **G** is characterized by evaluations v of the propositional variables in the real closed unit interval $[0,1]$ and the following truth functions for connectives:

$$v(A \wedge B) = \min(v(A), v(B)) \qquad v(A \vee B) = \max(v(A), v(B))$$

$$v(\bot) = 0 \qquad v(A \supset B) = \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ v(B) & \text{otherwise} \end{cases}$$

As usual, $\neg A$ can be defined as $A \supset \bot$. For sake of clarity we stick to the propositional level in the whole paper; but conjecture that our results can be extended to first-order, and even propositional, quantification.

Gödel-Dummett logic bears a special relation to intuitionistic logic **I**: it can be characterized not only by referring to the above truth functions over $[0,1]$, but also by imposing a *linearity condition* on intuitionistic Kripke structures or Heyting algebras. Indeed, as shown already in [5], Hilbert-type systems for **G** can be obtained by adding the linearity axiom—$(A \supset B) \vee (B \supset A)$—to any standard system for **I**.

In our context it is important that, in contrast to other fuzzy logics, convincing analytic proof systems have been presented for **G**. In particular, we refer to Avron's elegant hypersequent calculus **HLC** [3] for **G**. **HLC** contains Gentzen's sequent calculus **LI** for **I** as a sub-calculus, and simply adds an additional layer of information by allowing **LI**-sequents to live in the context of finite multisets of sequents (called *hypersequents*). Additional structural rules allow to manipulate sequents with respect to their contexts. The crucial new rule of the calculus **HLC**, is the so called communication rule (see Section VI), which is intended to model the 'exchange of information' between different sequents. To substantiate this latter intuition a 'computational interpretation' of hypersequents is needed. A first step in that direction was achieved in [4], where **HLC**-proofs are translated into a special natural deduction format. However, a convincing extraction of programs (e.g., in form of lambda terms) from hypersequent proofs still seems to be lacking.

In this paper, we introduce a version of parallel dialogue games to serve as a dynamic structure in which (analytic) hypersequent proofs for **G** can be interpreted faithfully. Besides providing a 'computational interpretation' for **G**, dialogue games are an interesting framework for investigating foundational issues and modeling proof search (as will be shown in a sequel to this paper).

II. Lorenzen style dialogue games

Logical dialogue games come in many forms and versions, nowadays. Here, we do not use more recent formulations in the style of Blass [2] or Abramsky [1], but rather refer directly to Paul Lorenzen's original idea (dating back to the late 1950s, see e.g., [13]) to identify logical validity of a formula A with the existence of a winning strategy for a *proponent* **P** in an idealized confrontational dialogue, in which **P** tries to uphold A against rational 'attacks' by an *opponent* **O**. Although the claim that this leads to an alternative characterization—or even: 'justification'—of *intuitionistic logic* was implicit already in Lorenzen's early essays, it took more then twenty years until the first rigorous, complete and error free proof of this central claim was published in [7]. Many variants of Lorenzen's original dialogue games have appeared in the literature since. (See, eg., [8], [11] for relevant references.) Here, we define a version of dialogue games that are:

1. well suited for demonstrating the close relation to analytic Gentzen-type systems;
2. easily shown to be equivalent to other versions of dialogue games for intuitionistic logic, that can be found in the literature;
3. straightforward to consider 'in parallel'.

Notation. An *atomic formula (atom)* is either a propositional variable or the 0-ary connective $\bot$ (*falsum*). As usual, *compound formulas* are built up from atoms using the connectives $\supset$, $\wedge$, $\vee$; $\neg A$ abbreviates $A \supset \bot$. In addition to formulas, the special signs ?, $\text{l}?$, $\text{r}?$ can be stated in a dialogue by the players **P** and **O**, as specified below.

Dialogue games are characterized by two sorts of rules: logical rules and structural rules.

The *logical rules* define how to attack a compound formula and how to defend against such an attack. They are summarized

Some of the results of this paper have been presented also in an invited talk of the first author at *LPAR 2002* in Tbilisi, Georgia. However, they have neither been published nor submitted for publication so far.

in the following table. (If **X** is the proponent **P** then **Y** refers to the opponent **O**, and vice versa.)

Logical dialogue rules:

X:	attack by **Y**	defense by **X**
$A \wedge B$	l? or r? (**Y** chooses)	A or B, accordingly
$A \vee B$	?	A or B (**X** chooses)
$A \supset B$	A	B

We will see below that atoms (including $\perp$) can be attacked too (by player **O**). Such an attack also consists in stating '?'. ($\perp$ is understood as an undefendable statement, as gets clear from the structural rule *Atom* and the winning condition $W\perp$, formulated below.)

A *dialogue* is a sequence of *moves*, which are either attacking or defending statements, in accordance with the logical rules. Each dialogue refers to a finite multiset of formulas, that are *initially granted* by **O**, and to an *initial formula* to be defended by **P**.

Moves can be viewed as state transitions. In any state of the dialogue the multiset of formulas, that have been either initially granted or stated by **O** so far, are called the *granted formulas* (at this state). The last formula that has been stated by **P** and that either already has been attacked or must be attacked in **O**'s next move is called *current formula*. With each state of a dialogue we thus associate a *dialogue sequent* $\Pi \vdash A$, where Π denotes the granted formulas and A the current formula.

Remark 1: Note that the current formula, in general, is *not* the last formula stated by **P**. (Since **P** may have stated formulas *after* the current formula that are not attacked by **O**.

Remark 2: We stipulate that each move carries the information (indices) necessary to reconstruct which formula is attacked or defended in which way (if there are different possibilities) in that move. However, we do not care about the exact way this information is coded.

Structural rules (*Rahmenregeln* in the diction of Lorenzen and his school) regulate the succession of moves. Quite a number of different systems of structural rules have been proposed in the literature (See e.g., [14], [8], [11]. In particular, [11] compares and discusses different systems.). The following rules, together with the winning conditions stated below, amount to a version of dialogues traditionally called *Ei*-dialogues (i.e., Felscher's *E*-dialogues combined with the so-called *ipse dixisti* rule; see, e.g., [11]).

Structural dialogue rules:

Start: The first move of the dialogue is carried out by **O** and consists in attack on the initial formula.

Alternate: Moves strictly alternate between player **O** and **P**.

Atom: Atomic formulas, including $\perp$, may be stated by both players, but can neither be attacked nor defended by **P**.

E: Each (but the first) move of **O** reacts directly to the immediately preceding move by **P**. I.e., if **P** attacks a granted formula then **O**'s next move either defends this formula or attacks the formula used by **P** to launch this attack. If, on the other hand, **P**'s last move was a defending one then **O** has to attack immediately the formula stated by **P** in that defense move.

Winning conditions (for P):

W: The game ends with **P** winning if **O** has attacked a formula that has already been granted (either initially or in a later move) by **O**.

$W\perp$: The game ends with **P** winning if **O** has granted $\perp$.

A *dialogue tree* τ *for* $\Pi \vdash C$ is a rooted, directed and labelled tree with nodes labelled by dialogue sequents and edges corresponding to moves, such that each branch[1] of τ is a dialogue with initially granted formulas Π and initial formula C. We thus identify the nodes of a dialogue tree with states of a dialogue. We distinguish **P**-nodes and **O**-nodes, according to whether it is **P**'s or **O**'s turn to move at the corresponding state.

A finite dialogue tree is called *winning strategy* (for **P**) if the following conditions are satisfied:

1. Every **P**-node has at most one successor node.
2. If a **P**-node is a leaf node, then the winning conditions for **P** are fulfilled at this node.
3. Every **O**-node has a successor node for each move by **O** that is a permissible continuation of the dialogue at this stage.

Remark 3: Winning strategies for a player in a non-cooperative two-person game are more commonly described as *functions* assigning a move for that player to each state of the game, taking into account all possible moves of the opponent. Observe that our tree form of a winning strategy just describes the corresponding function in a manner that makes the step-wise evolution of permissible dialogues more explicit.

Henceforth we use the following notation: For every compound formula $F = C \supset D$, F_p denotes C and F_c denotes D. If F is atomic then F_p is empty (and F_c remains undefined). F_{pp} is C_p if $F = C \supset D$.

As already mentioned, a dialogue game may be viewed as a state transition system, where moves in a dialogue correspond to transitions between **P**-nodes and **O**-nodes. A dialogue then is a possible trace in the system; and a winning strategy can be obtained by a systematic 'unraveling' of all possible traces.

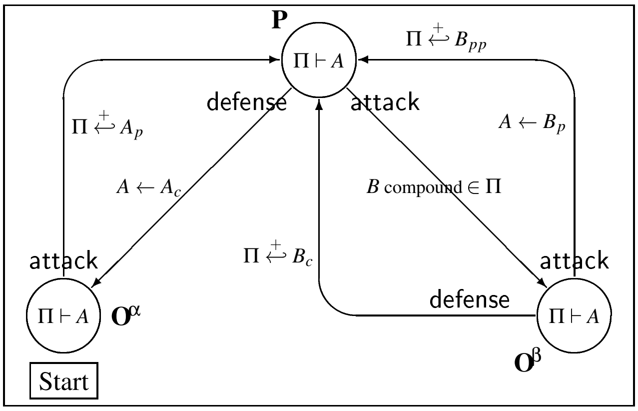
To illustrate the latter point, consider the implicational fragment of the language; i.e., the set of formulas not containing $\wedge$ or $\vee$. Figure 1 represents all permissible moves in a dialogue for this fragment. By labelling a transition with $\Pi \overset{+}{\hookleftarrow} F$ we denote that F is added to the multiset Π of granted formulas. $A \leftarrow C$ means that C replaces A as a result of the corresponding move.

Note that the encircled labels denote the dialogue sequent at the corresponding state. The edges from the **P**-node to the two **O**-nodes correspond to the principal choice of player **P**: either to defend the current formula or to attack a compound formula B among the granted formulas. (The fact that A_c is undefined if A is atomic means that in this case the transition from node **P** to node $\mathbf{O}^{\alpha}$ is not possible. This corresponds to the stipulation that atomic formulas cannot be defended by **P**, according to the structural rule *Atom*. However, remember that the dialogue is already in a winning state for **P** if the current formula A is among the granted formulas Π.)

On the other hand, according to the structural rule E, player **O** has no choice but to attack the last formula of **P** if **P**'s last move

[1] By a branch of a tree we mean a path starting at the root node that either ends in a leaf node or else is infinite.

Fig. 1. Dialogue as state transitions (⊃-fragment)



was a defending move (i.e., if we are in state $\mathbf{O}^\alpha$.) In state $\mathbf{O}^\beta$, however, **O** may either defend the attacked formula or (counter-)attack the formula used by **P** in launching the last attack.

(The fact that B_{pp} is empty if the premise B_p of B is an atom means that the atom B_p is attacked by **O** and thus becomes the current formula.)

The winning conditions have to be checked at state **P** only. If $\bot \in \Pi$ or $A \in \Pi$ then the game ends in that state with **P** winning.

Adding $\wedge$ and $\vee$ to the language amounts to adding further possible transitions (between the nodes **P** and $\mathbf{O}^\alpha$, and **P** and $\mathbf{O}^\beta$, respectively) that correspond to moves as specified by the logical rules.

III. Basic adequateness of dialogues

Quite a few proofs of the adequateness of dialogue games for characterizing intuitionistic logic can be found in the literature. Since we will build directly on such a proof—also in going beyond intuitionistic logic—we have to present our own version of it, which draws on ideas from [11], [12] and [7] but differs in some essential details.

We use the following variant of Gentzen's well known sequent calculus **LI** for intuitionistic logic. Sequents are objects of form $\Pi \longrightarrow C$, where Π denotes a multiset of formulas and C is a formula.

Remark 4: In standard formulations of **LI**, the right hand side of a sequent may also be empty. However, since we include $\bot$ among the formulas, and consider negation a derived connective, we will not have to consider this case.

As usual we use the notation F,Π for $\{F\} \cup \Pi$, etc.

Axioms:

$$\bot,\Pi \longrightarrow C \quad \text{and} \quad A,\Pi \longrightarrow A$$

Logical rules:

$$\frac{A,A\vee B,\Pi \longrightarrow C \qquad B,A\vee B,\Pi \longrightarrow C}{A\vee B,\Pi \longrightarrow C}\ (\vee,l)$$

$$\frac{\Pi \longrightarrow A_i}{\Pi \longrightarrow A_1 \vee A_2}\ (\vee_i,r) \qquad \frac{A_i,A_1\wedge A_2,\Pi \longrightarrow C}{A_1\wedge A_2,\Pi \longrightarrow C}\ (\wedge_i,l)$$

$$\frac{\Pi \longrightarrow A \qquad \Pi \longrightarrow B}{\Pi \longrightarrow A\wedge B}\ (\wedge,r) \qquad \frac{A,\Pi \longrightarrow B}{\Pi \longrightarrow A \supset B}\ (\supset,r)$$

$$\frac{A\supset B,\Pi \longrightarrow A \qquad B,A\supset B,\Pi \longrightarrow C}{A\supset B,\Pi \longrightarrow C}\ (\supset,l)$$

Structural rules:

$$\frac{\Pi \longrightarrow C}{A,\Pi \longrightarrow C}\ \text{(weakening)} \qquad \frac{A,A,\Pi \longrightarrow C}{A,\Pi \longrightarrow C}\ \text{(contraction)}$$

$$\frac{\Pi \longrightarrow A \quad A,\Pi \longrightarrow C}{\Pi \longrightarrow C}\ \text{(cut)}$$

We call this system $\mathbf{LI}'$. It is straightforward to check that a sequent is derivable in $\mathbf{LI}'$ if and only if it is derivable in Gentzen's **LI**. It follows that $\mathbf{LI}'$ is sound and complete for intuitionistic logic. As a corollary we have the following fact:

Proposition *1:* $A,\Gamma \longrightarrow A \supset B$ is provable in $\mathbf{LI}'$ only if $\Gamma \longrightarrow A \supset B$ is provable.

The proof of adequateness of dialogue games for intuitionistic logic consists in showing that winning strategies can be transformed into (analytic) $\mathbf{LI}'$ proofs and vice versa.

Theorem *1:* Every winning strategy τ for $\Gamma \vdash C$ can be transformed into an $\mathbf{LI}'$-proof $\pi(\tau)$ of $\Gamma \longrightarrow C$.

Proof: We prove by induction on the depth d of τ that for every **P**-node of τ there is an $\mathbf{LI}'$-proof of the $\mathbf{LI}'$-sequent corresponding to the dialogue sequent at this node. That this implies the theorem is obvious for the cases where C is either atomic, or a disjunction, or a conjunction; because, in those cases, the dialogue sequent at the **P**-node(s) immediately succeeding the root node is (are) identical to $\Gamma \vdash C$. In the case where $C = A \supset B$, the **P**-node succeeding the root carries $A,\Gamma \vdash A \supset B$ as dialogue sequent; and thus the theorem follows from Proposition 1.

The base case, $d = 1$, follows from the fact that the **P**-node (or, in case of C being a conjunction, two **P**-nodes) succeeding the root is a (are) leaf node(s). This implies that one of the winning conditions—$C \in \Gamma$ or $\bot \in \Gamma$—must hold. Consequently, the corresponding sequent $\Gamma \longrightarrow C$ is an axiom.

For $d > 1$ we have to distinguish cases according to the form of the current formula that is defended or the (compound) formula that is attacked by **P**. To keep the proof concise, we will only elaborate it for the implicational fragment of the language; it is straightforward to augment the proof to cover also conjunctions and disjunctions. (We refer to Figure 1 for a visualization of the relevant part of the winning strategy.)

P *defends* $A \supset B$*:* Let $A,\Pi \vdash A \supset B$ be the dialogue sequent at the current **P**-node. **P** moves from the **P**-node to the $\mathbf{O}^\alpha$-node by stating B. **O** has to reply with a move attacking B. We distinguish two cases:

1. If B is an atom then the attack consists in stating '?'. Thus we return to a **P**-node with dialogue sequent $A,\Pi \vdash B$. By the induction hypothesis there is an $\mathbf{LI}'$-proof of $A,\Pi \longrightarrow B$, which can be extended to a proof of $A,\Pi \longrightarrow A \supset B$ by rule $(\supset,r)$ and weakening.
2. If B is of form $B_p \supset B_c$ then **O** has to attack B by adding B_p to the granted formulas Π. Thus we return to a **P**-node with dialogue sequent $A,B_p,\Pi \vdash B$. By the induction hypothesis there is an $\mathbf{LI}'$-proof of $A,B_p,\Pi \longrightarrow B$. By Proposition 1 we obtain an $\mathbf{LI}'$-proof of $A,\Pi \longrightarrow B$. The required proof of $A,\Pi \longrightarrow A \supset B$ is obtained by applying rule $(\supset,r)$ and weakening.

P *attacks* $D \supset E$*:* Let $D \supset E,\Pi \vdash A$ be the dialogue sequent at the current **P**-node. **P**'s attack consists in stating D. (The move refers to the edge from node **P** to node $\mathbf{O}^\beta$ of the diagram.) The strategy then branches since **O** may either defend the implication or attack D.

1. If **O** chooses to attack D then D_p is added to the granted formulas if $D = D_p \supset D_c$. If D is atomic the multiset of granted formulas remains unchanged. In any case, D is the new current formula at the succeeding **P**-node. The corresponding dialogue sequent is

$$D_p, D \supset E, \Pi \vdash D \tag{1}$$

where D_p is empty if D is atomic.

2. If, on the other hand, **O** chooses to defend $D \supset E$ then it has to grant E. The current formula at the succeeding **P**-node remains A. The corresponding dialogue sequent is

$$E, D \supset E, \Pi \vdash A \tag{2}$$

By the induction hypothesis there are $\mathbf{LI}'$-proofs of the sequents corresponding to (1) and (2). By Proposition 1 we may assume that D_p in (1) is empty. Therefore we obtain a proof of $D \supset E, \Pi \longrightarrow A$ by combining the two proofs with an application of rule $(\supset, l)$. □

Remark 5: For proving the soundness of dialogue games (by this we mean that winning strategies only exist for intuitionistically valid sequents) it is in fact not necessary to refer to formal derivations. It rather suffices to check that intuitionistic validity transfers from the leaves of a dialogue tree upwards to the root. However for the following completeness proof the special form of the intuitionistic proofs is essential.

The 'weakening friendly' formulation of the axioms and rules of $\mathbf{LI}'$ allows to eliminate applications of the weakening rule. (Weakenings in $\mathbf{LI}'$-proofs can be moved upwards to the axioms, where they are obviously redundant.) Also the contraction rule becomes redundant if we disregard multiple occurrences of the same formula in the left hand side of a sequent. Most importantly, $\mathbf{LI}'$ is complete also without cut. Let as refer to a proof that does not contain any applications of structural rules as *strongly analytic*. The following proposition then sums up the just made observations.

Proposition *2:* There is a strongly analytic proof in $\mathbf{LI}'$ for $\Gamma \longrightarrow C$ if and only if $\Gamma' \longrightarrow C$ is provable in $\mathbf{LI}'$, where Γ' equals Γ if taken as set (i.e., if multiple memberships of the same element are discarded).

Theorem *2:* Every strongly analytic $\mathbf{LI}'$-proof π of $\Gamma \longrightarrow C$ can be transformed into a winning strategy for $\Gamma \vdash C$.

Proof: We proceed by induction on the depth of π. Again, we only show the theorem for the implicational fragment of the language.

If $\Gamma \longrightarrow C$ is an axiom the winning strategy (consisting of two nodes) is obvious. There are two cases to consider for the induction step.

π *ends with* $(\supset, r)$*:* The last part of π is of form

$$\dfrac{\begin{matrix}\vdots\\ A, \Gamma \longrightarrow B\end{matrix}}{\Gamma \longrightarrow A \supset B}\ (\supset, r)$$

By the induction hypothesis there is a winning strategy τ for $A, \Gamma \vdash B$. τ can be extended to a winning strategy for $\Gamma \vdash A \supset B$ as follows. We define a new root node; i.e., an $\mathbf{O}^{\alpha}$-node with dialogue sequent $\Gamma \vdash A \supset B$. To this root we attach an edge that leads to a new **P**-node. The corresponding move of **O** consists in granting A as an attack on $A \supset B$. Therefore the dialogue sequent at the new **P**-node is $A, \Gamma \vdash A \supset B$. We now only have to add an edge from this node to the root node of τ. This edge corresponds to **P** stating B in defense of $A \supset B$.

π *ends with* $(\supset, l)$*:* The last part of π is of form

$$\dfrac{\begin{matrix}\vdots\\ A \supset B, \Gamma \longrightarrow A\end{matrix} \qquad \begin{matrix}\vdots\\ B, A \supset B, \Gamma \longrightarrow C\end{matrix}}{A \supset B, \Gamma \longrightarrow C}\ (\supset, l)$$

By the induction hypothesis there is a winning strategy τ_1 for $A \supset B, \Gamma \vdash A$, and a winning strategy τ_2 for $B, A \supset B, \Gamma \vdash C$. Let τ_1^- be the tree, rooted in a **P**-node with dialogue sequent $A \supset B, C_p, A_p, \Gamma \vdash A$, that is obtained from τ_1 by removing its root and adding C_p to the granted formulas. We appeal to the general fact that a winning strategy for $\Pi \vdash F$ is also a winning strategy for $C, \Pi \vdash F$. Similarly let τ_2^- be the tree obtained form τ_2 that is rooted in a **P**-node with dialogue sequent $B, C_p, A \supset B, \Gamma \vdash C$. The construction of the winning strategy for $A \supset B, \Gamma \vdash C$ is illustrated in the following picture that refers to Figure 1.

$$\begin{tikzcd}
 & \mathbf{O}^{\alpha}\ A \supset B, \Gamma \vdash C \arrow[d, "\text{attack } on\ C"'] & \\
 & \mathbf{P}\ C_p, A \supset B, \Gamma \vdash C \arrow[d, "\text{attack } on\ A \supset B"'] & \\
 & \mathbf{O}^{\beta}\ C_p, A \supset B, \Gamma \vdash C \arrow[dl, "\text{attack } on\ A"'] \arrow[dr, "\text{defense } of\ A \supset B"] & \\
A_p, C_p, A \supset B, \Gamma \vdash A\ \mathbf{P} \arrow[d] & & \mathbf{P}\ B, C_p, A \supset B, \Gamma \vdash C \arrow[d] \\
\tau_1^- & & \tau_2^-
\end{tikzcd}$$

□

From now on we use the term **I**-*dialogues* to denote the dialogues, that we have just proven adequate for **I**.

IV. Parallel dialogue games

What happens to the winning powers of **P**, if we consider a game where dialogues may proceed in parallel? Of course, this question can only be answered once we have defined more precisely what we mean by 'parallel dialogue games'. Many options are open for exploration. Here, we propose and investigate just one particular form of parallelizing **I**-dialogues, that are characterized by the following features:

1. The logical rules as well as the structural rules of **I**-games remain unchanged. Indeed, ordinary **I**-game dialogues appear as sub-case of the (more general) parallel framework.
2. The proponent **P** may initiate additional **I**-dialogues by 'cloning' the dialogue sequent of one of the parallel **I**-dialogues in which it is **P**'s turn to move.
3. To win a set of parallel dialogues the proponent **P** has to win at least one of the component dialogues.
4. 'Communication' between parallel **I**-dialogues consists in **P**'s decision to merge two **I**-dialogues into one by taking the *union* of the granted formulas of the two dialogues as the granted formulas of the joint dialogue. **O**, in turn, can choose with which of the two current formulas of the merged components to continue the joint dialogue.

Features 1, 2, and 3 reflect basic decisions concerning 'parallelization'. In particular, it should be clear that we want to separate the level of individual dialogue moves strictly from the initiation of new dialogues and the interaction between dialogues. Moreover, we like to consider **P** as the (sole) 'scheduler' of parallel dialogues.

Feature 4 will be shown below to correspond closely to the central rule ('communication') of Avron's hypersequent calculus **HLC** [3] for **G**. In a sense, our parallel dialogues amount to a *computational interpretation* of (analytic) **HLC**-proofs. In particular, they are suited to illuminate Avron's bold claim that **G** (via **HLC**) allows to model communication between concurrent processes.

Before exploring 'communication' between **I**-dialogues, we will investigate parallel **I**-dialogues as specified by conditions 1, 2, and 3, alone. We will see (in Proposition 3, below) that this results in a game that does not change the winning powers of **P** over the (single) **I**-dialogue game.

Notation. A *parallel* **I**-*dialogue (P-***I***-dialogue)* is a sequence of nodes connected by moves. Each node ν is labelled by a *global state* $\Sigma(\nu)$. A global state is a non-empty finite set

$$\{\Pi_1 \vdash_{\iota 1} C_1, \ldots, \Pi_n \vdash_{\iota n} C_n\}$$

of *indexed* **I**-*dialogue sequents*. Each index ιk uniquely names one of the n elements, called *component dialogue sequents* or simply *components*, of the global state. In each of the components it is either **P**'s or **O**'s turn to move. We will speak of a **P**-component or an **O**-component, accordingly. We distinguish *internal* and *external* moves.

Internal moves combine single **I**-dialogue moves for some (possibly also none or all) of the components of the current global state. More exactly, an internal move from global state $\{\Pi_1 \vdash_{\iota 1} C_1, \ldots, \Pi_n \vdash_{\iota n} C_n\}$ to global state $\{\Pi'_1 \vdash_{\iota 1} C'_1, \ldots, \Pi'_n \vdash_{\iota n} C'_n\}$ consists in a set of indexed **I**-dialogue moves $\{\iota i_1 : \mathsf{move}_1, \ldots, \iota i_m : \mathsf{move}_m\}$ such that the indices ιi_j, $1 \leq j \leq m$, are pairwise distinct elements of $\{\iota 1, \ldots, \iota n\}$. $\Pi'_k \vdash_{\iota k} C'_k$ denotes the component corresponding to the result of move_k applied to the component indexed by ιk if $k \in \{i_1, \ldots, i_m\}$; otherwise $\Pi_k = \Pi'_k$ and $C_k = C'_k$.

External moves, in contrast to internal moves, may add or remove components of a global state, but do not change the local status (**P** or **O**) of existing components.

For now, we define only two external moves, called fork and cancel, respectively.

fork is a move by **P** and consists in duplicating one of the **P**-components of the current global state and assigning a new unique index to the added component.

cancel also is a **P**-move and consists in removing an arbitrary **P**-component from the global state.

Remark 6: fork corresponds to item 2 in the above list of basic features of our parallel dialogue games. By item 3 of the list, cancel does not affect the winning power of the proponent. (**P** cannot be forced to cancel, and therefore, in following a winning strategy, will only do so if **P** does not attempt to achieve the winning conditions at the removed component.)

The central condition in the definition of a *P*-**I**-dialogue is the following:

- for every index ι, the sequence of internal moves that refer to components indexed with ι is an **I**-dialogue.

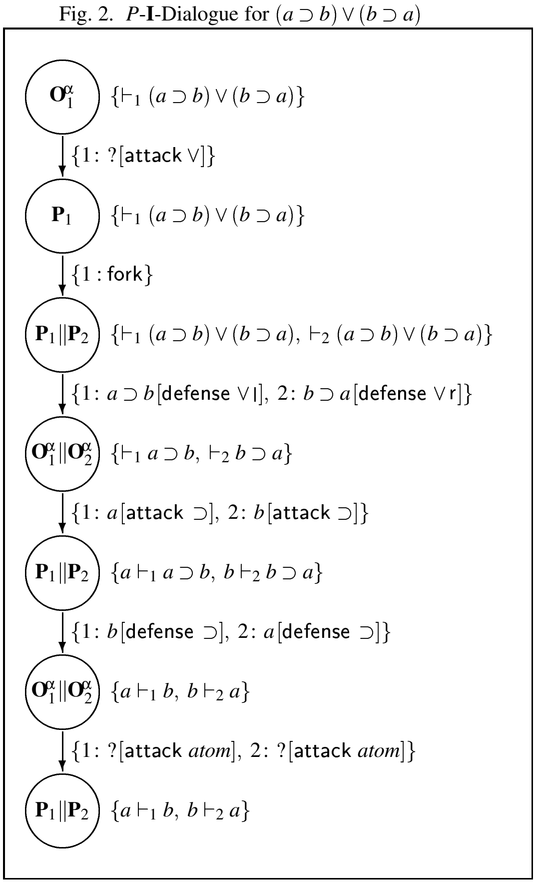
Observe that the *initial global state* $\Sigma(\nu)$—that is the state labelling the root node ν of a *P*-**I**-dialogue—consists of **O**-components only. We speak of a *P*-**I**-dialogue *for* $\Sigma(\nu)$ if ν is its root node. If $\Sigma(\nu)$ is of form $\{\vdash_\iota A\}$, we will speak of a *P*-**I**-dialogue *for A*.

There remains a trivial source of unfairness (to **P**) that we shall deal with right away: If the initial global state contains more than one component, then the opponent **O** might refuse to make the initial move for some of the components, spoiling the existence of a winning strategy for, e.g., $\{\vdash_{\iota 1} A \supset A, \vdash_{\iota 2} B\}$. (Remember that just one of the components has to satisfy the winning conditions for **P** to win the game.) We therefore require every *P*-**I**-dialogue to begin as follows:

- Every *P*-**I**-dialogue starts with an *initial segment*, which is a sequence of internal moves, each containing only first moves (by **O**) for the component dialogues, such that there is exactly one first **O**-move for each component of the initial global state. Note that, the initial segment ends in a global state that consists only of **P**-components.

Example 1: Figure 2 exhibits a *P*-**I**-dialogue for $(a \supset b) \vee (b \supset a)$, where a and b are atoms.

Fig. 2. *P*-**I**-Dialogue for $(a \supset b) \vee (b \supset a)$



Although alternative *P*-**I**-dialogues for $(a \supset b) \vee (b \supset a)$ are possible (in particular, by inserting empty internal move components) it should be clear that *all* such dialogues eventually have to lead to a state where player **P** is not winning, and where also no further move for **P** is available, that results in an essentially new global state. In the particular dialogue of Figure 2, **P** may only continue with a fork-move, which however does not change the state, if we identify dialogue sequents that only differ in their indices.

Our definition of parallel **I**-dialogues implies that the parallel version of the game may be viewed as a finite collection of state transition systems that are coordinated by referring to a global, discrete flow of time. At each time step some (possibly also none or all) of the component dialogues advance by one move. In a fork-move the component dialogues remain in their individual current states but a new dialogue, that copies the state of one of the old ones, is created. In a cancel-move one of the components (i.e., dialogues viewed as processes) is destroyed.

Observe that the definition of a P-**I**-dialogue game allows for considerable flexibility in 'implementing' the involved parallelism. We may, for example, require that *all* of the component dialogues have to advance at each time step; or, alternatively, that at most k dialogues may advance simultaneously (even if there are more than k components.) The latter option might, e.g., be understood as modeling a dialogue game were **P** and **O**, are not single persons, but rather consist of teams of k players each, and where each component dialogue is conducted by a different pair of opposite players. If, instead, we stick with a single proponent and a single opponent (i.e., $k = 1$) it seems natural to 'sequentialize' by dove-tailing the components of parallel moves. This motivates the following definition:

- A P-**I**-dialogue is called *sequentialized* if every internal move is a singleton (multi-)set.

In the proofs of Theorem 1 and 2 it was essential that full cycles of moves in a winning strategy—from a **P**-state to an **O**-state and back again to a **P**-state with an immediately responding move of **O**—correspond to a single inference step in $\mathbf{LI}'$. However, even in sequentialized P-**I**-dialogues such cycles may be interrupted, not only by internal moves that refer to other component dialogues, but also by external moves. We therefore define a P-**I**-dialogue to be *normal* if the following condition holds. Every internal move that contains a **P**-move, indexed with ιk,

- either is the last move in the component dialogue referred to by ιk,
- or else is immediately followed by another internal move with a ιk-indexed element.

Remark 7: In combination with structural rule E (see Section II), the conditions for normality can be understood as the stipulation that the proponent of a parallel dialogue game is the sole 'scheduler'. In other words—although **P** has no control over choices of **O** as long as they are immediate replies to her own previous move—**P** always determines at which dialogue component the game is to be continued.

Example 2: The P-**I**-dialogue of Figure 2 is already normal. It can easily be sequentialized, by replacing any move of form $\{1\colon \mathbf{P}\text{-}\mathsf{move}, 2\colon \mathbf{P}\text{-}\mathsf{move}'\}$ immediately followed by a move of form $\{1\colon \mathbf{O}\text{-}\mathsf{move}, 2\colon \mathbf{O}\text{-}\mathsf{move}'\}$ by the 'equivalent' sequence

$$\{1\colon \mathbf{P}\text{-}\mathsf{move}\},\ \{1\colon \mathbf{O}\text{-}\mathsf{move}\},\ \{2\colon \mathbf{P}\text{-}\mathsf{move}'\},\ \{2\colon \mathbf{O}\text{-}\mathsf{move}'\}$$

of four consecutive singleton moves.

Theorem *3:* Every finite P-**I**-dialogue δ for Σ can be translated into a sequentialized normal P-**I**-dialogue for Σ ending in the same global state as δ.

Proof: Sequentialization is easily achieved by replacing every internal move $\{\iota 1 : \mathsf{move}_1, \ldots, \iota n : \mathsf{move}_n\}$ by a sequence of internal moves $\{\iota 1 : \mathsf{move}_1\}, \ldots, \{\iota n : \mathsf{move}_n\}$. (Observe that, by the definition of an internal move, the indices ιi are pairwise different and therefore refer to different components of a global state.)

To obtain a normal dialogue, assume that δ is already sequentialized. Unless δ is already normal, it contains a subsequence of at least three moves $\{\iota 1 : \mathsf{move}_1\}, \{\iota 2 : \mathsf{move}_2\} \ldots, \{\iota n : \mathsf{move}_n\}$, where $\iota 1 = \iota n$, but $\iota i \neq \iota 1$ for all $2 \leq i < n$, and where move_n is an **I**-dialogue move by **O**, that directly reacts to move_1 by **P**. Clearly, reordering the sequence of moves into $\{\iota 1 : \mathsf{move}_1\}, \{\iota n : \mathsf{move}_n\}, \{\iota 2 : \mathsf{move}_2\}, \ldots, \{\iota[n{-}1] : \mathsf{move}_{n-1}\}$ results in the same final global state. Note that—disregarding proper notation—the moves $\{\iota 2 : \mathsf{move}_2\}$ to $\{\iota[n-1] : \mathsf{move}_{n-1}\}$ may actually also be external moves without affecting the result. The claim follows by repeating this rearrangement of moves as often as possible. □

Note [Important]. For the rest of the paper we will consider all parallel dialogues to be sequentialized and normal. Sequentialization implies that, just like for **I**-dialogues, we can speak of **P**-moves and **O**-moves of P-**I**-dialogues. (fork and cancel are **P**-moves.) Since the set parentheses are redundant in denoting moves of sequentialized dialogues, we will omit them from now on.

A P-**I**-dialogue tree τ for Σ is a rooted, directed tree with global states as nodes and edges labelled by (internal or external) moves such that each branch of τ is a P-**I**-dialogue for Σ.

A finite P-**I**-dialogue tree is called a *winning strategy* if the following condition is satisfied for every node ν:

(p) either ν has a single successor node, the edge to which is labelled by a **P**-move,

(o) or for each **O**-move that is a permissible continuation of the dialogue at global state $\Sigma(\nu)$ there is an edge leaving ν that is labelled by this move,

(l) or ν is a leaf node and at least one of the components of $\Sigma(\nu)$ fulfills the winning conditions.

Nodes satisfying *(p)* are called **P**-nodes; and nodes satisfying *(o)* are called **O**-nodes. Observe that, by normality, **P**-moves and **O**-moves strictly alternate in each branch, except for the initial segment (consisting of more than one consecutive **O**-nodes, in general) and external moves (which, in general, result in consecutive **P**-nodes.)

We have already observed that—with fork and cancel as the only additional rules—parallelization does not affect the 'winning power' of the proponent. More formally, we may state the following:

Proposition *3:* There exists a winning strategy for P-**I**-dialogues for $\{\Gamma \vdash C\}$ if and only if there exists a winning strategy for **I**-dialogues for $\Gamma \vdash C$.

To go beyond the realm of intuitionistic logic we have to allow some interaction between different component dialogues.

V. Communication between parallel dialogues

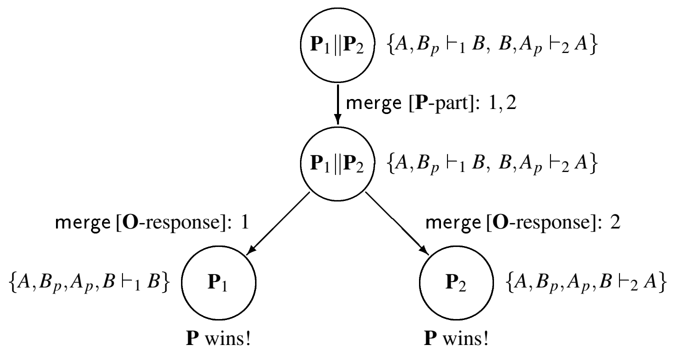
We now formalize the form of communication between **I**-dialogues that was indicated by feature 4 at the beginning of Section IV. It consists in a selection (for merging) by **P** of two **P**-components from the global state, and a consecutive choice by **O** of one of the two possible current statements for the merged component.

This results in the following additional external (two-part) dialogue rule merge.

> merge consists of two consecutive external moves:
> 1. [**P**-part] **P** picks two (indices of) **P**-components $\Pi_1 \vdash_{\iota 1} C_1$ and $\Pi_2 \vdash_{\iota 2} C_2$ from the current global state and indicates that $\Pi_1 \cup \Pi_2$ are the granted formulas of the resulting merged dialogue sequent.
> 2. [**O**-response] In response to this external **P**-move, **O** chooses either C_1 or C_2 as the current formula of the merged component, which is indexed by $\iota 1$ or $\iota 2$, correspondingly.

P-**G**-dialogues are defined exactly as *P*-**I**-dialogues, except for allowing also applications of merge. In particular, the notions of *normal* and *sequentialized* dialogues carry directly over from *P*-**I**-dialogues to *P*-**G**-dialogues.

Unlike the other external moves, merge increases the winning powers of the proponent: for *P*-**G**-dialogue games there exists a winning strategy for every instance of the linearity axiom $(A \supset B) \vee (B \supset A)$. We show this by referring to the *P*-**I**-dialogue of the previous example for $(a \supset b) \vee (a \supset b)$, as presented in Figure 2. It is not difficult to see that, even in the case of non-atomic instances of the linearity axiom, **P** can always force the dialogue to enter a global state $\{A, B_p \vdash_1 B,\ B, A_p \vdash_2 A\}$, where both components are **P**-components. Thus, using the merge-rule a winning strategy is obtained by matching the last node in Figure 2 with the first node of the following tree:



VI. Adequateness of *P*-G-dialogues for G

To match winning strategies of parallel dialogues with proofs, we have to switch from sequent to *hypersequent* calculi.

Hypersequent calculi arise by generalizing standard sequent calculi to refer to whole contexts of sequents instead of single sequents. In our context, a hypersequent is defined as a finite, non-empty multiset of **LI**-sequents, called *components*; written in form

$$\Gamma_1 \longrightarrow C_1 \mid \ldots \mid \Gamma_n \longrightarrow C_n.$$

The symbol "|" is intended to denote disjunction at the meta-level.

Like ordinary sequent calculi, hypersequent calculi consist in axioms as well as logical and structural rules. The latter are divided into *internal* and *external rules*. The internal structural rules deal with formulas within components, while the external ones manipulate whole components of a hypersequent. The standard **external structural rules** are external weakening and external contraction:

$$\frac{\mathcal{H}}{\Pi \longrightarrow C \mid \mathcal{H}}\ (E\text{-weak.}) \qquad \frac{\Pi \longrightarrow C \mid \Pi \longrightarrow C \mid \mathcal{H}}{\Pi \longrightarrow C \mid \mathcal{H}}\ (E\text{-contr.})$$

Axioms, logical rules and internal structural rules are essentially the same as in sequent calculi. The only difference is the presence of a side hypersequent $\mathcal{H}$, representing a (possibly empty) hypersequent. For instance, the hypersequent version of the **LI**$'$-rule $(\supset, l)$ is

$$\frac{A \supset B, \Pi \longrightarrow A \mid \mathcal{H} \quad B, A \supset B, \Pi \longrightarrow C \mid \mathcal{H}}{A \supset B, \Pi \longrightarrow C \mid \mathcal{H}}\ (\supset, l)$$

For hypersequent calculi it is possible to define additional external structural rules which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi compared to ordinary sequent calculi.

Let us denote with **HLI**$'$ the hypersequent version of **LI**$'$. A hypersequent calculus for Gödel-Dummett logic **G** is obtained by adding to **HLI**$'$ the following version of Avron's 'communication rule'[2]

$$\frac{\Pi_1, \Pi_2 \longrightarrow C_1 \mid \mathcal{H} \qquad \Pi_1, \Pi_2 \longrightarrow C_2 \mid \mathcal{H}}{\Pi_1 \longrightarrow C_1 \mid \Pi_2 \longrightarrow C_2 \mid \mathcal{H}}\ (\mathrm{com})$$

We use **HLG**$'$ to denote the resulting variant of Avron's **HLC**. An **HLG**$'$-proof is called *strongly analytic* if no internal structural rules have been applied.

Theorem *4:* **HLG**$'$ is sound and complete for **G**. Moreover, **HLG**$'$-proofs can be translated into strongly analytic proofs.

Proof: Follows from the soundness and cut-free completeness of **HLC** (see [3]). □

Let us denote by $\langle \mathcal{H} \rangle$ the global state $\{\Pi_1 \vdash_{\iota 1} C_1, \ldots, \Pi_n \vdash_{\iota n} C_n\}$ if $\mathcal{H}$ is the hypersequent $\Pi_1 \longrightarrow C_1 \mid \ldots \mid \Pi_n \longrightarrow C_n$. (We do not care about the names of indices as long as they are distinct.) Conversely, the hypersequent corresponding to a global state Σ is denoted by $[\Sigma]$.

Theorem *5:* Every winning strategy τ for sequentialized normal *P*-**G**-dialogues with initial global state Σ can be transformed into an **HLG**$'$-proof of $[\Sigma]$.

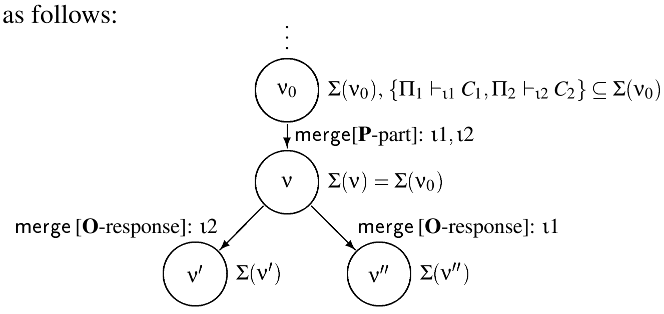
Proof: Again, we show by induction on the depth of τ that for every **P**-node of τ labelled with global state Σ', there is an **HLG**$'$-proof of $[\Sigma']$. Since the branches of τ are normal and sequential dialogues, edges of τ that correspond to *internal moves* are translated into corresponding inference steps using logical rules of **HLG**$'$, exactly as in the proof of Theorem 1. (Remember that the logical rules of **HLG**$'$ are identical to those of **LI**$'$ except for the context of side hypersequents.)

It remains to show that also *external moves* translate into **HLG**$'$-inferences. Suppose $\nu \longrightarrow \nu'$ is an edge of τ which corresponds to an external move emove, such that all edges nodes below ν' denote internal moves. There are three cases:

1. emove *is an instance of* fork: In this case the global state at $\Sigma(\nu')$ is like $\Sigma(\nu)$ except for an additional dialogue sequent $\Gamma \vdash_{\iota i} A$, where the index ιi is not yet used at ν, but where, for some ιj, $\Gamma \vdash_{\iota j} A$ is an element of $\Sigma(\nu)$. By the induction hypothesis there

[2] This rule is equivalent to the original one in [3]. It has been suggested by G. Mints.

as an **HLG**$'$-proof $\pi_{\nu'}$ of $[\Sigma(\nu')]$. Clearly, the required **HLG**$'$-proof of $[\Sigma(\nu)]$ is obtained from $\pi_{\nu'}$ by adding an appropriate application of external contraction (E-contr.) as the last inference.
2. emove *is an instance of* cancel: In this case $\Sigma(\nu')$ arises from $\Sigma(\nu)$ by removing a **P**-component. The argument is like in the case above, except for adding an appropriate application of external weakening (E-weak.).
3. emove *is an instance of* merge: The relevant part of τ looks as follows:



where $\Sigma(\nu') = \Sigma(\nu) - \{\Pi_1 \vdash_{\iota 1} C_1, \Pi_2 \vdash_{\iota 2} C_2\} \cup \{\Pi_1, \Pi_2 \vdash_{\iota 1} C_1\}$ and $\Sigma(\nu'') = \Sigma(\nu) - \{\Pi_1 \vdash_{\iota 1} C_1, \Pi_2 \vdash_{\iota 2} C_2\} \cup \{\Pi_1, \Pi_2 \vdash_{\iota 2} C_2\}$. By induction hypothesis there exist **HLG**$'$-proofs $\pi_{\nu'}$ and $\pi_{\nu''}$ of $[\Sigma(\nu')]$ and $[\Sigma(\nu'')]$, respectively. Clearly, $\pi_{\nu'}$ and $\pi_{\nu''}$ can be joined by an application of (com) to obtain the required proof of $[\Sigma(\nu_0)]$. $\square$

Theorem *6:* Every strongly analytic **HLG**$'$-proof π of the hypersequent $\mathcal{H}$ can be transformed into a winning strategy τ for sequentialized normal P-**G**-dialogues for $\langle \mathcal{H} \rangle$.

Proof: Since π is strongly analytic there are no applications of internal structural rules. The logical rules of **HLG**$'$ translate into full cycles of (internal) moves, exactly as in the proof of Theorem 2. It remains to show that also applications of external structural rules correspond to (external) P-**G**-dialogue moves. Without loss of generality we assume that the last inference of π is the only application of an external structural rule in π. There are three cases:
1. π ends with an external contraction (E-contr.): By induction hypothesis there exists a winning strategy τ' for, say, $\{\Pi \vdash_{\iota 1} C\} \cup \{\Pi \vdash_{\iota 2} C\} \cup \langle \mathcal{H} \rangle$, which has to be extended to one for $\{\Pi \vdash_{\iota 1} C\} \cup \langle \mathcal{H} \rangle$. This can be achieved by inserting a new edge corresponding to an appropriate instance of the fork-move immediately after the initial segment of each branch of τ'; and removing the first moves that refer to $\iota 2$. (Observe that the new edges connect **P**-nodes labelled with same global state; and therefore do not affect the rest of the strategy.)
2. π ends with external weakening (E-weak.): Similarly, as in case 1, the corresponding winning strategy is obtained by adding an appropriate instance of the cancel-move.
3. π ends in an application of communication rule:

$$\frac{\dfrac{\vdots}{\Pi_1, \Pi_2 \longrightarrow C_1 \mid \mathcal{H}} \qquad \dfrac{\vdots}{\Pi_1, \Pi_2 \longrightarrow C_2 \mid \mathcal{H}}}{\Pi_1 \longrightarrow C_1 \mid \Pi_2 \longrightarrow C_2 \mid \mathcal{H}} \ (\mathrm{com})$$

By induction hypothesis there exist winning strategies τ_1 and τ_2 for $\Sigma_1 = \{\Pi_1, \Pi_2 \vdash_{\iota 1} C_1\} \cup \langle \mathcal{H} \rangle$ and $\Sigma_2 = \{\Pi_1, \Pi_2 \vdash_{\iota 2} C_2\} \cup \langle \mathcal{H} \rangle$, respectively. We define a new dialogue tree with initial global state $\Sigma = \{\Pi_1 \vdash_{\iota 1} C_1\} \cup \{\Pi_2 \vdash_{\iota 2} C_2\} \cup \langle \mathcal{H} \rangle$ as follows: (1) We first construct an 'initial dialogue tree' τ_0, whose branches consist of all possible initial segments of dialogues for Σ. Observe that each possible initial segment of a dialogue for Σ_1 or for Σ_2 is contained in some branch of τ_0 as a subsequence. (2) Let τ^m be the tree (rooted in ν_0 and with leaf nodes ν' and ν''), that was presented in case 3 of the proof of Theorem 5. To each leaf node ν_i of τ_0 we attach a copy of τ^m by identifying ν_i with ν_0. Call the resulting tree τ_0^m. (3) Let ν_i' be the leaf node of τ_0^m that corresponds to ν' of the copy of τ^m attached to ν_i. We attach to ν_i' a copy of the subtree of τ_1, that is rooted at the last node of the initial segment for Σ_1 that is contained in the branch of τ_0 that ends in ν_i. We proceed analogously for ν'' (referring to τ_2 and Σ_2). Step (3) is repeated for all leaf nodes of τ_0^m.
It is straightforward to check that the resulting dialogue tree is indeed a winning strategy (for sequentialized normal dialogues) for Σ. $\square$

VII. Conclusion

We have shown that a certain way of parallelizing Lorenzen style dialogue games for intuitionistic logic is adequate for characterizing Gödel-Dummett logic. We like to consider this result only as a starting point for investigating many related topics; and therefore conclude with a list of such questions, that we intend to answer positively in future work:

- Are other intermediate logics characterizable in similar form?
- Can strategies be used to model *proof search*?
- Can sub-structural logics serve as a basis in exchange for **I**?
- Can game theoretic issues, like determinedness and 'information hiding', be fruitfully imported to dialogue games?

References

[1] S. Abramsky, R. Jagadeesan: Games and Full Completeness for Multiplicative Linear Logic. *J. Symbolic Logic*, 59(2) (1994), 543–574.
[2] A. Blass: A Game Semantics for Linear Logic. *Annals of Pure and Applied Logic*, 56(1992), 183–220.
[3] A. Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4(1991), 225-248.
[4] M. Baaz, A. Ciabattoni, C.G. Fermüller. A Natural Deduction System for Intuitionistic Fuzzy Logic. In: A. Di Nola, G. Gerla (eds), *Lectures on Soft Computing and Fuzzy Logic*, Physica-Verlag, 2000, 1–18.
[5] M. Dummett. A propositional calculus with denumerable matrix. *J. Symbolic Logic*, 24(1959), 97–106.
[6] J.M. Dunn, R.K. Meyer. Algebraic completeness results for Dummett's *LC* and its extensions. *Z. Math. Logik Grundlagen Math.*, 17 (1971), 225–230.
[7] W. Felscher: Dialogues, Strategies, and Intuitionistic Provability. *Annals of Pure and Applied Logic*, 28(1985), 217–254.
[8] W. Felscher: Dialogues as Foundation for Intuitionistic Logic. In: D. Gabbay and F. Günther (eds.), *Handbook of Philosophical Logic, III*, Reidel, 1986, 341–372.
[9] K. Gödel: Zum intuitionistischen Aussagenkalkül. *Anz. Akad. Wiss. Wien*, 69(1932), 65-66.
[10] P. Hájek, P.: *Metamathematics of Fuzzy Logic*. Kluwer, 1998.
[11] E.C.W. Krabbe: Formal Systems of Dialogue Rules. *Synthese*, 63(1985), 295–328.
[12] E.C.W. Krabbe: Dialogue Sequents and Quick Proofs of Completeness. In: J.Ph. Hoepelman (ed.), *Representation and Reasoning*. Max Niemeyer Verlag, 1988, 135–140.
[13] P. Lorenzen: Logik und Agon. In: *Atti Congr. Internat. di Filosofia*, Vol. 4 (Sansoni, Firenze, 1960), 187–194.
[14] S. Rahman: *Über Dialoge, Protologische Kategorien und andere Seltenheiten*. Europäische Hochschulschriften, Peter Lang, 1993.
[15] G. Takeuti, T. Titani: Intuitionistic fuzzy logic and intuitionistic fuzzy set theory. *J. Symbolic Logic*, 49(1984), 851-866.
[16] A. Visser: On the Completeness Principle: A Study of Provability in Heyting's Arithmetic. *Annals of Math. Logic*, 22(1982), 263-295.