Hintikka-style semantic games for fuzzy logics

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Abstract. Various types of semantics games for deductive fuzzy logics, most prominently for Łukasiewicz logic, have been proposed in the literature. These games deviate from Hintikka's original game for evaluating classical first-order formulas by either introducing an explicit reference to a truth value from the unit interval at each game state (as in [4]) or by generalizing to multisets of formulas to be considered at any state (as, e.g., in [12, 9, 7, 10]). We explore to which extent Hintikka's game theoretical semantics for classical logic can be generalized to a many-valued setting without sacrificing the simple structure of Hintikka's original game. We show that rules that instantiate a certain scheme abstracted from Hintikka's game do not lead to logics beyond the rather inexpressive, but widely applied Kleene-Zadeh logic, also known as 'weak Łukasiewicz logic' or even simply as 'fuzzy logic' [27]. To obtain stronger logics we consider propositional as well as quantifier rules that allow for random choices. We show how not only various extensions of Kleene-Zadeh logic, but also proper extensions Łukasiewicz logic arise in this manner.

1 Introduction

Fuzzy logics "in Zadeh's narrow sense" [34, 15], i.e. truth functional logics with the real unit interval as set of truth values, nowadays come in many forms and varieties. (We refer to the *Handbook of Mathematical Fuzzy Logics* [3] for an overview.) From an application oriented point of view, but also with respect to foundational concerns, this fact imparts enhanced significance to the problem of deriving *specific* logics from underlying *semantic principles of reasoning.* Among the various models that have been proposed in this vein are Lawry's voting semantics [22], Paris's acceptability semantics [28], re-randomising semantics [21], and approximation semantics [2, 29]. Of particular importance in our context is Robin Giles's attempt, already in the 1970s [12, 13] to justify Łukasiewicz logic, one of the most fundamental formalizations of deductive fuzzy logic, with respect to a game that models reasoning about dispersive experiments. While Giles explicitly acknowledged the influence of Paul Lorenzen's pioneering work on dialogical foundations for constructive logic [23, 24], he did not refer to Hintikka's game theoretic semantics [18, 20]. However, with the benefit of hindsight, one can classify Giles's game for Łukasiewicz logic as a *semantic game*, i.e. a game for evaluating a formula with respect to a given interpretation, guided by

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rules for the stepwise reduction of logically complex formulas into their subformulas. While this renders Giles's game closer to Hintikka's than to Lorenzen's game, Giles deviates in some important ways from Hintikka's concept, as we will explain in Section 2. Semantic games for Łukasiewicz logic that, arguably, are closer in their mathematical form to Hintikka's semantic game for classical logic have been introduced by Cintula and Majer in [4]. However, also these latter games exhibit features that are hardly compatible with Hintikka's motivation for introducing game theoretic semantics [18, 19] as foundational approach to logic and language. In particular, they entail an explicit reference to some (in general non-classical) truth value at every state of a game. The just presented state of affairs triggers a question that will guide the investigations of this paper: *To what extent can deductive fuzzy logics be modeled by games that remain close in their format, if not in spirit, to Hintikka's classic game theoretic semantics?*

The paper is organized as follows. In Section 2 we present (notational variants) of the mentioned semantic games by Hintikka, Cintula/Majer, and Giles in a manner that provides a basis for systematic comparison. In particular, we observe that so-called Kleene-Zadeh logic KZ, a frequently applied fragment of Łukasiewicz logic $\mathsf{Ł}$, is characterized already by Hintikka's classic game if one generalizes the set of possible pay-off values from $\{0, 1\}$ to the unit interval $[0, 1]$. In Section 3 we introduce a fairly general scheme of rules that may be added to Hintikka's game in a many-valued setting and show that each connective specified by such a rule is already definable in logic KZ. Adapting an idea from [8, 10], we then show in Sections 4 and 5 how one can go beyond KZ, while retaining essential features of Hintikka's original game format. In particular, we introduce in Section 4 a propositional 'random choice connective' π by a very simple rule. We show that this rule for π in combination with a rule for doubling the pay-off for the player who is currently in the role of the 'Proponent' leads to a proper extension of propositional Łukasiewicz logic. In Section 5 we indicate how, at the first-order level, various families of rules that involve a random selection of witnessing domain elements characterize corresponding families of fuzzy quantifiers. We conclude in Section 6 with a brief summary, followed by remarks on the relation between our 'randomized game semantics' and the 'equilibrium semantics' for IF-logic [25, 32] arising from considering incomplete information in Hintikka's game.

2 Variants of semantic games

Let us start by reviewing Hintikka's classic semantic game [18, 20]. There are two players, called *Myself* (or simply *I*) and *You*, here, who can both act either in the role of the *Proponent* $\mathbf{P}$ or of the *Opponent* $\mathbf{O}$[1] of a given first-order formula F, augmented by a variable assignment θ. Initially *I* act as $\mathbf{P}$ and *You*

[1] Hintikka uses *Nature* and *Myself* as names for the players and *Verfier* and *Falisifer* for the two roles. To emphasize out interest in the connection to Giles's game we use Giles's names for the players and Lorenzen's corresponding role names throughout the paper.

act as **O**. *My* aim — or, more generally, **P**'s aim at any state of the game — is to show that the initial formula is true in a given interpretation $\mathcal{M}$ with respect to θ. The game proceeds according to the following rules. Note that these rules only refer to the roles and the outermost connective of the *current formula*, i.e. the formula, augmented by an assignment of domain elements to free variables, that is at stake at the given state of the game. Together with a *role distribution* of the players, this *augmented formula* fully determines any state of the game.

($R^{\mathcal{H}}_{\wedge}$) If the current formula is $(F \wedge G)[\theta]$ then **O** chooses whether the game continues with $F[\theta]$ or with $G[\theta]$.
($R^{\mathcal{H}}_{\vee}$) If the current formula is $(F \vee G)[\theta]$ then **P** chooses whether the game continues with $F[\theta]$ or with $G[\theta]$.
($R^{\mathcal{H}}_{\neg}$) If the current formula is $\neg F[\theta]$, the game continues with $F[\theta]$, except that the roles of the players are switched: the player who is currently acting as **P**, acts as **O** at the the next state, and vice versa for the current **O**.
($R^{\mathcal{H}}_{\forall}$) If the current formula is $(\forall x F(x))[\theta]$ then **O** chooses an element c of the domain of $\mathcal{M}$ and the game continues with $F(x)[\theta[c/x]]$[2].
($R^{\mathcal{H}}_{\exists}$) If the current formula is $\exists x F(x)[\theta]$ then **P** chooses an element c of the domain of $\mathcal{M}$ and the game continues with $F(x)[\theta[c/x]]$.

Except for ($R^{\mathcal{H}}_{\neg}$), the players' roles remain unchanged. The game ends when an atomic (augmented) formula $A[\theta]$ is hit. The player who is currently acting as **P** *wins* and the other player, acting as **O**, *loses* if A is true with respect to θ in the given model $\mathcal{M}$. We associate pay-off 1 with winning and pay-off 0 with losing. We also include the truth constants $\top$ and $\bot$, with their usual interpretation, among the atomic formulas. The game starting with formula F and assignment θ is called the *$\mathcal{H}$-game for $F[\theta]$ under $\mathcal{M}$*.

Theorem 1 (Hintikka). *A formula F is true in a (classical) interpretation $\mathcal{M}$ with respect to the initial variable assignment θ (in symbols: $v^{\theta}_{\mathcal{M}}(F) = 1$) iff I have a winning strategy in the $\mathcal{H}$-game for $F[\theta]$ under $\mathcal{M}$.*

Our aim is to generalize Hintikka's Theorem to deductive fuzzy logics. As already mentioned in the introductions, contemporary mathematical fuzzy logic offers a plethora of logical systems. Here we focus on (extensions of) a system simply called 'fuzzy logic', e.g., in the well known textbook [27]. Following [1], we prefer to call this logic *Kleene-Zadeh logic*, or KZ for short. KZ is mostly considered only at the propositional level, where its semantics is given by extending an assignment $\mathcal{M}$ of atomic formulas to truth values in $[0, 1]$ as follows:

$$\begin{aligned}
&v_{\mathcal{M}}(F \wedge G) = \min(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G)),\\
&v_{\mathcal{M}}(F \vee G) = \max(v_{\mathcal{M}}(F), v_{\mathcal{M}}(G)),\\
&v_{\mathcal{M}}(\neg F) = 1 - v_{\mathcal{M}}(F),\\
&v_{\mathcal{M}}(\bot) = 0,\\
&v_{\mathcal{M}}(\top) = 1.
\end{aligned}$$

[2] $\theta[c/x]$ denotes the variable assignment that is like θ, except for assigning c to x.

At the first-order level an interpretation $\mathcal{M}$ includes a non-empty set D as domain. With respect to an assignment θ of domain elements to free variables, the semantics of the universal and the existential quantifier is given by

$$v_{\mathcal{M}}^{\theta}(\forall x F(x)) = \inf_{d\in D}(v_{\mathcal{M}}^{\theta[d/x]}(F(x))),$$
$$v_{\mathcal{M}}^{\theta}(\exists x F(x)) = \sup_{d\in D}(v_{\mathcal{M}}^{\theta[d/x]}(F(x))).$$

It is interesting to observe that neither the rules nor the notion of a state in an $\mathcal{H}$-game have to be changed in order to characterize logic KZ. We only have to generalize the possible pay-off values for the $\mathcal{H}$-game from $\{0,1\}$ to the unit interval $[0,1]$. More precisely, the pay-off for the player who is in the role of $\mathbf{P}$ when a game under $\mathcal{M}$ ends with the augmented atomic formula $A[\theta]$ is $v_{\mathcal{M}}^{\theta}(A)$.

If the pay-offs are modified as just indicated and correspond to the truth values of atomic formulas specified by a many-valued interpretation $\mathcal{M}$, we will speak of an $\mathcal{H}$-mv-game, where the *pay-offs match* $\mathcal{M}$. A slight complication arises for quantified formulas in $\mathcal{H}$-mv-games: there might be no element c in the domain of $\mathcal{M}$ such that $v_{\mathcal{M}}^{\theta[c/x]}(F(x)) = \inf_{d\in D}(v_{\mathcal{M}}^{\theta[d/x]}(F(x)))$ or no domain element e such that $v_{\mathcal{M}}^{\theta[e/x]}(F(x)) = \sup_{d\in D}(v_{\mathcal{M}}^{\theta[d/x]}(F(x)))$. A simple way to deal with this fact is to restrict attention to so-called witnessed models [17], where constants that witness all arising infima and suprema are assumed to exist. In other words: infima are minima and suprema are maxima in witnessed models. A more general solution refers to optimal payoffs up to some ϵ.

Definition 1. *Suppose that, for every $\epsilon > 0$, player $\boldsymbol{X}$ has a strategy that guarantees her a pay-off of at least $w - \epsilon$, while her opponent has a strategy that ensures that $\boldsymbol{X}$'s pay-off is at most $w + \epsilon$, then w is called the* value for $\mathbf{X}$ of the game*.*

This notion, which corresponds to that of an ϵ-equilibrium as known from game theory, allows us to state the following generalization of Theorem 1.

Theorem 2. *A formula F evaluates to $v_{\mathcal{M}}^{\theta}(F) = w$ in a KZ-interpretation $\mathcal{M}$ with respect to the variable assignment θ iff the $\mathcal{H}$-mv-game for $F[\theta]$ with pay-offs matching $\mathcal{M}$ has value w for Myself.*

A proof of Theorem 2 can (essentially[3]) be found in [10].

From the point of view of continuous t-norm based fuzzy logics, as popularized by Petr Hájek [15, 16], Kleene-Zadeh logic KZ is unsatisfying: while min is a t-norm, it's indicated residuum, which corresponds to implication in Gödel-Dummett logic is not expressible. Indeed, defining implication by $F \supset G =_{def} \neg F \vee G$ (in analogy to classical logic) in KZ, entails that $F \to F$ is not valid, i.e. $v_{\mathcal{M}}(F \to F)$ is not true in all interpretations.[4] In fact, formulas that do not contain truth constants are never valid in KZ. Besides Gödel-Dummett

[3] A variant of $\mathcal{H}$-games is used in [10] and KZ is called 'weak Łukasiewicz logic' there.

[4] We suppress the reference to a variable assignment θ when referring to propositional connectives.

logic, the most important fuzzy logic extending KZ arguably is Łukasiewicz logic $\mathsf{Ł}$. The language of $\mathsf{Ł}$ extends that of KZ by implication $\to$, strong conjunction $\otimes$, and strong disjunction $\oplus$. The semantics of these connectives is given by

$$\begin{aligned}
v_{\mathcal{M}}(F \to G) &= \min(1, 1 - v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G)),\\
v_{\mathcal{M}}(F \otimes G) &= \max(0, v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G) - 1),\\
v_{\mathcal{M}}(F \oplus G) &= \min(1, v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G)).
\end{aligned}$$

In fact all other propositional connectives could by defined in $\mathsf{Ł}$, e.g., from $\to$ and $\bot$, or from $\otimes$ and $\neg$, alone. However, neither $\to$ nor $\otimes$ nor $\oplus$ can be defined in KZ.[5] The increased expressiveness of $\mathsf{Ł}$ over KZ is particularly prominent at the first-order level: while in KZ there are only trivially valid formulas (which involve the truth constants in an essential manner), the set of valid first-order formulas in $\mathsf{Ł}$ is not even recursively enumerable, due to a classic result of Scarpellini [31].

It seems to be impossible to characterize full Łukasiewicz logic $\mathsf{Ł}$ by trivial extensions of the $\mathcal{H}$-game, comparable to the shift from $\mathcal{H}$-games to $\mathcal{H}$-mv-games. Before investigating, in Sections 4 and 5, how one can nevertheless generalize the $\mathcal{H}$-game to extensions of KZ, including $\mathsf{Ł}$, without changing the concept of a game state as solely determined by an (augmented) formula and a role distribution, we review two types of semantic games for $\mathsf{Ł}$ that deviate more radically from Hintikka's classic game theoretical semantics: explicit evaluation games, due to Cintula and Majer [4], and Giles's dialogue and betting game [12, 13].

In [4] Cintula and Majer present a game for $\mathsf{Ł}$ that conceptually differs from the $\mathcal{H}$-mv-game by introducing an explicit reference to a value $\in [0, 1]$ at every state of the game. They simply speak of an 'evaluation game'; but since all games considered in this paper are games for evaluating formulas with respect to a given interpretation, we prefer to speak of an *explicit evaluation game*, or $\mathcal{E}$-game for short. Like above, we call the players *Myself* (*I*) and *You*, and the roles **P** and **O**. In the initial state *I* am in the role of **P** and *You* are acting as **O**. In addition to the role distribution and the current (augmented) formula[6], also a *current value* $\in [0, 1]$ is included in the specification of a game state. We will not need to refer to any details of $\mathcal{E}$-games, but present the rules for $\oplus$, $\otimes$, $\neg$, and $\exists$ here, to assist the comparison with other semantic games:

($R^{\mathcal{E}}_{\otimes}$) If the current formula is $(F \otimes G)[\theta]$ and the current value[7] is r then **P** chooses a value $\bar{r} \leq 1 - r$ and **O** chooses whether to continue the game with $F[\theta]$ and value $r + \bar{r}$ or with $G[\theta]$ and value $1 - \bar{r}$.

($R^{\mathcal{E}}_{\oplus}$) If the current formula is $(F \oplus G)[\theta]$ and the current value is r then **P** chooses $\bar{r} \leq 1 - r$ and **O** chooses whether to continue with $F[\theta]$ and value $\bar{r}$ or with $G[\theta]$ and value $r - \bar{r}$.

($R^{\mathcal{E}}_{\neg}$) If the current formula is $\neg F[\theta]$ and the current value is r, then **O** chooses $\bar{r}$, where $0 < \bar{r} \leq r$, and the game continues with $F[\theta]$ and value $(1 - r) + \bar{r}$ after a role switch.

[5] Therefore KZ is sometimes called the 'weak (fragment of) Łukasiewicz logic'.

[6] I.e., the current formula, now over the language of $\mathsf{Ł}$, augmented by an assignment of domain elements to free variables.

[7] All values mentioned here have to be in $[0, 1]$.

($R_\exists^{\mathcal{H}}$) If the current formula is $\exists x F(x)[\theta]$ and the current value is r then **O** chooses $\bar{r} > 0$ and **P** picks an element c of the domain of $\mathcal{M}$ and the game continues with $F(x)[\theta[c/x]]$ and value $r - \bar{r}$.

The rules for $\wedge$, $\vee$, $\forall$ are analogous to the corresponding rules for the $\mathcal{H}$-mv-game: the current value remains unchanged. Cintula and Mayer [4] do not specify a rule for implication. However such a rule can be synthesized from the other rules, given the definability of $\rightarrow$ from the other connectives. As soon as the game reaches an augmented atomic formula $A[\theta]$ the game under interpretation $\mathcal{M}$ ends and the player in the current role of **P** wins (and the opposing player loses) if $v_{\mathcal{M}}^{\theta}(A) \geq r$. Otherwise the current **O** wins and the current **P** loses. Compared to Theorems 1 and 2, the adequateness theorem for the $\mathcal{E}$-game shows a somewhat less direct correspondence to the standard semantics of Ł.

Theorem 3 (Cintula/Mayer). *I have a winning strategy in the $\mathcal{E}$-game under $\mathcal{M}$ starting with $F[\theta]$ and value r iff $v_{\mathcal{M}}^{\theta}(F) \geq r$.*

A game based interpretation of Ł that arguably deviates even more radically from $\mathcal{H}$-games than $\mathcal{E}$-games was presented by Giles already in the 1970s [12, 13]. In fact Giles did not refer to Hintikka, but rather to the dialogue games suggested by Lorenzen [23, 24] as a foundation for constructive reasoning. Initially Giles proposed his game as a model of logical reasoning within theories of physics; but later he motivated the game explicitly as an attempt to provide "tangible meaning" for fuzzy logic [14]. We briefly review the essential features of Giles's game, in a variant called $\mathcal{G}$-game, that facilitates comparison with the other semantic games mentioned in this paper. Again we use *Myself* (I) and *You* as names for the players, and refer to the roles **P** and **O**. Unlike in $\mathcal{H}$-, $\mathcal{H}$-mv- or $\mathcal{E}$-games, a game state contains more that one current formula, in general. More precisely a state of a $\mathcal{G}$-game is given by

$$[F_1[\theta_1], \ldots, F_m[\theta_m] \mid G_1[\theta'_1], \ldots, G_n[\theta'_n]]\,,$$

where $\{F_1[\theta_1], \ldots, F_m[\theta_m]\}$ is the *multiset* of augmented formulas currently asserted by *You*, called *your tenet*, and $\{G_1[\theta'_1], \ldots, G_n[\theta'_n]\}$ is the multiset of augmented formulas currently asserted by *Myself*, called *my tenet*. At any given state an occurrence of a non-atomic augmented formula $H[\theta]$ is picked arbitrarily and distinguished as *current formula*.[8] If $H[\theta]$ is in my tenet then I am acting as **P** and *You* are acting as **O**. Otherwise, i.e. if $H[\theta]$ is in your tent, I am **O** and *You* are **P**. States that only contain atomic formulas are called *final*. At non-final states the game proceeds according to the following rules:

($R_\wedge^{\mathcal{G}}$) If the current formula is $(F \wedge G)[\theta]$ then the game continues in a state where the indicated occurrence of $(F \wedge G)[\theta]$ in **P**'s tenet is replaced by either $F[\theta]$ or by $G[\theta]$, according to **O**'s choice.

[8] It turns out that the powers of the players of a $\mathcal{G}$-game are not depended on the manner in which the current formula is picked at any state. Still, a more formal presentation of $\mathcal{G}$-games will employ the concepts of a regulation and of so-called internal states in formalizing state transitions. We refer to [7] for details.

($R^{\mathcal{G}}_{\vee}$) If the current formula is $(F \vee G)[\theta]$ then the game continues in a state where the indicated occurrence of $(F \vee G)[\theta]$ in **P**'s tenet is replaced by either $F[\theta]$ or by $G[\theta]$, according to **P**'s choice.

($R^{\mathcal{G}}_{\rightarrow}$) If the current formula is $(F \rightarrow G)[\theta]$ then the indicated occurrence of $(F \rightarrow G)[\theta]$ is removed from **P**'s tenet and **O** chooses whether to continue the game at the resulting state or whether to add $F[\theta]$ to **O**'s tenet and $G[\theta]$ to **P**'s tenet before continuing the game.

($R^{\mathcal{G}}_{\forall}$) If the current formula is $(\forall x F(x))[\theta]$ then **O** chooses an element c of the domain of $\mathcal{M}$ and the game continues in a state where the indicated occurrence of $(\forall x F(x))[\theta]$ in **P**'s tenet is replaced by $F(x)[\theta[c/x]]$.

($R^{\mathcal{G}}_{\exists}$) If the current formula is $(\exists x F(x))[\theta]$ then **P** chooses an element c of the domain of $\mathcal{M}$ and the game continues in a state where the indicated occurrence of $(\forall x F(x))[\theta]$ in **P**'s tenet is replaced by $F(x)[\theta[c/x]]$.

No rule for negation is needed if $\neg F$ is defined as $F \rightarrow \bot$. Likewise, rules for strong conjunction $\otimes$ and $\oplus$ can either be dispensed with by treating these connectives as defined from the other connectives or by introducing corresponding rules. (See [5, 7] for a presentation of rules for strong conjunction.) If no non-atomic formula is left to pick as current formula, the game has reached a final state

$$[A_1[\theta_1], \dots, A_m[\theta_m] \mid B_1[\theta'_1], \dots, B_n[\theta'_n]]\,,$$

where the $A_i[\theta_i]$ and $B_i[\theta'_i]$ are atomic augmented formulas. With respect to an interpretation $\mathcal{M}$ (i.e, an assignment of truth values to all atomic augmented formulas) the pay-off for *Myself* at this state is defined as

$$m - n + 1 + \sum_{1 \leq i \leq n} v^{\theta}_{\mathcal{M}}(B_i) - \sum_{1 \leq i \leq m} v^{\theta}_{\mathcal{M}}(A_i).$$

(Empty sums are identified with 0.) These pay-off values are said to *match* $\mathcal{M}$.

Just like for $\mathcal{H}$-mv-games, we need to take into account that suprema and infima are in general not witnessed by domain elements. Note that Definition 1 does not refer to any particular game. We may therefore apply the notion of the *value of a game* to $\mathcal{G}$-games as well. A $\mathcal{G}$-game where *my* tenet at the initial state consists of a single augmented formula occurrence $F[\theta]$, while *your* tenet is empty, is called a $\mathcal{G}$-game for $F[\theta]$. This allows us to express the adequateness of $\mathcal{G}$-games for Łukasiewicz logic in direct analogy to Theorem 2.

Theorem 4 (essentially Giles[9]). *A formula F evaluates to $v^{\theta}_{\mathcal{M}}(F) = w$ in a Ł-interpretation $\mathcal{M}$ with respect to the variable assignment θ iff the $\mathcal{G}$-game for $F[\theta]$ with pay-offs matching $\mathcal{M}$ has value w for Myself.*

At this point readers familiar with the original presentation of the game in [12, 13] might be tempted to protest that we have skipped Giles's interesting

[9] Giles [12, 13] only sketched a proof for the language without strong conjunction. For a detailed proof of the propositional case, where the game includes a rule for strong conjunction, we refer to [7].

story about betting money on the results of dispersive experiments associated with atomic assertions. Indeed, Giles proposes to assign an experiment E_A to each atomic formula A[10]. While each trial of an experiment yields either "yes" or "no" as its result, successive trials of the same experiment may lead to different results. However for each experiment E_A there is a known probability $\langle A \rangle$ that the result of a trial of E_A is negative. Experiment $E_\bot$ always yields a negative result; therefore $\langle \bot \rangle = 1$. Similarly $\langle \top \rangle = 0$. For each occurrence ('assertion') of an atomic formula in a player's final tenet, the corresponding experiment is run and the player has to pay one unit of money (say 1€) to the other player if the result is negative. Therefore Giles calls $\langle A \rangle$ the *risk* associated with A. For the final state $[A_1, \ldots, A_m \mid B_1, \ldots, B_n]$ the expected total amount of money that *I* have to pay to *You* (my total risk) is readily calculated to equal

$$\Big(\sum_{1 \leq i \leq m} \langle A_i \rangle - \sum_{1 \leq i \leq n} \langle B_i \rangle \Big) \text{€}.$$

Note that the total risk at final states translates into the pay-off specified above for $\mathcal{G}$-games via $v^\theta_\mathcal{M}(A) = 1 - \langle A \rangle$. To sum up: Giles's interpretation of truth values as inverted risk values associated with bets on dispersive experiments is totally independent from the semantic game for the stepwise reduction of complex formulas to atomic sub-formulas. In principle, one can interpret the pay-off values also for the $\mathcal{H}$-mv-game as inverted risk values and speak of bets on dispersive experiments at final states also there. The only (technically inconsequential) difference to the original presentation is that one implicitly talks about *expected* pay-off (inverted *expected* loss of money), rather than of certain pay-off when the betting scenario is used to interpret truth values.

Table 1 provides a summary of the general structure of the games reviewed in this section (where 'formula' means 'augmented formula' in the first-order case).

Table 1. Comparison of some semantic games

game	state determined by	pay-offs
$\mathcal{H}$-game	single formula + role distribution	bivalent
$\mathcal{H}$-mv-game	single formula + role distribution	many-valued
$\mathcal{E}$-game	single formula + role distribution + value	many-valued
$\mathcal{G}$-game	two multisets of formulas	many-valued

3 Generalized propositional rules for the $\mathcal{H}$-mv-game

At a first glimpse the possibilities for extending $\mathcal{H}$-mv-games to logics more expressive than KZ look very limited if, in contrast to $\mathcal{E}$-games and $\mathcal{G}$-games, we insist on *Hintikka's principle* that a state of the game is fully determined by a

[10] Giles ignores variable assignments, but stipulates that there is a constant symbol for every domain element. Thus only closed formulas need to be considered.

formula[11] and a distribution of the two roles (**P** and **O**) to the two players. One can come up with a more general concept of propositional game rules, related to those described in [6] for connectives defined by arbitrary finite deterministic and non-deterministic matrices. In order to facilitate a concise specification of all rules of that type, we introduce the following technical notion.

Definition 2. *An* n-selection *is a non-empty subset* S *of* $\{1, \ldots, n\}$*, where each element of* S *may additionally be marked by a* switch sign*.*

A game rule for an n-ary connective $\diamond$ in a *generalized* $\mathcal{H}$*-mv-game* is specified by a non-empty set $\{S_1, \ldots, S_m\}$ of n-selections. According to this concept, a round in a generalized $\mathcal{H}$-mv-game consists of two phases. The scheme for the corresponding game rule specified by $\{S_1, \ldots, S_m\}$ is as follows:

(Phase 1): If the current formula is $\diamond(F_1, \ldots, F_n)$ then **O** chooses an n-selection S_i from $\{S_1, \ldots, S_m\}$.
(Phase 2): **P** chooses an element $j \in S_i$. The game continues with formula F_j, where the roles of the players are switched if j is marked by a switch sign.

Remark 1. A variant of this scheme arises by letting **P** choose the n-selection S_i in phase 1 and **O** choose $j \in S_i$ in phase 2. But note that playing the game for $\diamond(F_1, \ldots, F_n)$ according to that role inverted scheme is equivalent to playing the game for $\neg \diamond (\neg F_1, \ldots, \neg F_n)$ using the exhibited scheme.

Remark 2. The rules $R^{\mathcal{H}}_{\wedge}$, $R^{\mathcal{H}}_{\vee}$, and $R^{\mathcal{H}}_{\neg}$ can be understood as instances of the above scheme:
– $R^{\mathcal{H}}_{\wedge}$ is specified by $\{\{1\}, \{2\}\}$,
– $R^{\mathcal{H}}_{\vee}$ is specified by $\{\{1, 2\}\}$, and
– $R^{\mathcal{H}}_{\neg}$ is specified by $\{\{1^*\}\}$, where the asterisk is used as switch mark.

Theorem 5. *In a generalized* $\mathcal{H}$*-mv-game, each rule of the type described above corresponds to a connective that is definable in logic* KZ*.*

Proof. The argument for the adequateness of all semantic games considered in this paper proceeds by backward induction on the game tree.

For (generalized) $\mathcal{H}$-mv-games the base case is trivial: by definition **P** receives pay-off $v_{\mathcal{M}}(A)$ and **O** receives pay-off $1 - v_{\mathcal{M}}(A)$ if the game ends with the atomic formula A.

For the inductive case assume that the current formula is $\diamond(F_1, \ldots, F_n)$ and that the rule for $\diamond$ is specified by the set $\{S_1, \ldots, S_m\}$ of n-selections, where $S_i = \{j(i, 1), \ldots, j(i, k(i))\}$ for $1 \leq i \leq m$ and $1 \leq k(i) \leq n$. Remember that the elements of S_i are numbers $\in \{1, \ldots, n\}$, possibly marked by a switch sign. For sake of clarity let us first assume that there are no switch signs, i.e. no role switches occur. Let us say that a player **X** *can force pay-off* w if **X** has a strategy that guarantees her a pay-off $\geq w$ at the end of the game. By the

[11] Since we focus on the propositional level, we will drop all explicit reference to variable assignments in Sections 3 and 4. However all statements remain valid if one replaces ‘formula’ by ‘formula augmented by a variable assignment’ throughout these sections.

induction hypothesis, **P** can force pay-off $v_{\mathcal{M}}(G)$ for herself and **O** can force pay-off pay-off $1 - v_{\mathcal{M}}(G)$ for himself if G is among $\{F_1, \ldots, F_n\}$ and does indeed occur at a successor state to the current one; in other words, if $G = F_{j(i,\ell)}$ for some $i \in \{1, \ldots, m\}$ and $\ell \in \{1, \ldots, k(i)\}$. Since **O** chooses the n-selection S_i, while **P** chooses an index number in S_i, **P** can force pay-off

$$\min_{1 \leq i \leq m} \max_{1 \leq \ell \leq k(i)} v_{\mathcal{M}}(F_{j(i,\ell)})$$

at the current state, while **O** can force pay-off

$$\max_{1 \leq i \leq m} \min_{1 \leq \ell \leq k(i)} (1 - v_{\mathcal{M}}(F_{j(i,\ell)})) = 1 - \min_{1 \leq i \leq m} \max_{1 \leq \ell \leq k(i)} v_{\mathcal{M}}(F_{j(i,\ell)}).$$

If both players play optimally these pay-off values are actually achieved. Therefore the upper expression corresponds to the truth function for $\diamond$. Both expressions have to be modified by uniformly substituting $1 - v_{\mathcal{M}}(F_{j(i,\ell)})$ for $v_{\mathcal{M}}(F_{j(i,\ell)})$ whenever $j(i,\ell)$ is marked by a switch sign in S_1 for $1 \leq i \leq m$ and $1 \leq k(i) \leq n$.

To infer that the connective $\diamond$ is definable in logic KZ it suffices to observe that its truth function, described above, can be composed from the functions $\lambda x(1-x)$, $\lambda x, y \min(x,y)$, and $\lambda x, y \max(x,y)$. But these functions are the truth functions for $\neg$, $\wedge$, and $\vee$, respectively, in KZ. □

4 Random choice connectives

In Section 2, following Giles, we have introduced the idea of expected pay-offs in a randomized setting. However, Giles applied this idea only to the interpretation of *atomic* formulas. For the interpretation of logical connectives and quantifiers in any of the semantic games mentioned in Section 2 it does not matter whether the players seek to maximize expected or certain pay-off or, equivalently, try to minimize either expected or certain payments to the opposing player. In [8, 10] we have shown that considering random choices of witnessing constants in quantifier rules for *Giles-style* games, allows one to model certain (semi-)fuzzy quantifiers that properly extend first-order Łukasiewicz logic. In this section we want to explore the consequences of introducing random choices in rules for propositional connectives context of *Hintikka-style* games.

The results of Section 3 show that, in order to go beyond logic KZ with Hintikka-style games, a new variant of rules has to be introduced. As already indicated, a particularly simple type of new rules, that does not entail any change in the structure of game states, arises from randomization. So far we have only considered rules where either **P** or **O** chooses the sub-formula of the current formula to continue the game with. In game theory one often introduces *Nature* as a special kind of additional player, who does not care what the next state looks like, when it is her time to move and therefore is modeled by a uniformly random choice between all moves available to *Nature* at that state. As we will see below, introducing *Nature* leads to increased expressive power of semantic games. In fact, to keep the presentation of the games simple, we prefer to leave the role of *Nature* only implicit and just speak of random choices, without attributing them officially to a third player. The most basic rule of the indicated type refers to a new propositional connective π and can be formulated as follows.[12]

($R_\pi^{\mathcal{R}}$) If the current formula is $(F\pi G)$ then a uniformly random choice determines whether the game continues with F or with G.

Remark 3. Note that no role switch is involved in the above rule: the player acting as **P** remains in this role at the succeeding state; likewise for **O**.

We call the $\mathcal{H}$-mv-game augmented by rule ($R_\pi^{\mathcal{R}}$) the *(basic) $\mathcal{R}$-game.* We claim that the new rule gives raise to the following truth function, to be added to the semantics of logic KZ:

$$v_{\mathcal{M}}(F\pi G) = (v_{\mathcal{M}}(F) + v_{\mathcal{M}}(G))/2.$$

$\mathsf{KZ}(\pi)$ denotes the logic arising from KZ by adding π. To assist a concise formulation of the adequateness claim for the $\mathcal{R}$-game we have to adapt Definition 1 by replacing 'pay-off' with 'expected pay-off'. In fact, since we restrict attention to the propositional level here, we can use the following simpler definition.

Definition 3. *If player $\boldsymbol{X}$ has a strategy that leads to an expected pay-off for her of at least w, while her opponent has a strategy that ensures that $\boldsymbol{X}$'s expected pay-off is at most w, then w is called the* expected value for **X** *of the game.*

Theorem 6. *A propositional formula F evaluates to $v_{\mathcal{M}}(F) = w$ in a $\mathsf{KZ}(\pi)$-interpretation $\mathcal{M}$ iff the basic $\mathcal{R}$-game for F with pay-offs matching $\mathcal{M}$ has expected value w for Myself.*

Proof. Taking into account that $v_{\mathcal{M}}(F)$ coincides with the value of the $\mathcal{H}$-mv-game matching $\mathcal{M}$ if F does not contain the new connective π, we only have to add the case for a current formula of the form $G\pi H$ to the usual backward induction argument. However, because of the random choice involved in rule ($R_\pi^{\mathcal{R}}$), it is now her *expected* pay-off that **P** seeks to maximize and **O** seeks to minimize.

Suppose the current formula is $G\pi H$. By the induction hypothesis, at the successor state σ_G with current formula G (the player who is currently) **P** can force[13] an expected pay-off $v_{\mathcal{M}}(G)$ for herself, while **O** can force an expected pay-off $1 - v_{\mathcal{M}}(G)$ for himself. Therefore the expected value for **P** for the game starting in σ_G is $v_{\mathcal{M}}(G)$ for **P**. The same holds for H instead of G. Since the choice between the two successor states σ_G and σ_H is uniformly random, we conclude that the expected value for **P** for the game starting with $G\pi H$ is the average of $v_{\mathcal{M}}(F)$ and $v_{\mathcal{M}}(G)$, i.e. $(v_{\mathcal{M}}(F)+v_{\mathcal{M}}(G))/2$. The theorem thus follows from the fact that I (*Myself*) am the initial **P** in the $\mathcal{R}$-game for F. $\square$

Since the function $\lambda x, y(x + y)/2$ cannot be composed solely from the functions $\lambda x(1 - x)$, $\lambda x, y \min(x, y)$, $\lambda x, y \max(x, y)$ and the values 0 and 1, we can make the following observation.

[12] A similar rule is considered in [33] in the context of partial logic.

[13] We re-use the terminology introduced in the proof of Theorem 5, but applied to *expected* pay-offs here.

Proposition 1. *The connective π is not definable in logic* KZ*.*

But also the following stronger fact holds.

Proposition 2. *The connective π is not definable in Łukasiewicz logic* $\mathsf{Ł}$*.*

Proof. By McNaughton's Theorem [26] a function $f : [0,1]^n \to [0,1]$ corresponds to a formula of propositional Łukasiewicz logic iff f is piecewise linear, where every linear piece has integer coefficients. But clearly the coefficient of $(x+y)/2$ is not an integer. $\square$

Remark 4. We may also observe that, in contrast to $\mathsf{Ł}$, not only $\overline{0.5} =_{def} \bot\pi\top$, but in fact every rational truth constant that has a finite representation in the binary system is definable in logic $\mathsf{KZ}(\pi)$,

Conversely to Proposition 2 we also have the following.

Proposition 3. *None of the connectives $\otimes$, $\oplus$, $\to$ of* $\mathsf{Ł}$ *can be defined in* $\mathsf{KZ}(\pi)$*.*

Proof. Let Ψ denote the set of all interpretations $\mathcal{M}$, where $0 < v_{\mathcal{M}}(A) < 1$ for all propositional variables A. The following claim can be straightforwardly checked by induction.

For every formula F of $\mathsf{KZ}(\pi)$ one of the following holds:
(1) $0 < v_{\mathcal{M}}(F) < 1$ for all $\mathcal{M} \in \Psi$, or
(2) $v_{\mathcal{M}}(F) = 1$ for all $\mathcal{M} \in \Psi$, or
(3) $v_{\mathcal{M}}(F) = 0$ for all $\mathcal{M} \in \Psi$.

Clearly this claim does not hold for $A \otimes B$, $A \oplus B$, and $A \to B$. Therefore the connectives $\otimes$, $\oplus$, $\to$ cannot be defined in $\mathsf{KZ}(\pi)$. $\square$

In light of the above propositions, the question arises whether one can come up with further game rules, that, like $(R^{\mathcal{R}}_{\pi})$, do not sacrifice what we above called *Hintikka's principle*, i.e., the principle that game state is determined solely by a formula and a role distribution. An obvious way to generalize rule $(R^{\mathcal{R}}_{\pi})$ is to allow for a (potentially) biased random choice:

$(R^{\mathcal{R}}_{\pi^p})$ If the current formula is $(F\pi^p G)$ then the game continues with F with probability p, but continues with G with probability $1-p$.

Clearly, π coincides with $\pi^{0.5}$. But for other values of p we obtain a new connective. However, it is straightforward to check that Proposition 3 also holds if replace π by π^p for any $p \in [0,1]$.

Interestingly, there is a fairly simple game based way to obtain a logic that properly extends Łukasiewicz logic by introducing a unary connective D that signals that the pay-off values for **P** is to be doubled (capped to 1, as usual) at the end of the game.

$(R^{\mathcal{R}}_{\mathrm{D}})$ If the current formula is $\mathrm{D}F$ then the game continues with F, but with the following changes at the final state. The pay-off, say x, for **P** is changed to $\max(1, 2x)$, while the the pay-off $1-x$ for **O** is changed to $1-\max(1, 2x)$.

Remark 5. Instead of explicitly capping the modified pay-off for **P** to 1 one may equivalently give **O** the opportunity to either continue that game with doubled pay-off for **P** (and inverse pay-off for **O** herself) or to simply end the game at that point with pay-off 1 for **P** and pay-off 0 for **O** herself.

Let us use $\mathsf{KZ}(\mathrm{D})$ for the logic obtained from KZ by adding the connective D with the following truth function to KZ:

$$v_{\mathcal{M}}(\mathrm{D}F) = \min(1, 2 \cdot v_{\mathcal{M}}(F)).$$

Moreover, we use $\mathsf{KZ}(\pi, \mathrm{D})$ to denote the extension of KZ with both π and D and call the $\mathcal{R}$-game augmented by rule $(R^{\mathcal{R}}_{\mathrm{D}})$ the D-*extended* $\mathcal{R}$-*game.*

Theorem 7. *A propositional formula F evaluates to $v_{\mathcal{M}}(F) = w$ in a $\mathsf{KZ}(\pi, \mathrm{D})$-interpretation $\mathcal{M}$ iff the* D*-extended $\mathcal{R}$-game for F with pay-offs matching $\mathcal{M}$ has expected value w for Myself.*

Proof. The proof of Theorem 6 is readily extended to the present one by considering the additional inductive case of $\mathrm{D}G$ as current formula. By the induction hypothesis, the expected value for **P** of the game for G (under the same interpretation $\mathcal{M}$) is $v_{\mathcal{M}}(G)$. Therefore rule $(R^{\mathcal{R}}_{\mathrm{D}})$ entails that the expected value for **P** of the game for $\mathrm{D}G$ is $\max(1, 2 \cdot v_{\mathcal{M}}(G))$. □

Given Proposition 3 and Theorem 7 the following simple observation is of some significance.

Proposition 4. *The connectives $\otimes$, $\oplus$ and $\to$ of $\mathsf{Ł}$ are definable in $\mathsf{KZ}(\pi, D)$.*

Proof. It is straightforward to check that the following definitions in $\mathsf{KZ}(\pi, D)$ match the corresponding truth functions for Ł: $G \oplus F =_{def} \mathrm{D}(G\pi F)$, $G \otimes F =_{def} \neg\mathrm{D}(\neg G\pi\neg F)$, $G \to F =_{def} \mathrm{D}(\neg G\pi F)$. □

Remark 6. Note that Proposition 4 jointly with Theorem 7 entails that one can provide game semantics for (an extension of) Łukasiewicz without dropping "Hintikka's principle" as done in $\mathcal{E}$-games and in $\mathcal{G}$-games.

Remark 7. The definitions mentioned in the proof of Proposition 4 give rise to corresponding additional rules for the D-extended $\mathcal{R}$-game. E.g., for strong disjunction we obtain:

$(R^{\mathcal{R}}_{\oplus})$ If the current formula is $G \oplus F$ then a random choice determines whether to continue the game with F or with G. But in any case the pay-off for **P** is doubled (capped to 1), while the pay-off for **O** remains inverse to that for **P**.

By further involving role switches similar rules for strong conjunction and for implications are readily obtained.

It remains to be seen whether these rules can assist in arguing for the plausibility of the corresponding connective in intended application scenarios. But in any case, it is clear that, compared to the sole specification of truth functions, the game interpretation provides an additional handle for assessing the adequateness of the Łukasiewicz connectives for formalizing reasoning with graded notions and vague propositions.

Like $(R^{\mathcal{R}}_{\pi})$, also rule $(R^{\mathcal{R}}_{\mathrm{D}})$ can be generalized in an obvious manner:

$(R^{\mathcal{R}}_{\mathrm{M}_c})$ If the current formula is $\mathrm{M}_c F$ then the game continues with F, but with the following changes at the final state. The pay-off, say x, for **P** is changed to $\max(1, c{\cdot}x)$, while the the pay-off $1-x$ for **O** is changed to $1-\max(1, c{\cdot}x)$.

The enrichment of $\mathsf{KZ}(\pi, \mathrm{D})$ by further instances of π^p and M_c leads to rather expressive propositional fuzzy logics, related to, e.g., to Rational Łukasiewicz Logic and to divisible MV-algebras [11]. But lack of space prevents us from taking up this route, here.

5 Random witnesses for quantifiers

The idea of allowing for random choices of witnessing elements in quantifier rules — instead of **O**'s choice of a witness for a universally quantified statement and **P**'s choice of a witness for an existentially quantified statement — has already been introduced in [8, 10]. But the corresponding rules there refer to Giles's game instead of the $\mathcal{H}$-game and make essential use of the possibility to add more than one formula to the state of a $\mathcal{G}$-game. Moreover, attention has been restricted to so-called semi-fuzzy quantifiers in a two-tiered language variant of Łukasiewicz logic, where the predicates in the scope of such a quantifiers are crisp. Here, we lift that restriction and moreover want to retain *Hintikka's principle* of game states as being determined by a single (augmented) formula and a current role distribution.

In picking a witness element randomly, we may in principle refer to any given distribution over the domain. However, as convincingly argued, e.g., in [30], the meaning of quantifiers must remain invariant under isomorphism, i.e., under permutations of domain elements, if those quantifier are to be conceived as *logical* particles. This principle entails that the random choice of witnessing elements has to refer to the *uniform* distribution over the domain. However, as is well known, only *finite domains* admit uniform distributions. The restriction to finite domains is moreover well justified by the intended applications that largely model linguistic phenomena connected to gradedness and vagueness. As a welcome side effect of this restriction, we may drop the more involved notion of a value of a game as arising from approximations of pay-offs (Definition 1) and, like in Section 3 for propositional logics, may define the value of a game as in Definition 3.

The rule for the simplest quantifier (denoted by $\mathit{\Pi}$) that involves a random witness element is as follows.

$(R^{\mathcal{R}}_{\mathit{\Pi}})$ If the current formula is $(\mathit{\Pi} x F(x))[\theta]$ then an element c from the (finite) domain of $\mathcal{M}$ is chosen randomly and the game continues with $F(x)[\theta[c/x]]$.

In analogy to case of $(R^{\mathcal{R}}_{\pi})$ a truth function for $\mathit{\Pi}$ can be extracted from this rule (where $|D|$ is the cardinality of the domain D of $\mathcal{M}$):

$$v^{\theta}_{\mathcal{M}}(\mathit{\Pi} x F(x)) = \sum_{c \in D} \frac{v^{\theta[c/x]}_{\mathcal{M}}(F(x))}{|D|}$$

By $\mathsf{KZ}(\Pi)$ we refer to the logic KZ augmented by the quantifier Π. The proof of the corresponding adequateness statement is analogous to that of Theorem 6 and is therefore left to the reader.

Theorem 8. *A formula F evaluates to $v_{\mathcal{M}}(F) = w$ in a $\mathsf{KZ}(\Pi)$-interpretation $\mathcal{M}$ iff the $\mathcal{R}$-game for F, extended by rule $(R^{\mathcal{R}}_{\Pi})$, with pay-offs matching $\mathcal{M}$ has expected value w for Myself.*

Remark 8. Like the propositional connective π, the quantifier Π can be seen as an 'averaging operator', that provides explicit access to the (uniform) average of the values of the sub-formulas or instances of a formula $F\pi G$ or $\Pi x F(x)$, respectively.

Remark 9. Obviously one may extend not just KZ, but also the extensions of KZ discussed in Section 3 with the random choice quantifier Π. This leads to first-order logics that are strictly more expressive than Łukasiewicz logic Ł.

In [10] it is demonstrated how random choices of witness elements allow for the introduction of different (infinite) families of semi-fuzzy quantifiers that are intended to address the problem to justify particular fuzzy models of informal quantifier expressions like 'few', 'many', or 'about half'. As already mentioned above, the corresponding quantifier rules in [10] (like those in [8]) employ Giles's concept of referring to multisets of formulas asserted by **P** and by **O**, respectively, at any given state of the game. However, even without sacrificing *Hintikka's principle* by moving to $\mathcal{G}$-games or to $\mathcal{E}$-games, one can come up with new quantifier rules. For example, one may introduce a family of quantifiers $\widehat{\Pi}^n$ by the following parameterized game rule:

$(R^{\mathcal{R}}_{\widehat{\Pi}^n})$ If the current formula is $(\widehat{\Pi}^n x F(x))[\theta]$ then n elements $c_1, \ldots, c_n$ from the domain of $\mathcal{M}$ are chosen randomly. **P** then chooses some $c \in \{c_1, \ldots, c_n\}$ and the game continues with $F(x)[\theta[c/x]]$.

A dual family of quantifiers is obtained by replacing **P** by **O** in rule $(R^{\mathcal{R}}_{\widehat{\Pi}^n})$.Yet another type of quantifiers arises by the following rule:

$(R^{\mathcal{R}}_{\widetilde{\Pi}^n})$ If the current formula is $(\widetilde{\Pi}^x F(x))[\theta]$ then an element c_1 from the domain of $\mathcal{M}$ is chosen randomly. **P** decides whether to continue the game with $F(x)[\theta[c_1/x]]$ or to ask for a further randomly chosen element c_2. This procedure is iterated until an element c_i, where $1 < i \leq n$ is accepted by **P**. (c_n has to be accepted if none of the earlier random elements was accepted.) The game then continues with $F(x)[\theta[c_i/x]]$.

Again, variants of this rule are obtained by replacing **P** with **O** in $(R^{\mathcal{R}}_{\widetilde{\Pi}^n})$, possibly only for certain $i \in \{1, \ldots, n\}$.

Truth functions corresponding to the above rules are readily computed by applying elementary principles of probability theory. We will not work out these examples here and leave the systematic investigation of logics arising from enriching the $\mathcal{H}$-mv-game or the $\mathcal{R}$-game in the indicated manner to future work.

6 Conclusion

We began our investigations by observing (in Section 2) that Hintikka's well known game semantics for classical first-order logic (here referred to as $\mathcal{H}$-game) can be straightforwardly generalized to the $\mathcal{H}$-mv-game, where the pay-off values are taken from the unit interval $[0,1]$ instead of just $\{0,1\}$. Following [1], we call the resulting basic fuzzy logic Kleene-Zadeh logic KZ. At least two alternative types of semantic games, called $\mathcal{E}$-game and $\mathcal{G}$-game here, can be found in the literature (see, e.g., [4, 12, 5, 9, 7]). These games provide alternative semantics for Łukasiewicz logic $\mathsf{Ł}$, which is considerably more expressive than KZ. Both, the $\mathcal{E}$-game and the $\mathcal{G}$-game, deviate quite drastically from the $\mathcal{H}$-mv-game (and therefore also from the $\mathcal{H}$-game) in their underlying concept of a game state. In this paper, we have explored the power of semantic games that adhere of 'Hintikka's principle', by which we mean the principle that each state of a game is determined by a single formula (possibly augmented by a variable assignment) and a role distribution (telling us who of the two players is currently acting as Proponent **P** and who is currently acting as Opponent **O**). In Section 3 we have shown that adding rules that instantiate a fairly general scheme of possible rules to the $\mathcal{H}$-game does not give rise to logics that are more expressible than KZ. However introducing random choices in game rules, either as an alternative or in addition to choices made by **P** or by **O**, leads to various proper extensions of KZ, as we have seen in Section 4 for propositional logics and in Section 5 for the first-order level. In particular, the combination of the basic random choice connective π with a unary connective that signals doubling of pay-offs for **P** (capped to 1) allowed us to characterize a logic, in which all connectives of $\mathsf{Ł}$ are definable. A more complete and systematic exploration of the rich landscape of new connectives and quantifiers that can be defined for 'randomized' $\mathcal{H}$-mv-games is an obvious topic for future research.

We conclude with a brief remark on the relation between our 'randomized game semantics' and 'equilibrium semantics' for $\mathcal{H}$-games with imperfect information. We have only considered games of perfect information in this paper: the players always know all previous moves and thus have full knowledge of the current state of the game. However, the full power of Hintikka's game semantics arises from admitting that players may not be aware of all previous moves. This leads to *Independence Friendly* logic (IF-logic), where occurrences of quantifiers and connectives in a formula may be 'slashed' with respect to other such occurrences to indicate that the moves in the game that refer to those slashed occurrences are unknown to the current proponent. E.g., the formula $F = (G \vee_{/\{\wedge\}} H) \wedge (H \vee_{/\{\wedge\}} G)$ refers to an $\mathcal{H}$-game, where the choice by **P** of either the conjunct $G \vee_{/\{\wedge\}} H$ or $H \vee_{/\{\wedge\}} G$ is unknown to **O** when **O** has to choose either the right or the left disjunct of the remaining current formula. In [25] and in [32], Sandu and his colleagues present so-called equilibrium semantics for IF-logic, where mixed strategies for $\mathcal{H}$-games with incomplete information induce intermediate expected pay-off values in $[0,1]$, even if each atomic formula is evaluated to either 0 or 1. It is readily checked that the corresponding value for the above formula F is $(v_{\mathcal{M}}(G) + v_{\mathcal{M}}(H))/2$, where $v_{\mathcal{M}}(G)$ and $v_{\mathcal{M}}(H)$ are the values for G and H, respectively. In other words, we can simulate the effect of the random choice that induces our new connective π by the IF-formula F, and vice versa: π simulates effects of imperfect knowledge in games with classical pay-offs. Clearly, the connections between equilibrium semantics and (extended) $\mathcal{R}$-games deserves to be explored in more detail in future work.

Finally, we suggest that the results of this paper — in addition to the earlier results of Giles [12–14], Cintula/Majer [4], as well as Fermüller and co-authors [5, 7, 8, 10] — may serve as a basis for discussing to what extent and in which manner the game semantic approach to fuzzy logic addresses the important challenge of deriving truth functions for fuzzy connectives and quantifiers from basic semantic principles and thus to guide the fuzzy modeler's task in many application scenarios.

References

1. S. Aguzzoli, B. Gerla, and V. Marra. Algebras of fuzzy sets in logics based on continuous triangular norms. In *Proceedings of the 10th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty (ECSQARU 2009)*, pages 875–886. Springer, 2009.
2. A.D.C. Bennett, J.B. Paris, and A. Vencovska. A new criterion for comparing fuzzy logics for uncertain reasoning. *Journal of Logic, Language and Information*, 9(1):31–63, 2000.
3. P. Cintula, P. Hájek, and C. Noguera, editors. *Handbook of Mathematical Fuzzy Logic*. College Publications, 2011.
4. P. Cintula and O. Majer. Towards evaluation games for fuzzy logics. In O. Majer, A.-V. Pietarinen, and T. Tulenheimo, editors, *Games: Unifying Logic, Language, and Philosophy*, pages 117–138. Springer, 2009.
5. C.G. Fermüller. Revisiting Giles's game. In O. Majer, A.-V. Pietarinen, and T. Tulenheimo, editors, *Games: Unifying Logic, Language, and Philosophy*, Logic, Epistemology, and the Unity of Science, pages 209–227. Springer, 2009.
6. C.G. Fermüller. On matrices, Nmatrices and games. *Journal of Logic and Computation*, page to appear, 2013.
7. C.G. Fermüller and G. Metcalfe. Giles's game and the proof theory of Łukasiewicz logic. *Studia Logica*, 92(1):27–61, 2009.
8. C.G. Fermüller and C. Roschger. Randomized game semantics for semi-fuzzy quantifiers. In S. et.al. Greco, editor, *Advances in Computational Intelligence*, volume 300 of *Communications in Computer and Information Science*, pages 632–641. Springer, 2012.
9. C.G. Fermüller and C. Roschger. From games to truth functions: A generalization of Giles's game. *Studia Logica*, 2013. to appear.
10. C.G. Fermüller and C. Roschger. Randomized game semantics for semi-fuzzy quantifiers. *Logic Journal of the IGPL*, to appear.
11. Brunella Gerla. Rational łukasiewicz logic and DMV-algebras. *arXiv preprint arXiv:1211.5485*, 2012.
12. R. Giles. A non-classical logic for physics. *Studia Logica*, 33(4):397–415, 1974.
13. R. Giles. A non-classical logic for physics. In R. Wojcicki and G. Malinkowski, editors, *Selected Papers on Łukasiewicz Sentential Calculi*, pages 13–51. Polish Academy of Sciences, 1977.

14. R. Giles. Semantics for fuzzy reasoning. *International Journal of Man-Machine Studies*, 17(4):401–415, 1982.
15. P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer Academic Publishers, 2001.
16. P. Hájek. Why fuzzy logic? In D. Jacquette, editor, *Blackwell Companion to Philosophical Logic*, pages 596–606. Wiley, 2002.
17. P. Hájek. On witnessed models in fuzzy logic. *Mathematical Logic Quarterly*, 53(1):66–77, 2007.
18. J. Hintikka. Language-games for quantifiers. In N. Rescher, editor, *Studies in logical theory*, pages 46–72. Blackwell, Oxford, 1968. Reprinted in [19].
19. J. Hintikka. *Logic, language-games and information: Kantian themes in the philosophy of logic*. Clarendon Press Oxford, 1973.
20. J. Hintikka and G. Sandu. Game-theoretical semantics. In *Handbook of Logic and Language*. Elsevier, 2010.
21. E. Hisdal. Are grades of membership probabilities? *Fuzzy Sets and Systems*, 25(3):325–348, 1988.
22. J. Lawry. A voting mechanism for fuzzy logic. *International Journal of Approximate Reasoning*, 19(3-4):315–333, 1998.
23. P. Lorenzen. Logik und Agon. In *Atti Congr. Internaz. di Filosofia (Venezia, 12-18 Settembre 1958), Vol. IV*. Sansoni, 1960.
24. P. Lorenzen. Dialogspiele als semantische Grundlage von Logikkalkülen. *Archiv für mathemathische Logik und Grundlagenforschung*, 11:32–55, 73–100, 1968.
25. A.L. Mann, G. Sandu, and M. Sevenster. *Independence-friendly logic: A game-theoretic approach*. Cambridge University Press, 2011.
26. R. McNaughton. A theorem about infinite-valued sentential logic. *Journal of Symbolic Logic*, 16(1):1–13, 1951.
27. H.T. Nguyêñ and E.A. Walker. *A first course in fuzzy logic*. CRC Press, 2006.
28. J.B. Paris. A semantics for fuzzy logic. *Soft Computing*, 1(3):143–147, 1997.
29. J.B. Paris. Semantics for fuzzy logic supporting truth functionality. In V. Novák and I. Perfilieva, editors, *Discovering the world with fuzzy logic*, pages 82–104. Physica-Verlag, 2000.
30. S. Peters and D. Westerståhl. *Quantifiers in language and logic*. Oxford University Press, USA, 2006.
31. B. Scarpellini. Die Nichtaxiomatisierbarkeit des unendlichwertigen Prädikatenkalküls von Łukasiewicz. *Journal of Symbolic Logic*, 27(2):159–170, 1962.
32. M. Sevenster and G. Sandu. Equilibrium semantics of languages of imperfect information. *Annals of Pure and Applied Logic*, 161(5):618–631, 2010.
33. Xuefeng Wen and Shier Ju. Semantic games with chance moves revisited: from IF logic to partial logic. *Synthese*, 190(9):1605–1620, 2013.
34. L.A. Zadeh. Fuzzy logic. *IEEE: Computer*, 21(4):83–93, 1988.