A Dialogue Game for Intuitionistic Fuzzy Logic Based on Comparisons of Degrees of Truth

Christian G. Fermüller Norbert Preining
Institut 185, Logic and Theory Group
Vienna University of Technology
Favoritenstr. 9, 1040 Vienna, Austria

E-mail: `[chrisF,preining]@logic.at`

Abstract: A dialogue game for fuzzy logic, based on the comparison of truth degrees, is presented. It is shown that the game is adequate for $G_\infty^\triangle$, i.e., intuitionistic fuzzy logic enriched by the projection operator $\triangle$. Any given counter-model to a formula can be used to construct a winning strategies for one of the players, called Opponent. Conversely, counter-models can be extracted from each winning strategy for Opponent. Winning strategies for the other player, Proponent, correspond to proofs of validity. The systematic construction of so-called complete dialogue trees can be viewed as tableau style proof search procedure.

Keywords: foundations of fuzzy logic, dialogue games, model construction

1 Introduction

Fuzzy logic, in general, is based on the observation that truth often seems to come in *degrees*. Suppose that Chris is 175 cm tall and weighs 66kg, and suppose that Norbert is 183cm tall and weighs 73kg. Consider the statements:

C = 'Chris is a big man'

and

N = 'Norbert is a big man'.

In many context it will be inadequate to evaluate these statements as either being absolutely true or absolutely false. However, it seems acceptable to call N 'truer' than C. Similarly, we may reasonably claim that N is at least as true as C. (If putting 'true' into comparative mode causes offense, one may replace 'truer' by 'true to a higher degree', which in turn may be explained, e.g., as: 'stated to be true with a higher level of confidence'.)

Motivated by such examples, one usually models degrees of truth (truth values) by real numbers from the closed unit interval $[0,1]$, with 1 referring to 'absolute truth' and 0 referring to absolute falsity. The natural order $<$ on $[0,1]$ is intended to reflect the order on all possible degrees of truth. Moreover, one frequently assumes that the semantics of conjunction is modeled by an appropriate t-norm; and that the other connectives can be derived from this t-norm as well, following some general principles on defining truth functional connectives.

It turns out, that all such t-norm based logics can be represented as ordinal sums of just three basic t-norm based logics, namely Łukasiewicz logic [19], Product logic [17] and intuitionistic fuzzy logic, also called Gödel logic G_∞, respectively. (See, e.g., [16] for details.) From these three logics, intuitionistic fuzzy logic G_∞ can be singled out as the only one where the definition of truth functions can be stated without any reference to arithmetical operations on truth values, but solely in reference to the order of truth values, including the endpoints 0 and 1.

Given the importance of G_∞ it is not surprising that many proof systems for it have been presented in the literature (see, e.g., [1, 2, 3, 4, 12, 8, 20].) In [14] (see also [13]) we have presented a characterization of Gödel logic (i.e., intuitionistic fuzzy logic) in terms of parallelized Lorenzen style dialogue games. Among other things, the results of [14] provide an alternative computational interpretation for Avron's hypersequent calculus **HLC** [2] for logic G_∞. However, the dialogue game of [14] does not relate in any direct manner to the underlying semantics of G_∞ in terms of comparisons of degrees of truth.

Motivated by the challenge to provide a formal foundation for (analytic) reasoning in intuitionistic fuzzy logic that refers directly to its semantics in terms of comparisons of degrees of truth, we present a dialogue game for $G_\infty^\triangle$ that is quite different from the ones described in [14] and [13], but is related to the proof systems presented in [8] and [7]. We will show that our dialogue game not only is sound and complete for $G_\infty^\triangle$, but moreover provides a frame for computionally adequate, tableau style proof search in intuitionistic fuzzy logic.

The paper is organized as follows. After shortly reviewing the basic definitions for logic G_∞ we look (in Section 3) at a way to express simple statements, called *assertions*, that express the relative truth of formulas. This will also motivate us to extend G_∞ to logic $G_\infty^\triangle$ by including a natural projection operator in the set of connectives. In Section 4 we present our dialogue game for $G_\infty^\triangle$. We also provide a formal definition of *dialogue trees*. Winning strategies for the two players – Proponent **P** and Opponent **O** – will formally appear as special cases of dialogue trees. In Section 5 we show how to construct a winning strategy for **O** in a game starting with **P**'s claim that a formula F is valid, given a counter-model to F. The converse operation, i.e., the extraction of a counter-model from a given winning strategy for **O** is presented in Section 6. In Section 7 we show that winning strategies for **P** can be viewed as proofs of the validity of the formula which **P** states at the beginning of the dialogue. Rather than simply relying on Zermelo's celebrated theorem [23] about the determinedness of zero-sum two-person games of finite depth, we present a self-contained proof of this central fact. In Section 9 we explain the relation between winning strategies for **P** and derivations in an extended version of the proof system $\mathbf{R}G_\infty^\triangle$ presented in [7]. We conclude by pointing out open problems and tasks for future work.

2 Intuitionistic fuzzy logic G_∞

The logic G_∞ arguably is one of the most interesting non-classical logics. It is known under various different names, including Gödel logic (see, e.g., [16]), Dummett's **LC** (after [10]), Gödel-Dummett logic, and intuitionistic fuzzy logic (see [21]). This corresponds to the fact that it naturally turns up in different fields in logic and computer science. Already in the 1930's Gödel [15] used it (implicitly) to shed light on aspects of intuitionistic logic; later Dunn and Meyer [11] pointed out its relevance for relevance logic; Visser [22] employed it in investigations of the provability logic of Heyting arithmetic; and eventually it was recognized as one of the most useful species of 'fuzzy logic' (see [16, 21]).

Considered as a fuzzy logic, (propositional) G_∞ is characterized by *interpretations* v of the propositional variables in the real closed unit interval $[0,1]$, and the following extension of interpretations to compound formulas:

$$v(A \wedge B) = \min(v(A), v(B)) \qquad v(\bot) = 0$$
$$v(A \vee B) = \max(v(A), v(B)) \qquad v(\top) = 1$$
$$v(A \supset B) = \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ v(B) & \text{otherwise} \end{cases}$$

$\neg A$ is defined as $A \supset \bot$; i.e., $v(\neg A) = 1$ if $v(A) = 0$ and $v(\neg A) = 0$ in all other cases. Propositional variables, as well as $\bot$ and $\top$ are called *atoms*. All other formulas are *compound*. As usual, a formula F is called *valid* if $v(F) = 1$ for all interpretations v. If $v(F) < 1$ then v is called a *counter-model* to F.

As was pointed out, e.g., in [16], G_∞ is one of just three fundamental fuzzy logics, when 'fuzzy logics' are stipulated to be truth-functional, t-norms based logics. All other t-norm based logics can be represented as defined piecewise on subintervals of $[0,1]$, such that on each interval the logic is identical to either Gödel logic, or Łukasiewicz logic [19], or Product logic [17]. (See [16] for a detailed exposition.)

3 Comparing degrees of truth

As we have just seen, G_∞ only refers to the order relation on $[0,1]$ and its endpoints for the definition of truth functions. In contrast, the corresponding definitions for Łukasiewicz logic and Product logic refer to arithmetical operations on $[0,1]$. However, we cannot fully express the natural order on truth degrees within the object language of G_∞ itself. More exactly (as can be easily checked by induction on the complexity of formulas) we have:

Proposition 1 *In G_∞ there exists no formula F, with occurrences of propositional variables A and B, such that for all interpretations: $v(F) = 1$ if $v(A) < v(B)$ and $v(F) = 0$ otherwise. The same holds if $<$ is replaced by $\leq$.*

Observe that $F = (B \supset A) \supset B$ 'almost' expresses $<$ in the sense that: $v(F) = 1$ iff either $v(A) < v(B)$ or $v(A) = v(B) = 1$. However the reference to the special case $v(A) = v(B) = 1$ cannot be eliminated. This motivates the following definitions of truth functions for additional unary connectives $\triangle$ and $\bigtriangledown$, as follows (compare [5]):

$$v(\triangle A) = \begin{cases} 1 & \text{if } v(A) = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$v(\bigtriangledown A) = \begin{cases} 1 & \text{if } v(A) \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

$\bigtriangledown A$ can be defined in G_∞ as $\neg\neg A [= (A \supset \bot) \supset \bot]$; whereas $\triangle A$ cannot be defined in G_∞.

To enhance the expressibility of G_∞ we extend it to $G_\infty^\triangle$ by including the $\triangle$-connective (like in [5]). Moreover we will consider statements, called *assertions*, of form $A < B$ and $A \leq B$, for formulas A, B of $G_\infty^\triangle$, denoting $v(A) < v(B)$ and $v(A) \leq v(B)$, respectively. We do not introduce $<$ and $\leq$ as connectives in $G_\infty^\triangle$ (i.e., $<$ and $\leq$ will not occur in nested form), but we will make assertions our basic units of reasoning.

A finite set of assertions $\{A_1 \lhd_1 B_1, \ldots, A_n \lhd_n B_n\}$, where $\lhd_i \in \{<, \leq\}$ for $1 \leq i \leq n$, is said to be *satisfied* by a valuation v if $v(A_i) \lhd_i v(B_i)$ holds for all $1 \leq i \leq n$. The set is called *unsatisfiable* if no such valuation exists.

4 Comparison based dialogues

Already in the 1950s Paul Lorenzen (see, e.g., [18]) and his students developed an important approach to face the challenge of justifying and deriving logical rules from first principles about correct reasoning. Lorenzen's main idea is to identify logical validity a formula F with the existence of a winning strategy for a *proponent* in an idealized confrontational dialogue game, in which **P** tries to uphold F against corresponding attacks on F by an *opponent* **O**. (The roles of **P** and **O** may switch in the course of a dialogue.)

In [14] (see also [13]) we have presented a characterization of Gödel logic (i.e., intuitionistic fuzzy logic) in terms of parallelized Lorenzen style dialogue games. Among other things, the results of [14] provide an alternative computational interpretation for Avron's hypersequent calculus **HLC** [2] for logic G_∞. However, the relation between the rules of the game and the truth degree based semantics of G_∞ (as defined in Section 2) was left unclear. In particular, no (direct) connection between counter-models to F and winning strategies for the opponent in a dialogue game for F was provided. It remains open whether any direct connection of this kind exists at all for Lorenzen style dialogue games.

Here, we aim at a characterization of logical validity for intuitionistic fuzzy logic based on the comparison of truth degrees. For this purpose, we define a quite different type of game, which is founded on the idea that any logical connective $\circ$ of G_∞ can be characterized via an adequate response by a player **X** to player **Y**'s attack on **X**'s claim that a statement of form $(A \circ B) \lhd C$ or $C \lhd (A \circ B)$ holds, where $\lhd$ is either $<$ or $\leq$. (The resulting dialogue rules are related to the calculi of [8, 7].)

An assertion $F \lhd G$ is *atomic* if both, F and G, are atoms; otherwise it is a *compound assertion*. Atomic assertions of form $A < A$, $A < \bot$, $\top < A$ or $\top \leq \bot$ are called *elementary contradictions*. An *elementary order claim* is a set of two assertions of form $\{E \lhd_1 F, F \lhd_2 G\}$, where E, F, and G are atoms, and $\lhd_1, \lhd_2 \in \{<, \leq\}$.

Following the tradition alluded to above, we call the player that initially claims the validity of a chosen formula the *Proponent* **P**, and the player that tries to refute this claim the *Opponent* **O**. The dialogue game proceeds in rounds as follows:

1. A dialogue starts with **P**'s claim that a formula F is valid. **O** answers to this move by contradicting this claim with the assertion $F < \top$.

2. Each following round consists in two steps:

   (i) **P** either attacks a compound assertion or an elementary order claim, contained in the set of assertions that have been made by **O** up to this state of the dialogue, but that have not yet been attacked by **P**.

   (ii) **O** answers to the attack by adding a set of assertions according to the rules of Table 1 (for compound assertions) and Table 2 (for elementary order claims).

3. The dialogue ends with **P** as winner if **O** has asserted an elementary contradiction. Otherwise, **O** wins if there is no further possible attack for **P**.

Remark. Instead of considering the rules of Table 1 and 2 as derived from the truth functions for $G_\infty^\triangle$, one may argue that the dialogue rules are derived from fundamental principles about reasoning in a truth functional, order based fuzzy logic.

Consider the example of conjunction. We contend that anyone who claims '$A \wedge B$ is at least as true as C' (for any concrete statements A, B, and C) has to be prepared to defend the claim that 'A is at least as true as C' and the claim that 'B is at least as true as C'. On the other hand, the claim that 'C is at least as true as $A \wedge B$', arguably, should be supported either by 'C is at least as true as A' or by 'C is at least as true as B'. (Likewise, if we replace 'at least as true' by 'truer than'.) One may then go on to argue

Table 1: Rules for connectives

P attacks:	**O** asserts as answer:
$A \wedge B \lhd C$	$\{A \lhd C\}$ or $\{B \lhd C\}$
$C \lhd A \wedge B$	$\{C \lhd A, C \lhd B\}$
$A \vee B \lhd C$	$\{A \lhd C, B \lhd C\}$
$C \lhd A \vee B$	$\{C \lhd A\}$ or $\{C \lhd B\}$
$A \supset B < C$	$\{B < A, B < C\}$
$C < A \supset B$	$\{C < B\}$ or $\{A \leq B, C < \top\}$
$A \supset B \leq C$	$\{\top \leq C\}$ or $\{B < A, B \leq C\}$
$C \leq A \supset B$	$\{A \leq B\}$ or $\{C \leq B\}$
$\triangle A < C$	$\{A < \top, \bot < C\}$
$C < \triangle A$	$\{\top \leq A, C < \top\}$
$\triangle A \leq C$	$\{A < \top\}$ or $\{\top \leq C\}$
$C \leq \triangle A$	$\{\top \leq A\}$ or $\{C \leq \bot\}$

In the first four lines, $\lhd$ denotes either $<$ or $\leq$, consistently throughout each line. Assertions, which involve a choice of **O** in the answer (indicated by 'or') are called *or-type* assertions. All other assertions are of *and-type*.

Table 2: Rules for elementary order claims

P attacks:	**O** asserts as answer:
$\{A \leq B, B \leq C\}$	$\{A \leq C\}$
$\{A < B, B \lhd C\}$	$\{A < C\}$
$\{A \lhd B, B < C\}$	$\{A < C\}$

where $\lhd$ is either $<$ or $\leq$.

that this reading of the rules for $\wedge$ in Table 1 completely determines correct reasoning about assertions of this form. Form this assumption, one can *derive* that $v(A \wedge B) = \min(v(A), v(B))$ is the only adequate definition for the semantics of conjunction in this setting.

The case for disjunction is very similar. Implication, as usual, is more controversial. However, it is easy to see that there are hardly any reasonable alternatives to our rules, if the truth of any assertion involving a formula $A \supset B$ should only depend on the relative degree of truth of A and B (but should not depend on the result of an arithmetical operations that had to be performed on the values of A and B, respectively). In fact, it can be shown that the constraints for the truth function of implication, that are expressed by the rules in Table 1, are unavoidable if the resulting fuzzy logic is to enjoy Craig's interpolation property. (See [16].)

To assist precise argumentation we introduce the following formal notions for dialogues and winning strategies.

A *dialogue tree* (on F) is a finitely branching, downward rooted tree. Along each branch the nodes of the tree alternate strictly between **P**-*nodes* and **O**-*nodes*. The **P**-nodes and **O**-nodes correspond to dialogue game moves by **P** and **O**, respectively. Every **P**-node (except the root node) is labeled either by a single compound assertion or by an elementary order claim, that **P** chooses to attack in the corresponding state of the dialogue. Every **O**-nodes is labeled by the set of as-yet-unattacked assertions that **O** has made up to the corresponding state of the dialogue. We use $\Lambda(\rho)$ to denote the *label* of node ρ. The root is a **P**-node, labeled with the formula F; it has a single child node ρ, which is an **O**-node with $\Lambda(\rho) = \{F < \top\}$.

The labels for the remaining nodes are defined inductively as follows:

- If ω is an **O**-node then every child node π of ω is a **P**-node. $\Lambda(\pi)$ is either a compound assertion $\in \Lambda(\omega)$ or an elementary order claim $\subseteq \Lambda(\omega)$.

- If π is a **P**-node then every child node of π is an **O**-node. In the following let ω' be the parent node of π:

  - If $\Lambda(\pi)$ is a compound assertion of and-type, then π has a single child node ω with $\Lambda(\omega) = (\Lambda(\omega') - \{\Lambda(\pi)\}) \cup \alpha(\pi)$, where $\alpha(\pi)$ is the answer to the attack on $\Lambda(\pi)$ by **P**, according to Table 1.

  - If $\Lambda(\pi)$ is a compound assertion of or-type, then π has either ω_1 or ω_2 or both, as child nodes, where $\Lambda(\omega_i) = (\Lambda(\omega') - \{\Lambda(\pi)\}) \cup \beta_i(\pi)$, for $i \in \{1,2\}$; $\beta_1(\pi)$ and $\beta_2(\pi)$ stand the two sets of assertions, respectively, that **O** may state according to Table 1 as answer to **P**'s attack on $\Lambda(\pi)$.

  - If $\Lambda(\pi)$ is an elementary order claim (contained in $\Lambda(\omega')$) that has not yet been attacked by **P**, then π has a single child node ω with $\Lambda(\omega) = \Lambda(\omega') \cup \alpha(\pi)$, where $\alpha(\pi)$ is the answer to the attack on the order claim $\Lambda(\pi)$ by **P**, according to Table 2.

Observe that the branches of a dialogue tree correspond to *dialogues*, as defined informally earlier.

Proposition 2 *Every dialogue tree is finite.*

Proof: Since dialogue trees are finitely branching the claim follows from the following observations:
(1) No new propositional variables are introduced by the rules of Table 1 and 2.

(2) In each move by player **O** that answers to **P**'s attack on a compound assertion $G \lhd F$, this assertion is replaced by (either one or two) assertions, which are strictly smaller, if we measure the size of an assertion by the number of occurrences of connectives in it.
(3) There are only finitely many different atomic assertions that can occur in a dialogue tree. Since each elementary order claim can only be attacked once in a dialogue, there are only finitely many such attacks possible in each dialogue (branch of the tree). $\square$

An **O**-node ω, whose label $\Lambda(\omega)$ contains an elementary contradiction, is called a *winning node for* **P**. If $\Lambda(\omega)$ consists of atomic assertions only, but does not contain an elementary contradiction, then it is a *winning node for* **O** if all elementary order claims $\subseteq \Lambda(\omega)$ have already been attacked by **P**.

Note that the winning conditions for **P** and **O** are complementary in the following sense. If a dialogue cannot be continued at an **O**-node ω – i.e., if $\Lambda(\omega)$ consists only of atomic assertions and is closed under applications of rules for elementary order claims – then ω is either a winning node for **P** or for **O**, but not for both players simultaneously.

A *winning strategy for* **O** is a dialogue tree, where all leaf nodes are winning nodes for **O**, and which satisfies the following conditions:

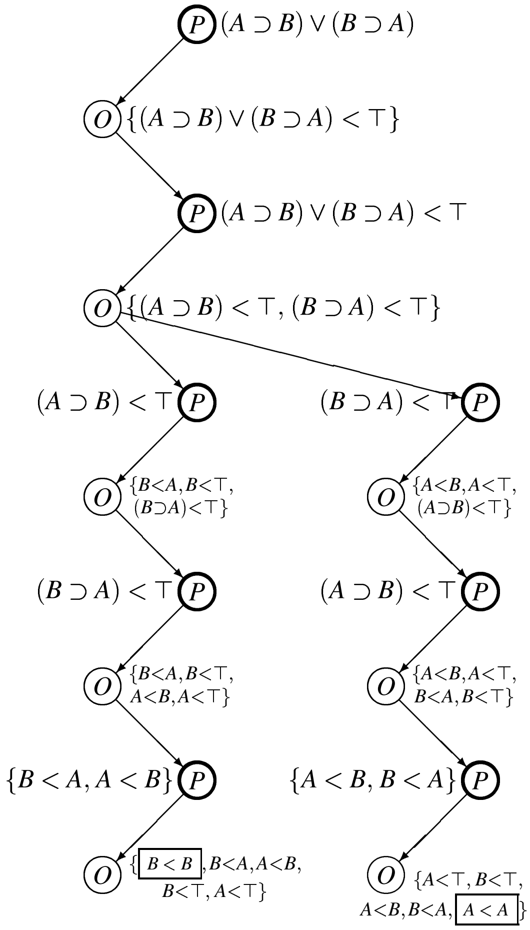
1. Each **P**-node has exactly one child node.
2. Each non-leaf **O**-node ω has a child **P**-node for each $F \lhd G \in \Lambda(\omega)$, as well as for each elementary order claim $\subseteq \Lambda(\omega)$, that has not yet been attacked by **P**. (The child nodes are labeled accordingly.)

A *winning strategy for* **P** is a dialogue tree, which satisfies the following conditions:

1. Each **O**-node is either a winning node for **P**, or has exactly one child node.
2. Each **P**-node labeled by an assertion of or-type has exactly two child nodes; all other **P**-nodes have exactly one child node.

Winning strategies for a player **X** in a zero-sum two-person game are more commonly described as *functions* assigning a move by **X** to each state of the game, taking into account all possible moves of the other player. Observe that our 'tree form' of a winning strategy just describes the corresponding function in a manner that makes the step-wise evolution of permissible dialogues more explicit.

We illustrate the concept of a dialogue tree by sketching a winning strategy for **P** on the formula $(A \supset B) \vee (B \supset A)$:



Since all dialogues are finite (Proposition 2) and the winning conditions for **P** and **O** are complementary, it already follows from Zermelo's well-known Theorem on two-person games [23] that for every F there is either a winning strategy for **P** or a winning strategy for **O**. However, we want to relate winning strategies for **O** to counter-models, and winning strategies for **P** to proofs in calculi for $G^{\triangle}_{\infty}$, more explicitly.

5 Winning strategies for Opponent induced by counter-models

We show the soundness of our dialogue game with respect to logic $G^{\triangle}_{\infty}$ in the following form: any given counter-model to a formula F can be used by player **O** to choose his answer to any attack by **P** in such a way that **O** is guaranteed to win all corresponding dialogues starting with **P**'s claim that F is valid.

Theorem 1 *For every counter-model to a formula F of $G^{\triangle}_{\infty}$ one can construct a winning strategy for* **P** *in the dialogue game on F.*

Proof: Let v be the given counter-model, i.e., an interpretation for which $v(F) < 1$. We construct a dialogue tree T for F, that is, a winning strategy for **P**, as follows.

The root of T is a **P**-node labeled by F, followed by an **O**-node labeled by $\{F < \top\}$. The child nodes of all **O**-nodes in T are already determined by the definition of a winning strategy for **P**: there is one child node for each possible move by **P**.

It remains to pick a single child **O**-node ω to each **P**-node π, and to show that $v(F) \lhd v(G)$ for all assertions $F \lhd G \in \Lambda(\pi)$. The proof is by induction on the length of a path in T from the root node to ω. We proceed by cases according to the form of $\Lambda(\pi)$ (i.e., the compound assertion or elementary order claim attacked by **P** in the move corresponding to π). Let ω' denote the parent node of π. We only present some of the cases; the others are very similar.

$\Lambda(\pi) = A \wedge B < C$**:** By induction hypothesis $\Lambda(\omega')$ is satisfied by v. Since $\Lambda(\pi) \in \Lambda(\omega')$, we have: $v(A \wedge B) < v(C)$. Since $v(A \wedge B)$ is defined as $\min(v(A), v(B))$ it follows that either
(i) $v(A) < v(C)$, or
(ii) $v(B) < v(C)$,
(or both). In case *(i)* we set $\Lambda(\omega) = (\Lambda(\omega') - \{A \wedge B < C\}) \cup \{A < C\}$. In case *(ii)* we set $\Lambda(\omega) = (\Lambda(\omega') - \{A \wedge B < C\}) \cup \{B < C\}$. (If both, *(i)* and *(ii)*, hold we can choose freely among these two options.) In any case, $\Lambda(\omega)$ is satisfied by v.

$\Lambda(\pi) = A \vee B \leq C$**:** By induction hypothesis $\Lambda(\omega')$ is satisfied by v. Since $\Lambda(\pi) \in \Lambda(\omega')$, we have: $v(A \vee B) \leq v(C)$. Since $v(A \vee B) = \max(v(A), v(B))$ this implies $v(A) \leq v(C)$ as well as $v(B) \leq v(C)$. We set $\Lambda(\omega) = (\Lambda(\omega') - \{A \vee B \leq C\}) \cup \{A \leq C, B \leq C\}$. It follows that $\Lambda(\omega)$ is satisfied by v.

$\Lambda(\pi) = C < A \supset B$**:** By induction hypothesis $\Lambda(\omega')$ is satisfied by v. Since $\Lambda(\pi) \in \Lambda(\omega')$, we have: $v(C) < v(A \supset B)$. By the definition of the truth function for implication, we have either
(i) $v(C) < v(B)$, or
(ii) $v(A) \leq v(B)$ and $v(C) < 1$,
(or both). In case *(i)* we set $\Lambda(\omega) = (\Lambda(\omega') - \{C < A \supset B\}) \cup \{C < B\}$. In case *(ii)* we set $\Lambda(\omega) = (\Lambda(\omega') - \{C < A \supset B\}) \cup \{A \leq B, C < \top\}$. (If both, *(i)* and *(ii)*, hold we can choose freely among these two options.) In any case, $\Lambda(\omega)$ is satisfied by v.

$\Lambda(\pi) = \triangle A < C$**:** By induction hypothesis $\Lambda(\omega')$ is satisfied by v. Since $\Lambda(\pi) \in \Lambda(\omega')$, we have: $v(\triangle A) < v(C)$. By the definition of $\triangle$, this implies that $v(A) < 1$ as well as $0 < v(C)$. We set $\Lambda(\omega) = (\Lambda(\omega') - \{\triangle A < C\}) \cup \{A < \top, \bot < C\}$. It follows that $\Lambda(\omega)$ is satisfied by v.

$\Lambda(\pi) = \{A < B, B \leq C\}$**:** By induction hypothesis $\Lambda(\omega')$ is satisfied by v. Since $\Lambda(\pi) \in \Lambda(\omega')$, we have: $v(A) < v(B)$ and $v(B) \leq v(C)$. This implies $v(A) < v(C)$. Therefore we set $\Lambda(\omega) = \Lambda(\omega') \cup \{A < C\}$, which is satisfied by v.

We have thus shown that all sets of assertions that label **O**-nodes are satisfied by v. In particular also all labels at leaf nodes of T are satisfiable. Therefore, they cannot contain an elementary contradiction, and consequently are winning nodes for **O**. □

6 Extracting counter-models from winning strategies for Opponent

We complete the proof of the adequateness of our dialogue game by showing the converse of Theorem 1.

Theorem 2 *Every leaf node of a winning strategy T for* **O** *on F induces a counter-model to F.*

Proof: Let ω be a leaf node of T. By definition of a winning strategy for **P**, $\Lambda(\omega)$ is a non-empty set of atomic assertions that does not contain an atomic contradiction. Moreover, $\Lambda(\omega)$ is closed with respect to applications of rules for order claims (see Table 2). Let us call all sets of assertions fulfilling these conditions *consistently saturated*.

Let Γ be a consistently saturated set of atomic assertions. We first prove by induction on the number p_Γ of propositional variables in Γ that there is an interpretation v which satisfies Γ.

$p_\Gamma = 0$**:** Since Γ is not empty, but does not contain $\bot < \bot$, or $\top < \top$, or $\top < \bot$, we have $\Gamma = \{\bot < \top\}$, which clearly is satisfied by all interpretations.

$p_\Gamma > 0$**:** Let A be a propositional variable occurring in Γ and let Γ^{-A} be the set of assertions obtained from Γ by removing all assertions of form $A \lhd F$ or $F \lhd A$ ($\lhd \in \{<, \leq\}$) from it, and adding $\bot < \top$ to it (if it is not yet present). By induction hypothesis, there is an interpretation v that satisfies Γ^{-A}. v can be extended to include an assignment of a truth value to A as follows. We define

$$P^{\leq A} = \{B \mid B \leq A \in \Gamma \text{ or } B < A \in \Gamma, A \neq B\} \cup \{\bot\},$$

$$P^{\geq A} = \{B \mid A \leq B \in \Gamma \text{ or } A < B \in \Gamma, A \neq B\} \cup \{\top\}.$$

$P^{=A}$, defined as $P^{\leq A} \cap P^{\geq A}$, is not empty in general. However, from the fact that Γ is closed with respect to applications of rules for order claims, it follows that $\{B \leq C, C \leq B\} \subseteq \Gamma$ if $B, C \in P^{=A}$. Consequently, $v(B) = v(C)$ for all $B, C \in P^{=A}$. Obviously, we can extend v to assign the (unique) value $v(B)$ to A, if $B \in P^{=A}$. If, however, $P^{=A}$ is empty then any r fulfilling

$$\max\{v(B) | B \in P^{\leq A}\} < r < \min\{v(B) | B \in P^{\geq A}\}$$

can be assigned to A. That such values r exist is, again, guaranteed by the fact that Γ is consistently saturated.

Since for any leaf node ω in T, the set of assertions $\Lambda(\omega)$ labeling ω is, by definition, consistently saturated, we have shown how to extract interpretations for all such $\Lambda(\omega)$ from T. It remains to show that these interpretations satisfy other labels of **O**-nodes in T too. To this aim we apply backward induction (from the leaves up to the root of T).

Let π be a **P**-node of T, but not its root, and let ω' be its parent (**O**-)node. By definition of a winning strategy for **O**, π has a single child (**O**-)node ω. We show that if $\Lambda(\omega)$ is satisfied by the interpretation v, then v also satisfies $\Lambda(\omega')$. The transition from ω' to ω via π corresponds to a single round in the dialogue game. The required property of the labels is easily checked by inspection of the rules of Table 1 and 2. We present only a few cases for illustration.

$\Lambda(\pi) = C < A \wedge B$: By definition of a dialogue tree we have $\Lambda(\omega) = (\Lambda(\omega') - \{C < A \wedge B\}) \cup \{C < A, C < B\}$. If v satisfies $\Lambda(\omega)$ then, in particular, $v(C) < v(A)$ and $v(C) < v(B)$. Therefore also $v(C) < \min(v(A), v(B)) = v(A \wedge B)$; which implies that v satisfies $\Lambda(\omega')$.

$\Lambda(\pi) = C < A \supset B$: We have either $\Lambda(\omega) = (\Lambda(\omega') - \{C < A \supset B\}) \cup \{C < B\}$ or $\Lambda(\omega) = (\Lambda(\omega') - \{C < A \supset B\}) \cup \{A \leq B, C < \top\}$. In the first case note that $v(A \supset B)$ either equals 1 or $v(B)$. Therefore $v(C) < v(B)$ implies that $v(C) < v(A \supset B)$, as required for v to satisfy $\Lambda(\omega')$. For the second case it suffices to note that $v(A) \leq v(B)$ implies $v(A \supset B) = 1$. Therefore $v(C) < v(A \supset B)$ follows from the assumption that $v(C) < v(\top) = 1$.

If $\Lambda(\pi)$ is an elementary order claim then the claim follows trivially since, in that case, we have $\Lambda(\omega') \subseteq \Lambda(\omega)$.

It follows that the label of the **O**-node ω_0 that is the child node of the root node of T is satisfied by all interpretations that satisfy a label at a leaf node of T. Since $\Lambda(\omega_0) = \{F < \top\}$, all such interpretations are counter-models to F. □

Remark. Instead of presenting single interpretations for a formula F as mappings from the propositional variables into reals $\in [0,1]$, one may specify classes of interpretations by order constraints of form

$$0 \bowtie_0 v(A_1) \bowtie \ldots \bowtie_n v(A_n) \bowtie_n 1,$$

where $\bowtie_i \in \{<,=\}$, for $0 \leq i \leq n$, and where $A_1, \ldots, A_n$ enumerates the propositional variables occurring in F (compare [9].)

It is not difficult to see that one can extract such specifications from the leaf nodes of a winning strategy T for **O**, too. In fact, the labels at all leaf nodes of T jointly represent a complete specification of *all* counter-models to the formula F at the root node.

7 Winning strategies for Proponent as proofs of validity

We call a dialogue tree T *complete* if all child nodes to a node ρ in T reflect all possible moves to continue the dialogue at the corresponding state. In other words, complete dialogue trees are defined exactly as winning strategies for **P** at all **P**-nodes, and exactly as winning strategies for **O** at all **O**-nodes. Since a complete dialogue tree takes into account all possible moves for both players, there is a *unique* complete dialogue tree T^c_F for each formula F.

Let us say that a dialogue tree T *contains* a dialogue tree T' if T' can be obtained from T by removing some nodes from T, together with the subtrees rooted at the removed nodes. It follows from Theorem 1 that T^c_F contains a winning strategy for **O** whenever F is not valid. On the other hand, Theorem 2 implies that no winning strategy for **O** can be contained in T^c_F if F is valid. We show that, in the latter case, T^c_F contains a winning strategy for **P**.

Theorem 3 *A complete dialogue tree T^c_F contains a winning strategy for* **P** *iff it does not contain a winning strategy for* **O**.

Proof: By definition, every leaf node of T^c_F is either a winning node for **P** or a winning node for **O**, but no leaf node can be a winning node for both players. It follows that T^c_F cannot contain winning strategies for both players simultaneously.

Since T^c_F is finite (by Proposition 2) we can apply Zermelo's Theorem [23] – implying that all two-person zero-sum games of finite depth are determined – to obtain a proof of the theorem. However, to keep this exposition self-contained, we reproduce the central argument here. More exactly, we show how to extract a winning strategy for either **P** or **O** from T^c_F. First, color all winning nodes for **O** black and all winning nodes for **P** white. Then repeat the following until all nodes of T^c_F are colored: for all nodes ρ that are not yet colored, but where all child nodes are already colored,

- if ρ is a **P**-node and all child nodes of ρ are white then color ρ white, otherwise color ρ black,
- if ρ is an **O**-node and all child nodes of ρ are black then color ρ black, otherwise color ρ white.

It is easy to see that if a node in T^c_F is colored black then this means that the corresponding state of the dialogue game is such that **O** can win the dialogue for every choice of moves by **P**. The same holds for white nodes with **O** and **P** interchanged. In other words: if the root of T^c_F is black then T^c_F contains a winning strategy for **O**. It can be extracted from T^c_F by removing all white nodes and additionally removing all, but one, black nodes that succeed any given **P**-node (if there is more than one child node to this **P**-node). If the root of T^c_F is white, a winning strategy for **P** can be extracted analogously. □

Theorems 1, 2 and 3 jointly imply:

Corollary 1 *A formula F is valid in $G^{\triangle}_{\infty}$ iff there exists a winning strategy for* **P** *on F.*

Proof:

If: By Theorem 3, the existence of a winning strategy for **P** implies that the complete dialogue tree T^c_F does not contain a winning strategy for **O**. Since, by definition, T^c_F contains all possible strategies we conclude that no winning strategy for **O** exists for a game on F. By Theorem 1 it follows that no counter-model to F exists. But this implies that F is valid.

Only if: If F is valid then there is no counter-model to F. Hence, by Theorem 2 and the definition of a complete dialogue tree, no winning strategy for **O** is contained in T^c_F. By Theorem 3 this implies that there exists a winning strategy for **P** on F. □

8 Construction of complete dialogue trees as proof search

It is well known that deciding validity in logic G_∞ is a co-NP-complete problem (see, e.g., [16]). The complexity level does not increase if the $\triangle$-operator is added. (This can also be seen from Proposition 3, below.) In other words, the asymptotic complexity of (propositional) fuzzy logic is the same as for classical logic.

Some important calculi for G_∞ (and, by extension, $G^{\triangle}_{\infty}$) are based on the fact that G_∞ can be obtained from intuitionistic logic by adding the *linearity axiom* $(A \supset B) \vee (B \supset A)$. In particular, Avron's elegant hypersequent calculus **HLC** for G_∞ contains Gentzen's celebrated sequent calculus **LJ** for intuitionistic logic as a sub-calculus and adds an additional layer of context which allows to form rules that correspond to the linearity axiom. However, the validity problem for intuitionistic logic is PSPACE-complete (and therefore expected to be strictly more complex than that for G_∞). As a consequence, proof search for intuitionistic fuzzy logic that is based on Avron's calculus and related systems can hardly be expected to be computationally adequate.

Returning to our dialogue game, observe that the systematic construction of the complete dialogue tree for F can be viewed as tableau style proof search for F. In fact, we could have presented the rules of Tables 1 and 2 as tableau construction rules, and consequently refer to the definition of a dialogue tree as the specification of a tableau system. (We prefer the dialogue game format mainly for its more explicit reference to fundamental principles of correct reasoning.)

We have already seen in Proposition 2 that all dialogue trees are finite. Therefore, Corollary 1 implies that the systematic construction of T^c_F constitutes a *decision procedure* for logic $G^{\triangle}_{\infty}$.

To demonstrate, that – in contrast to other systems mentioned above – this form of proof search is computionally adequate we show that each branch of a complete dialogue tree T^c_F is of polynomial length in the size of F. (By the size of a formula we mean the number of symbols occurring in it; by the length of a branch of a tree we mean the number of nodes in it.) Remember that each branch of a dialogue tree corresponds to a single dialogue. T^c_F is just a systematic representation of all possible dialogues on F.

Proposition 3 *For all $G^{\triangle}_{\infty}$-formulas F each branch of T^c_F can be constructed in (deterministic) polynomial time in the size of F.*

Proof: Since a single move in the dialogue game can clearly be recorded unambiguously in polynomial time, it suffices to show that each branch b of T^c_F is of polynomial length, with respect $\|F\|$, the number of symbols occurring in F.

We use $SF(F)$ to denote the set of all subformulas of F united with $\{\bot, \top\}$. $aSF(F)$ is used to denote the set of propositional variables occurring in F, again united with $\{\bot, \top\}$.

Observe that the label $\Lambda(\pi)$ at a **P**-node π of b is either of form $G \lhd H$ or of form $\{A \lhd_1 B, B \lhd_2 C\}$, where $G, H \in SF(F)$, $A, B, C \in aSF(F)$, and $\lhd, \lhd_1, \lhd_2 \in \{<, \leq\}$. Since there are at most $2|SF(F)|^2$ labels of the first type and at most $4|aSF(F)|^3$ labels of the second type, the claim follows from $|aSF(F)| \leq |SF(F)| \leq \|F\| + 2$. □

9 Dialogue states as relational sequents

We have already remarked in Section 4 that our comparison based dialogue game is related to the so-called sequent of relations systems introduced in [8] for a range of logics including G_∞, and further investigated and extended to $G_\infty^\triangle$ in [6, 7].

In fact, the sequent of relation system $\mathbf{RG}_\infty^\triangle$ is also based on finite sets of assertions. However, the sets of assertions, called *relational sequents* in $\mathbf{RG}_\infty^\triangle$, are written in form

$$F_1 \lhd_1 F_2 \mid \ldots \mid F_n \lhd_n F_{n+1}$$

and interpreted as *disjunctions* (rather then conjunctions) of assertions. Still, the logical rules of $\mathbf{RG}_\infty^\triangle$ are systematically related to those of Table 1, above. E.g., the $\mathbf{RG}_\infty^\triangle$-rule for implication at the left argument place of '$\leq$' in a relational sequent is:

$$\frac{\top \leq C \mid B < A \mid \mathcal{H} \qquad B \leq C \mid \mathcal{H}}{(A \supset B) \leq C \mid \mathcal{H}} \ (\supset{:}\leq{:}\, l)$$

where $\mathcal{H}$ denotes a so-called side-sequent, i.e., a (possible empty) set of assertions. The $\mathbf{RG}_\infty^\triangle$-rule for introducing implication at the right hand side of '$<$' is

$$\frac{C < \top \mid \mathcal{H} \qquad A \leq B \mid C < B \mid \mathcal{H}}{C < (A \supset B) \mid \mathcal{H}} \ (\supset{:}<{:}\, r)$$

To see the correspondence to the dialogue game, we define the *dual* of an assertion $F \lhd G$ as $[F \lhd G]^d = G \lhd^d F$ for $\lhd \in \{<, \leq\}$, where $<^d = \leq^d$ and $\leq^d = <$. The $\mathbf{RG}_\infty^\triangle$-rule for introducing an assertion $F \lhd G$ in a relational sequent can be obtained from the possible answers by **O** (according to Table 1) to **P**'s attack on assertion $[F \lhd G]^d$, by dualizing the answered assertions and placing them in the context of a side-sequent as appropriate. E.g., the two premises of rule $(\supset{:}<{:}\, r)$ correspond to the dual versions of the two possible answers $\{\top \leq C\}$ and $\{B < A, B \leq C\}$ by **O** to an attack on $[C < (A \supset B)]^d = A \supset B \leq C$ by **P**.

The close correspondence between $\mathbf{RG}_\infty^\triangle$ and our dialogue game breaks down at the level of axioms. The axioms of $\mathbf{RG}_\infty^\triangle$ are the relational sequents that are of one of the following forms:

(a) $A_1 \lhd_n A_n \mid \ldots \mid A_3 \lhd_2 A_2 \mid A_2 \leq A_1$, where $\lhd_i \in \{<, \leq\}$ and the case $n = 1$ is defined as $A_1 \leq A_1$,

(b) $A_n \leq A_{n-1} \mid A_{n-1} < A_{n-2} \mid \ldots \mid A_1 < \top$, where the case $n = 1$ is defined as $A_1 \leq \top$,

(c) $\bot < A_n \mid \ldots \mid A_3 < A_2 \mid A_2 \leq A_1$, where the case $n = 1$ is defined as $\bot \leq A_1$,

(d) $\bot < A_1 \mid A_1 < A_2 \mid \ldots \mid A_n < \top$, where the case $n = 0$ is defined as $\bot < \top$.

In $\mathbf{RG}_\infty^\triangle$ axioms need not be atomic, i.e., the formulas occurring in axioms can be compound.

To be able to transform proofs in system $\mathbf{RG}_\infty^\triangle$ into winning strategies for **P** in the dialogue game, we have to enrich $\mathbf{RG}_\infty^\triangle$ with new rules, which we call *expansion* rules:

$$\frac{A \leq B \mid \mathcal{H}}{A \leq C \mid C \leq B \mid \mathcal{H}} \ (\text{expand}_{\leq)}$$

$$\frac{A < B \mid \mathcal{H}}{A \lhd_1 C \mid C \lhd_2 B \mid \mathcal{H}} \ (\text{expand}_{<)}$$

where $\lhd_1, \lhd_2 \in \{<, \leq\}$, but at least one of the two is $<$. It is easy to see that all axioms of $\mathbf{RG}_\infty^\triangle$ can be derived from the singleton sequents, called *reduced axioms*, that are of form $A \leq A$, $A \leq \top$, $\bot \leq A$, or $\bot < \top$ respectively, using expansion rules only. These reduced axioms are the dual forms of the elementary contradictions that determine the winning conditions for the dialogue game. Moreover, note that the expansion rules correspond (via duality) to the dialogue game rules for elementary order claims, if only atomic assertions are considered.

$\mathbf{RG}_\infty^\triangle$ also contains the following 'structural' rule:

$$\frac{\mathcal{H}}{A \mid \mathcal{H}} \ \text{weakening}$$

However, this rule is not needed if we extend the definition of axioms to include all those relational sequents that contain an original axiom as subset.[1]

Summarizing, there is a one-to-one correspondence (via duality) between winning strategies for **P** in our dialogue game and proofs from reduced axioms in a weakening-free version of $\mathbf{RG}_\infty^\triangle$ with expansion rules, where the axioms and the exhibited assertions in the expansion rules are atomic. The results of this paper imply the soundness and completeness of this special version of $\mathbf{RG}_\infty^\triangle$.

[1] The original version of system $\mathbf{RG}_\infty^\triangle$ also contains a contraction rule. However, this rule disappears in our version of $\mathbf{RG}_\infty^\triangle$ since we consider relational sequents as sets and not as multi-sets.

10 Open questions and future work

We have demonstrated that a simple dialogue game, based on comparisons of degrees of truth, is adequate for intuitionistic fuzzy logic. We conclude by posing a few open questions, which hopefully stimulate further research on the topic:

- Can the dialogue games be modified in a straightforward manner to support reasoning also in *finitely* valued Gödel logics and related logics?
- Can one lift the game from propositional to the first-order level?
- Is there a direct relation between winning strategies for **P** and proofs in other calculi for G_∞ (like, e.g., those in [1, 4, 12, 20])?

Finally we mention that we plan to implement the proof search procedure via systematic construction of dialogue trees, that was indicated in Section 8. The proof search should also be able to proceed in an interactive mode. In particular, a graphic user interface should enable to actually play the dialogue game against the computer, alternatively as **P**(roponent) or **O**(pponent) of a formula.

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