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From Games to Truth Functions: A Generalization of Giles's Game

Abstract. Motivated by aspects of reasoning in theories of physics, Robin Giles defined a characterization of infinite valued Łukasiewicz logic in terms of a game that combines Lorenzen-style dialogue rules for logical connectives with a scheme for betting on results of dispersive experiments for evaluating atomic propositions. We analyze this game and provide conditions on payoff functions that allow us to extract many-valued truth functions from dialogue rules of a quite general form. Besides finite and infinite valued Łukasiewicz logics, also Meyer and Slaney's Abelian logic and Continuous Hoop Logic turn out to be characterizable in this manner.

Keywords: dialogue games, evaluation games, many-valued logic, truth functionality

1. Introduction and overview

Already in the 1970s Robin Giles [10, 11] combined dialogue rules for the systematic reduction of arguments involving logically complex statements to simpler statements with a scheme for betting on the results of dispersive experiments and proved that the resulting game is sound and complete for (infinite-valued) Łukasiewicz logic. While Giles explicitly referred to Paul Lorenzen's dialogical semantics for intuitionistic logic [14, 15], his game arguably should be thought of as a special form of an *evaluation game*, rather than a Lorenzen-style game for characterizing validity: in devising optimal strategies it is essential that the players know the payoff values associated with atomic statements. On the other hand, Giles's game is also not just a variant of Hintikka's evaluation game for classical first order logic. Like in Lorenzen's dialogue game, more than just one sub-formula of the originally asserted formula has to be considered in general at any particular state of the game. However, in contrast to Lorenzen's setup, no strict regulation on the successions of moves has to be imposed on the two players. These and a number of other features render Giles's game an interesting object of study, independently from the renewed interest in Łukasiewicz logic in the context of t-norm based fuzzy logics [12, 16].

In [8] and [7] a connection between analytic proofs in so-called hypersequent calculi for Łukasiewicz logic and winning strategies for the proponent of a formula in Giles's game has been investigated. In [3] and [6] it is explained how that connection can be generalized to cover also the two other fundamental t-norm based fuzzy logics, Gödel logic and Product logic. This generalization however comes at a price: not only does one have to augment the dialogue rule for implication in a somewhat problematic manner, but one also has to distinguish between two different types of game states, indicating whether for the evaluation of final game states a strict ($<$) or a non-strict comparison ($\leq$) between the values of atomic formulas is to be used. More importantly from our current point of view, one also loses the direct correspondence between payoff values and truth values that one can observe about Giles's original game.*

Here, we are not interested in the relation between proof theory and game based semantics for many-valued logics, but ask to what extent the neat interpretation of truth values in Łukasiewicz logics as payoff values resulting from optimal dialogue game strategies can be extended to other many-valued logics. The aim is to stick as closely as possible to the elegant structure of Giles's game, while at the same time replacing Giles's very particular payoff scheme and his specific rules for (a selection of) logical connectives with general conditions on viable payoff functions and on the format of dialogue rules. Our results imply that one can indeed extract a truth functional semantics from any Giles-style game that satisfies some rather weak conditions. It also turns out that, in spite of the generality of the game format, only a rather narrow family of logics can be characterized in this manner: the only prominent members of this family are, besides all finite and the infinite valued Łukasiewiz logics ($\mathbf{Ł}_\infty$, $\mathbf{Ł}_n$), Continuous Hoop Logic **CHL** [4], and Meyer and Slaney's Abelian Logic **A** [17].

The paper is organized as follows. Section 2 describes our base camp: Giles's original game for Łukasiewicz logic. In Section 3 we add rules for so-called 'strong conjunction' and isolate the role of a 'principle of limited liability'. Section 4 presents general conditions on suitable payoff functions, while Section 5 introduces a general format for logical dialogue rules, that allows us to lift, in Section 6, payoffs from final game states to arbitrary ones. In Section 7 we show how a number of known logics emerge as concrete instances of our general framework. Section 8 concludes with a brief summary and an outlook on topics for further investigation.

*If one is willing to pay the indicated price, however, one obtains a game that corresponds to a quite remarkable (hypersequent) calculus with uniform rules for all three fundamental t-norm based fuzzy logics. This proof system enjoys, among other desirable properties, cut elimination, invertible rules, reduction to atomic axioms, unrestricted permutability of logical rules, and moreover supports efficient proof search [3, 6].

2. Giles's game for Łukasiewicz logic

In [10] and, in more detail, in [11] Robin Giles sets out to determine a logic for reasoning about physical theories with dispersive experiments, meaning that repeated trials of the same experiment may yield different results. (The most familiar examples arise from quantum mechanics.) To provide 'tangible meaning' of logical connectives Giles refers to Lorenzen's dialogue games for intuitionistic and classical logic [14, 15]. More precisely, Giles employs Lorenzen's so-called particle rules to reduce arguments involving logically complex assertions to arguments about atomic assertions. To evaluate the latter Giles assigns an dispersive experiments to each atomic proposition and lets the players bet on the corresponding results. We follow Giles in referring to the players as *me* and *you*, respectively.

Let us first review the betting phase of the game, largely ignoring Giles's motivation pertaining to the philosophy of science. Each atomic proposition p is associated with an experiment (test) E_p, which has a fixed probability $\pi(E_p)$ of yielding a positive result. Giles identifies this (subjective) probability with a player's expectation that a trial of E_p will end positively and cashes out this interpretation by the following betting scheme. I promise to pay to you a fixed amount of money, say 1€, for each of my assertions of p, where a corresponding trial of E_p yields a negative result. Likewise, you have to pay 1€ to me for each of your assertions that does not pass the associated test. Note that it matters whether we assert the same proposition just once or more often. A final game state at which $[p_1, \ldots, p_n]$ is the *multiset* of atomic assertions made by you and $[q_1, \ldots, q_m]$ is the multiset of atomic assertions made by me is denoted by

$$[p_1, \ldots, p_n \mid q_1, \ldots, q_m] \,.$$

Let us define the *risk value* of p by $\langle p \rangle^r = 1 - \pi(E_p)$. We can then specify the expected total amount of money (in €) that I have to pay to you at the exhibited state by

$$\langle p_1, \ldots, p_n \mid q_1, \ldots, q_m \rangle^r = \sum_{1 \leq i \leq m} \langle q_i \rangle^r - \sum_{1 \leq j \leq n} \langle p_j \rangle^r \,.$$

We call this number briefly my *risk* associated with that state. Note that the risk can be negative, i.e., the risk values of the relevant propositions may be such that I expect a net payment by you to me.

As an example consider the state $[p, p \mid q]$, where you have asserted p twice and I have asserted q once. Three trials of experiments are involved in the corresponding evaluation: two trials of E_p, one for each of your assertions and one trial of E_q to test my assertion. If $\langle p \rangle^r = 0.2$, i.e., if the probability that the experiment E_p yields a positive result is 0.8, and $\langle q \rangle^r = 0.5$ then $\langle p, p \mid q \rangle^r = 0.1$. This means that my expected loss of money according to our betting scheme is 0.1€. If, on the other hand, $\langle p \rangle^r = \langle q \rangle^r = 0.5$, then $\langle p, p \mid q \rangle^r = -0.5$, which means that I expect an (average) gain of 0.5€.

In order to evaluate logically complex assertions, Giles defines dialogue rules for the reduction of disjunctive, conjunctive, implicative, and negated statements to their sub-statements. In Giles's diction—except for changing gender and currency—these rules are as follows:

- She who asserts $A \vee B$ undertakes to assert either A or B at her own choice.
- She who asserts $A \wedge B$ undertakes to assert either A or B at her opponent's choice.
- She who asserts $A \supset B$ agrees to assert B if her opponent asserts A.
- She who asserts $\neg A$ agrees to pay 1€ if her opponent asserts A.

The last rule mixes reduction of formulas with final evaluation. To retain a strict separation between the dialogue phase of the game (reducing arguments involving complex assertions to arguments about simpler assertions) and the betting phase (evaluation of final game states by reference to dispersive experiments) we introduce the propositional constant $\bot$ and stipulate that it refers to an experiment that always yields a negative result and therefore corresponds to the (certain) payment of 1€ by any player asserting this formula. By defining $\neg A$ as abbreviation for $A \supset \bot$ the above negation rule thus becomes redundant. As will get clear below, the risk associated with an assertion of $\neg A$ is inverse to the risk for A.

Giles also considers rules for evaluating quantified formulas. However, these rules involve some complications that we will not have to deal with since we are only interested in propositional logic here. At that level, Giles's main result can be formulated as follows.

THEOREM 1 (Giles). *For every assignment of risk values to propositional variables I have a strategy for avoiding positive expected loss of money in the game starting with my assertion of a formula F if and only if F is valid in Łukasiewicz logic $\mathbf{Ł}_\infty$.*

To render this paper self-contained, we still have to formally specify Łukasiewicz logic $\mathbf{Ł}_\infty$. Formulas of $\mathbf{Ł}_\infty$ are built up from propositional variables and the propositional constant $\bot$ using the connectives $\neg$, $\wedge$, $\vee$, $\supset$, and augmented by so-called strong conjunction &. The corresponding semantics extends any assignment (*valuation*) v of values in $[0,1]$ to propositional variables to arbitrary formulas as follows:

$$\begin{array}{ll} v(\bot) = 0 & v(\neg A) = 1 - v(A) \\ v(A \wedge B) = \min(v(A), v(B)) & v(A \& B) = \max(0, v(A) + v(B) - 1) \\ v(A \vee B) = \max(v(A), v(B)) & v(A \supset B) = \min(1, 1 - v(A) + v(B)) \end{array}$$

A formula F is called *valid* in $\mathbf{Ł}_\infty$ if $v(F) = 1$ for all valuations.

The connection between Giles's dialogue rules and the valuation function for the corresponding connective actually is tighter than the above formulation of Theorem 1 reveals. In fact, the game can be seen as an *evaluation game* where risk value assignments correspond to valuations via $\langle p \rangle^r = 1 - v(p)$ for all p: the minimal risk r at a final state that I can enforce by an optimal strategy for a game starting with my assertion of F turns out to be $1 - v(F)$.

3. Strong conjunction and the principle of limited liability

The attentive reader will have noticed we have included so-called 'strong conjunction' (&) in defining Łukasiewicz logic $\mathbf{Ł}_\infty$, while this connective is not considered by Giles. From the point of view of contemporary mathematical fuzzy logic [12, 16] the clause $v(A\&B) = \max(0, v(A) + v(B) - 1)$ is central indeed, since the latter function is one of three fundamental t-norms—the others being min and multiplication—which determine fuzzy logics in a canonical way.† However, in fact all connectives of $\mathbf{Ł}_\infty$ can be defined from $\supset$ and $\bot$ as follows: $\neg A =_{def} A \supset \bot$, $A\&B =_{def} \neg(A \supset \neg B)$, $A \wedge B =_{def} A\&(A \supset B)$, $A \vee B =_{def} ((A \supset B) \supset B) \wedge ((B \supset A) \supset A)$.

Nevertheless, from Giles's own perspective of providing 'tangible meaning' of logical connectives, it is somewhat odd to consider exclusively a dialogue rule for conjunction, where only one of the conjuncts has to be defended. One might rather be tempted to add the following rule:

- She who asserts $A \wedge' B$ undertakes to assert both A and B.

However, it is easy to see that this has undesirable effects. E.g., there is no strategy for avoiding positive risk when initially asserting $\bot \supset (\bot \wedge' \bot)$.

†A t-norm is a commutative, associative, and monotonic function on $[0,1]$ with 1 as neutral element. Following Hájek [12], any continuous t-norm uniquely determines a (fuzzy) logic by interpreting it as truth function for conjunction and taking its residuum as truth function for implication.

More profoundly, one cannot any longer limit one's risk associated with asserting a single formula by 1€. Therefore Giles defends his choice of conjunction rule, originating with Lorenzen [14], by referring to what he calls the *principle of limited liability*. One may interpret the challenge to find an intuitively convincing rule for conjunction as an effect of dropping the contraction rule. In this view the principle of limited liability is triggered by 'going sub-structural' and it is not surprising that in fact a second, different form of conjunction arises. Indeed, as pointed out in [8], one can formulate a simple rule that is adequate for strong conjunction & in $\mathbf{Ł}_\infty$:

- She who asserts $A \& B$ undertakes to assert either both A and B, or else to assert $\bot$.

Remember that asserting $\bot$ obliges one to pay the agreed upon maximal 'fine' of 1€ for asserting a statement that cannot be verified by a corresponding trial of a dispersive experiment ($\langle \bot \rangle^r = 1$). In this sense our rule, too, is motivated by the principle of limited liability.

With hindsight one can detect yet another form of the principle of limited liability already at play in Giles's rule for implication: instead of attacking $A \supset B$ by asserting A to force the opponent to assert B, a player may choose not to attack $A \supset B$ at all. Since the risk associated with A may be higher than the risk associated with B, the latter choice (no attack) amounts to an option that limits my risk originating with your assertion of $A \supset B$—a risk that I would not be able to avoid if the rules of the game required that every asserted implication is to be attacked. We will return to this issue in Section 7 in connection with Abelian logic $\mathbf{A}$.

For later reference, let us formulate both relevant forms of limiting risk in a slightly more abstract form:

Limited liability for defense (LLD): A player can always choose to just assert $\bot$ in reply to an attack by her opponent.

Limited liability for attack (LLA): A player can always declare not to attack an occurrence of a formula that has been asserted by her opponent.

From this general perspective on the principle of limited liability, which does not refer to particular connectives, it might seem unsatisfying to invoke LLD only for implication and LLA only for strong conjunction. However it is straightforward to check that the proof of Theorem 1 as presented in [8] remains essentially unchanged if LLD and LLA are uniformly imposed on all specific dialogue rules. To understand this fact it suffices to observe that a player can never decrease her expected loss (risk) by asserting $\bot$ in reply to an attack on any statement that is not a strong conjunction, nor can one's risk be decreased by not attacking an opponent's assertion, except for attacks on implicative statements.

4. General payoff functions

So far we have only dealt with concepts that are closely connected to Giles's game for Łukasiewicz logic $\mathbf{Ł}_\infty$ as presented in [10, 11, 8]. In the following we aim at a more general framework that, in contrast to the game variants described in [3] and [6], nevertheless preserves some essential and arguably quite desirable features of the original game. We will not any longer talk about specific rules for particular logical connectives. Moreover, we will also look at the evaluation of final (atomic) game states from a wider perspective that is neither dependent on philosophical motivations regarding proper forms of reasoning in physics nor on a specific logical target language.

We will stick to Giles's convention of referring to the two players as *you* and *me*, respectively.

DEFINITION 1 (Tenet). The *tenet* Γ of a player (me or you) is the finite multiset $[\phi_1, \ldots, \phi_n]$ of formulas asserted by that player at a given state of the game. A tenet is *atomic* if all formulas in Γ are atomic.

We will denote atomic tenets by lower Greek letters $\gamma, \delta, \ldots$ and arbitrary tenets by upper Greek letters $\Gamma, \Delta, \ldots$. Moreover, we write $[\Gamma, \Delta]$ to denote the union of the multisets Γ and Δ as well as $[\Gamma, \phi]$ instead of $[\Gamma, [\phi]]$, etc.

DEFINITION 2 (Game state). A *(game) state* $[\Gamma \mid \Delta]$ consists of two tenets Γ and Δ, where Γ is *your* tenet and Δ is *my* tenet. A game state is *atomic* if Γ and Δ are atomic.

Ignoring all specific details of Giles's story about risky bets on dispersive experiments, we see that the proposed betting scheme boils down to an ordinary payoff function (in the game theoretic sense), i.e., an assignment of real numbers to all final states of the game. Probabilities ('risk') and amounts of money to be paid by either me or you only serve as an—interesting, but in principle dispensable—*interpretation* of those real numbers. This observation motivates the formulation of general principles for assigning payoff values to atomic states.

We will only be interested in *my* payoff and may thus simply speak of 'the payoff' associated with an atomic state. (More precisely, we can think of *your* payoff for the same state as directly inverse to mine. In other words, the game is zero-sum. This is codified in Payoff Principle 2, below.)

DEFINITION 3 (Payoff). A *payoff* function assigns a value $\in \mathbb{R}$ to every atomic game state. The payoff of the game state $[\gamma \mid \delta]$ is denoted as $\langle \gamma \mid \delta \rangle$.

PAYOFF PRINCIPLE 1 (Context independence). A payoff function $\langle \cdot \mid \cdot \rangle$ is *context independent* if for all atomic tenets $\gamma, \delta, \gamma', \delta', \gamma''$, and δ'' the following holds: If $\langle \gamma' \mid \delta' \rangle = \langle \gamma'' \mid \delta'' \rangle$ then $\langle \gamma, \gamma' \mid \delta', \delta \rangle = \langle \gamma, \gamma'' \mid \delta'', \delta \rangle$.

Context independence entails that the payoff for a state $[\gamma, \gamma' \mid \delta, \delta']$ is solely determined by the payoffs of its sub-states $[\gamma \mid \delta]$ and $[\gamma' \mid \delta']$. This property will be crucial to achieve a truth functional (compositional) semantics.

PROPOSITION 1. *Let $\langle \cdot \mid \cdot \rangle$ be a context independent payoff function and let $G = [\gamma, \gamma' \mid \delta, \delta']$ be an atomic game state. Then there exists an associative and commutative binary operation $\oplus$ on $\mathbb{R}$ such that $\langle G \rangle = \langle \gamma \mid \delta \rangle \oplus \langle \gamma' \mid \delta' \rangle$.*

PROOF. Assume that $\langle \gamma \mid \delta \rangle = \langle \gamma'' \mid \delta'' \rangle = x$ and $\langle \gamma' \mid \delta' \rangle = \langle \gamma''' \mid \delta''' \rangle = y$. Then $\langle \gamma'', \gamma''' \mid \delta'', \delta''' \rangle = \langle \gamma, \gamma''' \mid \delta, \delta''' \rangle = \langle \gamma, \gamma' \mid \delta, \delta' \rangle$ by applying context independence twice. Thus we may write $\langle \gamma, \gamma' \mid \delta, \delta' \rangle = x \oplus y$. Associativity and commutativity of $\oplus$ directly follow from the fact that tenets are multisets. ∎

Remark. We will call $\oplus$ as specified in Proposition 1 the *aggregation function corresponding to* $\langle \cdot \mid \cdot \rangle$. In Giles's original game the function $\oplus$ is ordinary addition, which motivates our notation.

PAYOFF PRINCIPLE 2 (Symmetry). A payoff function $\langle \cdot \mid \cdot \rangle$ is *symmetric* if $\langle \gamma \mid \delta \rangle = - \langle \delta \mid \gamma \rangle$ for all atomic tenets γ and δ.

If $\langle \cdot \mid \cdot \rangle$ is context independent and symmetric then the payoff of an arbitrary atomic game state can be decomposed as follows:

$$\begin{aligned} &\langle p_1, \dots, p_n \mid q_1, \dots, q_m \rangle \\ &= \langle p_1 \mid \rangle \oplus \dots \oplus \langle p_n \mid \rangle \oplus \langle \mid q_1 \rangle \oplus \dots \langle \mid q_m \rangle \\ &= - \langle \mid p_1 \rangle \oplus \dots \oplus - \langle \mid p_n \rangle \oplus \langle \mid q_1 \rangle \oplus \dots \oplus \langle \mid q_m \rangle . \end{aligned}$$

Note that symmetry implies that $\langle \gamma \mid \gamma \rangle$=0. In other words, the payoff is 0 in an atomic state where your tenet is identical to mine. Moreover, this shows that one could focus on single tenets instead of two-sided states.

PROPOSITION 2. *Let $\langle \cdot \mid \cdot \rangle$ be a context independent and symmetric payoff function. Then*
(i) $-$ distributes over the corresponding aggregation function $\oplus$, i.e., for all payoff values x and y, $-(x \oplus y) = -x \oplus -y$, and
(ii) $-$ is inverse to $\oplus$, i.e., $x \oplus -x = 0$ holds for all values x.

PROOF.
(i) Let $[\gamma_1 \mid \delta_1]$ and $[\gamma_2 \mid \delta_2]$ be two atomic states such that $\langle \gamma_1 \mid \delta_1 \rangle = x$ and $\langle \gamma_2 \mid \delta_2 \rangle = y$. Then

$$\begin{aligned}
-(x \oplus y) &= -(\langle \gamma_1 \mid \delta_1 \rangle \oplus \langle \gamma_1 \mid \delta_2 \rangle) && \text{by definition of } x, y \\
&= -\langle \gamma_1, \gamma_2 \mid \delta_1, \delta_2 \rangle && \text{by Proposition 1} \\
&= \langle \delta_1, \delta_2 \mid \gamma_1, \gamma_2 \rangle && \text{by Payoff Principle 1 (symmetry)} \\
&= \langle \delta_1 \mid \gamma_1 \rangle \oplus \langle \delta_2 \mid \gamma_2 \rangle && \text{by Proposition 1} \\
&= -\langle \gamma_1 \mid \delta_1 \rangle \oplus -\langle \gamma_2 \mid \delta_2 \rangle && \text{by Payoff Principle 1 (symmetry)} \\
&= -x \oplus -y && \text{by definition of } x, y.
\end{aligned}$$

(ii) Let $[\gamma \mid \delta]$ be an atomic game state such that $\langle \gamma \mid \delta \rangle = x$. Then

$$\begin{aligned}
x \oplus -x &= \langle \gamma \mid \delta \rangle \oplus -\langle \gamma \mid \delta \rangle && \text{by definition of } x \\
&= \langle \gamma \mid \delta \rangle \oplus \langle \delta \mid \gamma \rangle && \text{by Payoff Principle 1 (symmetry)} \\
&= \langle \gamma, \delta \mid \gamma, \delta \rangle && \text{by Proposition 1} \\
&= \langle \gamma \mid \gamma \rangle \oplus \langle \delta \mid \delta \rangle && \text{by Proposition 1} \\
&= 0 \oplus 0 && \text{by Payoff Principle 1 (symmetry)} \\
&= 0 && \text{by Proposition 1.}
\end{aligned}$$

■

Note that every context independent and symmetric payoff function induces via its aggregation function a totally ordered abelian group with (some subset of) the reals as base set and with 0 as neutral element.

Given Proposition 2 we can rewrite the decomposition of the payoff for an atomic state $[p_1, \ldots, p_n \mid q_1, \ldots, q_m]$ as

$$\langle p_1, \ldots, p_n \mid q_1, \ldots, q_m \rangle = \bigoplus_{1 \le i \le m} \langle \mid q_i \rangle \oplus - \bigoplus_{1 \le j \le n} \langle \mid p_i \rangle .$$

PAYOFF PRINCIPLE 3 (Monotonicity). A payoff function $\langle \cdot \mid \cdot \rangle$ is *monotone* if for all tenets γ,δ,γ', δ', γ'', and δ'' the following holds: if $\langle \gamma' \mid \delta' \rangle \le \langle \gamma'' \mid \delta'' \rangle$ then $\langle \gamma, \gamma' \mid \delta', \delta \rangle \le \langle \gamma, \gamma'' \mid \delta'', \delta \rangle$.

PROPOSITION 3. *Let $\langle \cdot \mid \cdot \rangle$ be a monotone and context independent payoff function and $\oplus$ the corresponding aggregation function. Then for all payoff values x, y, and z:*
(i) if $y \le z$ then $x \oplus y \le x \oplus z$,
(ii) min *and* max *distribute over $\oplus$, i.e.,* $\min(x \oplus y, x \oplus z) = x \oplus \min(y, z)$ *and* $\max(x \oplus y, x \oplus z) = x \oplus \max(y, z)$.

PROOF. *(i)* Let $G = [\gamma \mid \delta]$, $G' = [\gamma' \mid \delta']$, and $G'' = [\gamma'' \mid \delta'']$ be three atomic states such that $\langle G \rangle = x$, $\langle G' \rangle = y$, and $\langle G'' \rangle = z$. Then the premise $y \le z$

amounts to $\langle \gamma' \mid \delta' \rangle \leq \langle \gamma'' \mid \delta'' \rangle$ and $x \oplus y \leq x \oplus z$ to $\langle \gamma \mid \delta \rangle \oplus \langle \gamma' \mid \delta' \rangle \leq \langle \gamma \mid \delta \rangle \oplus \langle \gamma'' \mid \delta'' \rangle$ or, equivalently, to $\langle \gamma, \gamma' \mid \delta, \delta' \rangle \leq \langle \gamma, \gamma'' \mid \delta, \delta'' \rangle$, which is just an instance of Payoff Principle 3.

(ii) We only consider the equation for min; the argument for max is analogous. Assume that $y \leq z$ holds. Then, by *(i)*, $x \oplus y \leq x \oplus z$ holds for all x and thus also $\min(x \oplus y, x \oplus z) = x \oplus y = x \oplus \min(y, z)$. On the other hand, if $z \leq y$ then $x \oplus z \leq x \oplus y$ and thus also $\min(x \oplus y, x \oplus z) = x \oplus z = x \oplus \min(y, z)$. ■

Definition 4 (Discriminating). We call a payoff function $\langle \cdot \mid \cdot \rangle$ *discriminating* if it is context independent, symmetric, and monotone.

We will see in Section 6 that under some very general conditions on the form of logical dialogue rules, to be investigated in the next section, discriminating payoff functions can be extended to arbitrary game states.

5. A general format of logical dialogue rules

We now turn our attention to logical connectives and look for dialogue rules that regulate the stepwise reduction of states with logically complex assertions to final atomic states. We assume perfect information, which in particular implies that the two players have common knowledge of the payoff values. Since we strive for full generality, we will not consider conjunction, disjunction, implication, etc., separately, but rather specify a generic format of dialogue rules for arbitrary n-ary connectives ($n \geq 1$). It turns out that two simple and general 'dialogue principles', in combination with discriminating payoff functions, suffice to guarantee that a truth functional semantics can be extracted from the corresponding game.

Dialogue Principle 1 (Decomposition). A (dialogue) rule for an n-ary connective $\diamond$ is *decomposing* if in any corresponding round of the game exactly one occurrence of a compound formula $\diamond(A_1, \ldots, A_n)$ is removed from the current state and (possibly zero) occurrences of $A_1, \ldots, A_n$ and of propositional constants are added to obtain the successor state. (See below for a step-by-step description of what is meant by 'round' here.)

The decomposition principle entails that each occurrence of a formula can be attacked at most once: it is simply removed from the state in the corresponding round of the game. Moreover, an attack may or may not involve sub-formulas of the attacked formula occurrence (and/or propositional constants) to be asserted by the attacking player. For example, in Giles's original game attacking $A \supset B$ requires the attacker to assert A (see Section 2). We require the reply to any attack to follow at once. In our example of an attack to $A \supset B$ in Giles's original game this means that an assertion of B will be added to the tenet of the attacked player. In general, the attacking player may choose between one of several available forms of attacking a particular formula, as witnessed by the rule for (weak) conjunction in the original game. Likewise, as exemplified in Giles's rule for disjunction, a rule may also involve a choice on the side of the defending player. Consequently, every round of the game may be thought of as consisting of a sequence of three consecutive moves (we only consider the case where you attack one of the formulas asserted by me, the other case is dual):

1. You pick an occurrence of a compound formula $\diamond(A_1, \ldots, A_n)$ from my current tenet for attack (or possibly for dismissal, see below).
2. You choose the form of attack (if there is more than one form available).
3. I choose the way in which I want to reply to the given attack on the indicated occurrence of $\diamond(A_1, \ldots A_n)$ (if such a choice is possible).

The corresponding rule may be depicted as shown in Figure 1. That there is a forest rather than a single tree rooted in $[\Gamma \mid \Delta, \diamond(A_1, \ldots, A_n)]$ reflects the fact that *you* may choose between different forms of attack for formulas of the form $\diamond(A_1, \ldots, A_n)$. In contrast, the branching in the trees corresponds to *my* possible choices in defending against your particular attack.

$$[\Gamma \mid \Delta, \diamond(A_1, \ldots, A_n)] \qquad\qquad \ldots\ldots \qquad\qquad [\Gamma \mid \Delta, \diamond(A_1, \ldots, A_n)]$$

$$[\Gamma, X^1_1 \mid \Delta, Y^1_1] \cdots [\Gamma, X^1_{k_1} \mid \Delta, Y^1_{k_1}] \qquad [\Gamma, X^m_1 \mid \Delta, Y^m_1] \cdots [\Gamma, X^m_{k_m} \mid \Delta, Y^m_{k_m}]$$

where X^i_j and Y^i_j, for $1 \leq j \leq k_i$, $1 \leq i \leq m$, are multisets of zero or more occurrences of the formulas $A_1, \ldots, A_n$ and of propositional constants.

Figure 1. Generic dialogue rule for your attack of my assertion of $\diamond(A_1, \ldots, A_n)$

To illustrate this dialogue rule format by a concrete example, consider the case of your attack on my assertion of $A \supset B$ in a variant of Giles's game where both forms of the principle of limited liability, LLD and LLA, are imposed (see Section 3). The resulting version of the implication rule is depicted in Figure 2.

The right (degenerate) tree in Figure 2 corresponds to your declaration not to attack the exhibited occurrence of $A \supset B$ at all. We treat this case as a special form of attack, where the 'attacked' formula occurrence is simply

$$
\begin{tikzcd}[column sep=small]
 & {[\Gamma \mid \Delta, A \supset B]} \arrow[dl, no head] \arrow[dr, no head] & & {[\Gamma \mid \Delta, A \supset B]} \arrow[d, no head] \\
{[\Gamma, A \mid \Delta, B]} & & {[\Gamma \mid \Delta, \bot]} & {[\Gamma \mid \Delta]}
\end{tikzcd}
$$

Figure 2. Implication rule (your attack) with two-fold principle of limited liability

removed to obtain the successor state. The first tree indicates a choice by me (i.e., the defending player): I may either according to LLD assert $\perp$ in reply to your attack or else assert B in exchange for your assertion of A.

The second principle that we want to maintain in generalizing Giles's game is player neutrality, i.e., role duality: you and me have the very same obligations and rights in attacking or defending a particular type of formula.

Dialogue Principle 2 (Duality). A rule $\delta_\diamond$ for my (your) assertion of a formula of the form $\diamond(A_1, \ldots, A_n)$ is called *dual* to the rule $\delta'_\diamond$ for your (my) assertion of $\diamond(A_1, \ldots, A_n)$, if $\delta_\diamond$ is obtained from $\delta'_\diamond$ by just switching the roles of the players.

We will say that a dialogue game *has dual rules* if for every dialogue rule of the game there is dual rule.

Figure 3 depicts the generic dialogue that is dual to that in Figure 1. Note that now *I* am the one who, in attacking your assertion of $\diamond(A_1, \ldots, A_n)$, is free to pick a tree of the forest, whereas the branching in the tree now refers to *your* choices when defending against my attack.

$$
\begin{tikzcd}[column sep=tiny]
 & {[\Gamma, \diamond(A_1,\ldots,A_n) \mid \Delta]} \arrow[dl, no head] \arrow[dr, no head] & & \cdots\cdots & & {[\Gamma, \diamond(A_1,\ldots,A_n) \mid \Delta]} \arrow[dl, no head] \arrow[dr, no head] & \\
{[\Gamma, Y^1_1 \mid \Delta, X^1_1]} & \cdots\cdots & {[\Gamma, Y^1_{k_1} \mid \Delta, X^1_{k_1}]} & & {[\Gamma, Y^m_1 \mid \Delta, X^m_1]} & \cdots\cdots & {[\Gamma, Y^m_{k_m} \mid \Delta, X^m_{k_m}]}
\end{tikzcd}
$$

where m, k_i, X^i_j, and Y^i_j are defined in Figure 1.

Figure 3. Generic dialogue rule dual to that in Figure 1

Note that since the format of decomposing rules allows for a choice between different types of attacks as well as corresponding replies, we may speak without loss of generality of *the* dialogue rule for a connective $\diamond$ if the game has dual rules.

6. Lifting payoffs to valuations of general states

Following the well-known game theoretic principle of backward induction, the maximal payoff value that I can enforce at a game state S—for short: my *enforceable payoff* at S—amounts to the minimum of enforceable payoffs at the successor states of S if it is *your* turn to move at S as well as to the maximum of enforceable payoffs at the successor states if it is *my* turn to move at S. Correspondingly, the function $\langle \cdot \mid \cdot \rangle$ that denotes my enforceable payoff at an arbitrary state in our dialogue games (where a round involves a move by both of us in turn) is induced by the corresponding payoff function for atomic game states and by the following *min-max conditions* for non-atomic game states:

$$\langle \Gamma \mid \diamond(A_1, \dots A_n), \Delta \rangle = \min_{1 \le i \le m} \max_{1 \le j \le k_i} \left\langle \Gamma, X^i_j \mid \Delta, Y^i_j \right\rangle \tag{1}$$

$$\langle \diamond(A_1, \dots A_n), \Gamma \mid \Delta \rangle = \max_{1 \le i \le m} \min_{1 \le j \le k_i} \left\langle \Gamma, Y^i_j \mid \Delta, X^i_j \right\rangle \tag{2}$$

where m, k_i, X^i_j, and Y^i_j are defined as in Figure 1. We call this function the *extended payoff* function.[‡]

In Section 4 we have defined context independence, symmetry, and monotonicity for payoff functions which, by definition, refer only to atomic game states. However, by inspecting Definitions 1, 2, and 3 it is obvious that neither these properties, nor those expressed in Propositions 1, 2, and 3 depend on the atomicity of the formulas in a corresponding tenet. Therefore we can speak without ambiguity of context independence, symmetry, and monotonicity for arbitrary functions from general states to real numbers, not just for proper payoff functions.

THEOREM 2. *Let $\eth$ be a dialogue game with a discriminating payoff function and decomposing dual rules. Then the extended payoff function denoting my enforceable payoff is context independent, symmetric, and monotone.*

PROOF. Given a discriminating payoff function $\langle \cdot \mid \cdot \rangle$ with corresponding aggregation function $\oplus$, we define a function v from (arbitrary) game states

[‡]It can easily be checked that the above min-max conditions define a unique extension of any discriminating payoff function to arbitrary game states if the dialogue rules are dual and discriminating. As pointed out in [8] (for Giles's game) this fact implies that the order of rule applications is irrelevant: we arrive at the same enforceable payoff, independently of the specific formula occurrence that is picked by you or me for attack at any given state.

to the real numbers inductively as follows:

$$\begin{aligned}
&(a)\quad v([\mid p]) = \langle \mid p\rangle\\
&(b)\quad v([\mid \Delta]) = \bigoplus\nolimits_{B\in\Delta} v([\mid B]))\\
&(c)\quad v([\Gamma \mid \Delta]) = v([\mid \Delta]) \oplus - v([\mid \Gamma])\\
&(d)\quad v([\mid \diamond(A_1,\dots A_n)]) = \min_{1\le i\le m}\max_{1\le j\le k_i} v(\left[X^i_j \mid Y^i_j\right])
\end{aligned}$$

where m, k_i, X^i_j, and Y^i_j are defined as in Figure 1.

We prove that v indeed calculates my enforceable payoff, i.e., it coincides with $\langle\cdot\mid\cdot\rangle$ on atomic states and fulfills the min-max conditions. Moreover we show that it is context independent, symmetric, and monotone.

It is straightforward to check that $v([\gamma \mid \delta])$ indeed coincides with $\langle\gamma \mid \delta\rangle$ for all atomic states $[\gamma \mid \delta]$. Taking our clue from this observation we will from now on usually write $\langle\Gamma \mid \Delta\rangle$ instead of $v([\Gamma \mid \Delta])$, even if the tenets Γ and Δ are not atomic.

The symmetry of $v([\cdot \mid \cdot])$ immediately follows from its definition, where (here as well as further on) we freely exploit the commutativity and associativity of $\oplus$.

$$\begin{aligned}
(-v([\Gamma \mid \Delta] =) - \langle\Gamma \mid \Delta\rangle &= -(\langle\mid \Delta\rangle \oplus - \langle\mid \Gamma\rangle) && \text{by definition of } v \text{ (c)}\\
&= -\langle\mid \Delta\rangle \oplus \langle\mid \Gamma\rangle && \text{by Proposition 2(i)}\\
&= \langle\Delta \mid \Gamma\rangle && \text{by definition of } v \text{ (c)}
\end{aligned}$$

Note that the definition of v directly entails that, just like the payoff at atomic states, the enforceable payoff at arbitrary states can also be obtained from the enforceable payoffs for sub-states by applying $\oplus$: we will refer to *merging* of and *partitioning*, respectively. More precisely:

$$\begin{aligned}
\langle\Gamma,\Gamma' \mid \Delta',\Delta\rangle &= \langle\mid \Delta',\Delta\rangle \oplus - \langle\mid \Gamma,\Gamma'\rangle && \text{by definition of } v \text{ (c)}\\
&= (\langle\mid \Delta'\rangle \oplus \langle\mid \Delta\rangle) \oplus -(\langle\mid \Gamma'\rangle \oplus \langle\mid \Gamma\rangle) && \text{by definition of } v \text{ (b)}\\
&= \langle\mid \Delta'\rangle \oplus \langle\mid \Delta\rangle \oplus - \langle\mid \Gamma'\rangle \oplus - \langle\mid \Gamma\rangle && \text{by Proposition 2}\\
&= \langle\Gamma' \mid \Delta'\rangle \oplus \langle\Gamma \mid \Delta\rangle && \text{by definition of } v \text{ (c).}
\end{aligned}$$

Given this fact, it is easy to see that $\langle\cdot \mid \cdot\rangle$ is context independent. Let $[\Gamma' \mid \Delta']$, $[\Gamma'' \mid \Delta'']$ be two game states such that $\langle\Gamma' \mid \Delta'\rangle = \langle\Gamma'' \mid \Delta''\rangle$. Then for arbitrary tenets Γ and Δ

$$\begin{aligned}
\langle\Gamma,\Gamma' \mid \Delta',\Delta\rangle &= \langle\Gamma' \mid \Delta'\rangle \oplus \langle\Gamma \mid \Delta\rangle && \text{by partitioning}\\
&= \langle\Gamma'' \mid \Delta''\rangle \oplus \langle\Gamma \mid \Delta\rangle && \text{by assumption}\\
&= \langle\Gamma,\Gamma'' \mid \Delta'',\Delta\rangle && \text{by merging.}
\end{aligned}$$

Monotonicity also straightforwardly carries over from atomic to arbitrary game states. Let $[\Gamma' \mid \Delta']$, $[\Gamma'' \mid \Delta'']$ be two game states such that $\langle \Gamma' \mid \Delta' \rangle \leq \langle \Gamma'' \mid \Delta'' \rangle$. Then for arbitrary tenets Γ and Δ

$$\begin{aligned} \langle \Gamma, \Gamma' \mid \Delta', \Delta \rangle = \langle \Gamma' \mid \Delta' \rangle \oplus \langle \Gamma \mid \Delta \rangle & \quad \text{by partitioning} \\ \leq \langle \Gamma'' \mid \Delta'' \rangle \oplus \langle \Gamma \mid \Delta \rangle & \quad \text{by assumption and Proposition 3(i)} \\ = \langle \Gamma, \Gamma'' \mid \Delta, \Delta'' \rangle & \quad \text{by merging.} \end{aligned}$$

It remains to check that the min-max conditions are satisfied. For states of the form $[\Gamma \mid \Delta, \diamond(A_1, \dots A_n)]$ we obtain min-max condition (1) as follows:

$$\begin{aligned} & \langle \Gamma \mid \Delta, \diamond(A_1, \dots A_n) \rangle \\ &= \langle \Gamma \mid \Delta \rangle \oplus \langle \mid \diamond(A_1, \dots A_n) \rangle && \text{by partitioning} \\ &= \langle \Gamma \mid \Delta \rangle \oplus \min\nolimits_{1 \leq i \leq m} \max\nolimits_{1 \leq j \leq k_i} \left(\left\langle X_j^i \mid Y_j^i \right\rangle \right) && \text{by definition of } v \text{ (d)} \\ &= \min\nolimits_{1 \leq i \leq m} \max\nolimits_{1 \leq j \leq k_i} \left(\langle \Gamma \mid \Delta \rangle \oplus \left\langle X_j^i \mid Y_j^i \right\rangle \right) && \text{by Proposition 3(ii)} \\ &= \min\nolimits_{1 \leq i \leq m} \max\nolimits_{1 \leq j \leq k_i} \left(\left\langle \Gamma, X_j^i \mid Y_j^i, \Delta \right\rangle \right) && \text{by merging.} \end{aligned}$$

The dual min-max condition (2) exploits the symmetry of $\langle \cdot \mid \cdot \rangle$:

$$\begin{aligned} & \langle \Gamma, \diamond(A_1, \dots A_n) \mid \Delta \rangle \\ &= - \langle \Delta \mid \Gamma, \diamond(A_1, \dots A_n) \rangle && \text{by symmetry} \\ &= - \min\nolimits_{1 \leq i \leq m} \max\nolimits_{1 \leq j \leq k_i} \left(\left\langle \Delta, X_j^i \mid Y_j^i, \Gamma \right\rangle \right) && \text{by min-max condition (1)} \\ &= \max\nolimits_{1 \leq i \leq m} \min\nolimits_{1 \leq j \leq k_i} \left(- \left\langle \Delta, X_j^i \mid Y_j^i, \Gamma \right\rangle \right) && \text{by Proposition 3(ii)} \\ &= \max\nolimits_{1 \leq i \leq} \min\nolimits_{1 \leq j \leq k_i} \left(\left\langle Y_j^i, \Gamma \mid \Delta, X_j^i \right\rangle \right) && \text{by symmetry,} \end{aligned}$$

where m, k_i, X_j^i, and Y_j^i are defined as in Figure 1. ■

Remark. The duality of dialogue rules is used only indirectly in the above proof: it is reflected in the corresponding duality of the two min-max conditions and in the symmetry of the extended payoff function.

COROLLARY 1. *Let* $\mathfrak{D}$ *be a game with discriminating payoff function and decomposing dual rules. Then for each connective* $\diamond$ *there is a function* $f_\diamond$ *such that* $\langle \mid \diamond(A_1, \dots A_n) \rangle = f_\diamond(\langle \mid A_1 \rangle, \dots, \langle \mid A_n \rangle)$ *for all formulas* $A_1, \dots, A_n$*, where* $\langle \cdot \mid \cdot \rangle$ *denotes the extended payoff function of Theorem 2.*

PROOF. Applying min-max condition (1) as well as context independence and symmetry, we obtain

$$\begin{aligned}
&\langle| \diamond(A_1,\ldots,A_n)\rangle\\
&= \min_{1\le i\le m}\max_{1\le j\le k_i}\left\langle X^i_j \mid Y^i_j\right\rangle\\
&= \min_{1\le i\le m}\max_{1\le j\le k_i}\left(\left\langle| Y^i_j\right\rangle \oplus \left\langle X^i_j |\right\rangle\right)\\
&= \min_{1\le i\le m}\max_{1\le j\le k_i}\left(\left\langle| Y^i_j\right\rangle \oplus - \left\langle| X^i_j\right\rangle\right)\\
&= \min_{1\le i\le m}\max_{1\le j\le k_i}\left(\bigoplus\nolimits_{Y\in Y^i_j}\langle| Y\rangle \oplus - \bigoplus\nolimits_{X\in X^i_j}\langle| X\rangle\right),
\end{aligned}$$

where $\oplus$ is the aggregation function corresponding to $\langle\cdot \mid \cdot\rangle$; m, k_i, Y^i_j, and X^i_j obviously again refer to the dialogue rule for $\diamond(A_1,\ldots A_n)$ as exhibited in Figure 1. Note that the X^i_js and Y^i_js are multisets containing only the formulas $A_1,\ldots,A_n$ and propositional constants, which of course are evaluated to constant real numbers. Therefore that last expression defines the required function $f_\diamond$. ■

To emphasize that $f_\diamond$ is of type $\mathbb{R}^n \mapsto \mathbb{R}$ it can be rewritten as

$$f_\diamond(x_1,\ldots,x_m) = \min_{1\le i\le n}\max_{1\le j\le k_i}\left(\bigoplus_{y\in\overline{Y^i_j}} y \oplus - \bigoplus_{x\in\overline{X^i_j}} x\right),$$

where $\overline{Y^i_j}$ is a multiset of real numbers defined with respect to the multiset of formulas Y^i_j as follows: $\overline{Y^i_j} = \{\overline{A} \mid A \in Y^i_j\}$, where $\overline{A} = x_i$ when $A = A_i$ for $1 \le i \le n$ and $\overline{A} = \langle| A\rangle$ if A is a propositional constant.

The duality of the rules entails $\langle\diamond(A_1,\ldots A_n) |\rangle = -\langle| \diamond(A_1,\ldots A_n)\rangle = -f_\diamond(\langle| A_1\rangle,\ldots,\langle| A_n\rangle)$. By identifying payoff values with truth values we may thus claim to have extracted a unique truth function for $\diamond$ from a given payoff function and any decomposing dialogue rule for $\diamond$. However, as we will see in the next section, standard truth functions for many affected logics usually are based on different sets of truth values. To obtain those truth functions from an appropriate game we have to use certain bijections between payoff values and truth values, as explained in Section 7.

7. Which logics are captured?

Revisiting Giles's game

To illustrate the emergence of concrete logics as instances of the general framework for games presented in Sections 4 to 6 we should first check whether Giles's original game for Łukasiewicz logic is indeed covered. While the assignment of risk $\langle \cdot \mid \cdot \rangle^r$ to atomic states, as defined in Section 2, amounts to a discriminating payoff function (according to Definition 4), the connection to the standard truth functional semantics for $\mathbf{Ł}_\infty$ becomes clearer when we convert risk, that is to be minimized, to payoff, that is to be maximized, and set

$$\begin{aligned}\langle p_1, \ldots, p_n \mid q_1, \ldots, q_m \rangle &= -\langle p_1, \ldots, p_n \mid q_1, \ldots, q_m \rangle^r \\ &= -\sum_{1 \le i \le m} \langle q_i \rangle^r + \sum_{1 \le j \le n} \langle p_j \rangle^r \\ &= -\sum_{1 \le i \le m} -\langle \mid q_i \rangle + \sum_{1 \le j \le n} -\langle \mid p_j \rangle \\ &= \sum_{1 \le i \le m} \langle \mid q_i \rangle - \sum_{1 \le j \le n} \langle \mid p_j \rangle \,.\end{aligned}$$

Clearly, the aggregation function corresponding to $\langle \cdot \mid \cdot \rangle$ is ordinary addition. Figure 4 presents the dialogue rules in the format defined in Section 5. Because of duality—which is obvious from Giles's generic presentation of the rules—we only have to consider your attacks on my assertions explicitly.

$$\begin{array}{c}[\Gamma \mid \Delta, A \supset B]\\ |\\ [\Gamma, A \mid \Delta, B]\end{array} \qquad \begin{array}{c}[\Gamma \mid \Delta, A \supset B]\\ |\\ [\Gamma \mid \Delta]\end{array}$$

(a) Implication

$$\begin{array}{ccc} & [\Gamma \mid \Delta, A \& B] & \\ \swarrow & & \searrow \\ [\Gamma \mid \Delta, A, B] & & [\Gamma \mid \Delta, \bot]\end{array}$$

(b) Strong Conjunction

$$\begin{array}{c}[\Gamma \mid \Delta, A \wedge B]\\ |\\ [\Gamma \mid \Delta, A]\end{array} \qquad \begin{array}{c}[\Gamma \mid \Delta, A \wedge B]\\ |\\ [\Gamma \mid \Delta, B]\end{array}$$

(c) Conjunction

$$\begin{array}{ccc} & [\Gamma \mid \Delta, A \vee B] & \\ \swarrow & & \searrow \\ [\Gamma \mid \Delta, A] & & [\Gamma \mid \Delta, B]\end{array}$$

(d) Disjunction

Figure 4. Giles's game with strong conjunction (your attack/my defense)

Note that discriminating payoff functions have 0 as neutral element. If we want to match the functions $f_\supset$, $f_\&$, $f_\wedge$, and $f_\vee$ extracted from these dialogue rules according to Corollary 1 with standard truth functions over $[0, 1]$ we still have to add 1 to the payoff. It is straightforward to check that, modulo that transformation, the functions extracted from the rules in Figure 4 indeed coincide with the standard truth functions for $\mathbf{Ł}_\infty$, reviewed in Section 2. We only illustrate the case for implication. From the rule for my assertion of $A \supset B$, which gives you a choice between asserting A to force me to assert A or else to declare that you will not attack this assertion at all, we obtain the following instance of min-max condition (1):

$$\langle | \, A \supset B \rangle = \min(\langle A \mid B \rangle, \langle | \rangle) = \min(0, \langle | \, B \rangle - \langle | \, A \rangle).$$

Adding 1 yields the truth function $v(A \supset B) = 1 + \langle | \, A \supset B \rangle = \min(1, 1 + \langle | \, B \rangle + 1 - (\langle | \, A \rangle + 1)) = \min(1, 1 - v(A) + v(B))$. The truth function for the other connectives are obtained in the same manner.

Finite valued Łukasiwicz logics

Instead of considering arbitrary risk (and therefore also arbitrary truth values) from $[0,1]$, one may restrict the set of permissible risk values (equivalently: truth values) to $V_n = \{\frac{i}{n-1} \mid 1 \leq i < n\}$, for some $n \geq 2$. Since V_n is closed with respect to addition, subtraction, as well as min and max, truth functions for all *finite valued* Łukasiewicz logics $\mathbf{Ł}_n$ are obtained just like those for $\mathbf{Ł}_\infty$.

Note that by this observation we have also covered classical logic, which coincides with $\mathbf{Ł}_2$. This means that classical logic can be modeled by a version of Giles's game where the experiments that determine the payoffs are not dispersive: every atomic proposition p is simply true or false, entailing a determinate payment of 1€ for every assertion of p in case it is false. For every assignment of risk values 0 or 1 to atomic formulas I have a strategy for avoiding (net) payment in a game starting with my assertion of a formula A, if A is true under that assignment; on the other hand, if A is false, my best strategy limits my payment to you to 1€.

Continuous hoop logic

A more interesting case is continuous hoop logic **CHL** [4]. The truth value set of **CHL** is $(0,1]$; correspondingly the propositional constant $\perp$, along with negation ($\neg$) is removed from the language. The truth functions for implication and strong conjunction are given as

$$v(A \& B) = v(A) \cdot v(B) \qquad v(A \supset B) = \begin{cases} \frac{v(A)}{v(B)} & \text{if } v(A) \geq v(B) \\ 1 & \text{else.} \end{cases}$$

At first sight it is unclear how to obtain these truth functions from dialogue rules in our framework. However remember that in the game for Łukasiewicz logics—assuming that Giles's "risk values" have already been translated into payoff values by multiplying with -1—we still had to shift payoff values by 1 to obtain the standard truth function $\tilde{\diamond}$ from the function $\oplus_\diamond$ that can be extracted from the dialogue rule for the connective $\diamond$. It will be helpful to visualize the general form of this relation, as follows:

$$\begin{array}{ccc} \mathcal{V}_{\text{payoff}} & \xrightarrow{f_\diamond} & \mathcal{V}_{\text{payoff}} \\ \mu\uparrow & & \downarrow\sigma \\ \mathcal{V}_{\text{truth}} & \xrightarrow{\tilde{\diamond}} & \mathcal{V}_{\text{truth}} \end{array}$$

In the case of $\mathbf{Ł}_\infty$ we have $\mathcal{V}_{\text{truth}} = [0,1]$, $\mathcal{V}_{\text{payoff}} = [-1,0]$, $\mu(x) = x-1$, and $\sigma(x) = x+1$. In **CHL** we have $\mathcal{V}_{\text{truth}} = (0,1]$. If we set $\mu(x) = \log(x)$ and accordingly $\mathcal{V}_{\text{payoff}} = (-\infty,0]$ and $\rho(x) = \exp(x)$, then the implication rule of Giles's game (see Figure 4) yields the truth function for implication in **CHL**. In the same manner addition $(+)$ over $(-\infty,0]$ maps into multiplication $(\cdot)$ over $(0,1]$. However, the function $f_\&$ extracted from the dialogue rule for & of Giles's game (with risk inverted into payoff) is $\&(x,y) = \max(-1, x-1+y-1)$ rather than the required $+$. (Note that the Łukasiewicz t-norm that models & in the standard semantics for $\mathbf{Ł}_\infty$ is obtained by adding $+1$. i.e. by applying σ, as explained above.) To obtain a dialogue rule for & such that $f_\& = +$, we have to drop the option to reply to an attack on $A\&B$ by asserting $\bot$, instead of asserting A and B. In other words we simply drop the principle of limited liability LLD from the original rule for strong conjunction.

Abelian logic

So far we have only considered logics where the set of truth values is a proper subset of $\mathbb{R}$ and where we had to explicitly transform payoff values into truth values and vice versa. But there is an interesting and well studied logic, namely Slaney and Meyer's Abelian logic **A** [17, 18, 9] which coincides with one of Casari's logics for modeling comparative reasoning in natural language [1, 2], where arbitrary real valued payoffs in a Giles-style game can be directly interpreted as truth values. The truth value set of **A** indeed is $\mathbb{R}$. The truth functions for implication $(\supset)$ is subtraction and the truth function for strong conjunction $(\&)$ is addition over $\mathbb{R}$. In addition, max and min serve as truth functions for disjunction $(\vee)$ and weak conjunction $(\wedge)$, respectively.

The game based characterization of **A** is particularly simple: just drop both forms of the principle of limited liability, LLA and LLD, from Giles's game. In other words: every assertion made by the opposing player, including those of the form $A \supset B$, has to be attacked, moreover the only permissible reply to attack an $A \& B$ is to assert both A and B. (The latter rule has already been used for **CHL**, above.) The functions that can be extracted from the resulting dialogue rules according to Corollary 1 are precisely those mentioned above: $f_\supset = -$, $f_\& = +$, $f_\wedge = \min$, and $f_\vee = \max$.

Alternative aggregation functions

In all the above examples, the aggregation function $\oplus$ corresponding to the respective payoff function has been addition ($+$). This raises the question, whether in fact $\oplus$ always has to be $+$. This question is of some interest, since every truth function that can be directly extracted from a Giles-style game is built up from $\oplus$, $-$, min, max, and constant real numbers corresponding to propositional constants. (By 'directly extracted' we mean: disregarding further transformations—like $+1$ for $\mathbf{Ł}_\infty$, and exp for **CHL**—that we may want to apply to map payoffs into standard truth values for particular logics.)

To settle this question in the negative it suffices to check that for any assignment v of reals to atomic propositions

$$\langle \gamma \mid \delta \rangle = \sqrt[3]{\sum_{q \in \delta} v(q)^3} - \sqrt[3]{\sum_{p \in \gamma} v(p)^3}$$

is a discriminating payoff function with $\oplus(x, y) = \sqrt[3]{x^3 + y^3}$ as corresponding aggregation function. However, we do not know of any many-valued logic in the literature where definitions of truth functions involve this or other possible aggregation functions different from $+$.

Finally, one may ask whether for any aggregation function the ordered group $G = (\mathbb{R}; \leq, \oplus, 0, -)$ is isomorphic to $(\mathbb{R}; \leq, +, 0, -)$. A partly positive answer is provided by noting that G is archimedean. This is essentially due to monotonicity (Payoff Principle 3) and the standard order $\leq$ on the base set $\mathbb{R}$. Therefore Hölder's Theorem [13] entails that G is isomorphic to a *sub*group of $(\mathbb{R}; \leq, +, 0, -)$.

8. Conclusion

Taking Giles's characterization of Łukasiewicz logic $\mathbf{Ł}_\infty$ in terms of a dialogue game with final betting scheme as a starting point, we have defined a general concept of 'Giles-style' dialogue games for many-valued logics. We have shown that quite general conditions on payoff functions (context independence, symmetry, and monotonicity) and on the format of logical dialogue rules (decomposition and duality) guarantee that a truth functional semantics for a corresponding logic can be extracted from the game. It can easily be checked by providing simple counter examples, that in fact the three mentioned payoff principles and the two mentioned dialogue principles are not only (jointly) sufficient, but (individually) necessary for the extraction of truth functions.

We emphasize that the inverse of Corollary 1 does not hold: many, if not most interesting logics with a truth functional semantics defined over (a subset of) the real numbers as set of truth values cannot be characterized by a Giles-style game in the sense of this paper. This throws interesting light on the alternative generalizations of Giles's original game that have been presented in [3], and explored in more detail in [6] and [5]. There, in order to arrive at a characterization of Gödel logic as well as Product logic, we considered two different types of states that may occur in a given game. These types of states correspond to strict and non-strict comparison of real numbers ($</\leq$), respectively. The results of the current paper can be interpreted as demonstrating that such deviations are unavoidable, at least when other desirable features of Giles's game are kept in place.

These observations trigger a host of questions for further investigation: Can the range of logics that are extractable from a Giles-style game be characterized concisely? What kind of extensions and variations of the game are needed to characterize other important many-valued logics in a similar manner? Can the correspondence between payoff values and truth values be maintained even if the truth functions are not continuous (like for $\mathbf{Ł}_\infty$, **CHL**, **A**) or result from continuous functions by restriction to finite subsets of $\mathbb{R}$ (like for $\mathbf{Ł}_n$)? Perhaps most interestingly: can the translation of Giles's dialogue rules into logical rules of a cut-free (hypersequent) calculus, described in [8], be generalized to other variants and types of games?

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