

Resolution Refinements for Cut-Elimination based on Reductive Methods

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- Clarify (some of) the **essential differences** between CERes (cut-elimination by resolution) and reductive methods of cut-elimination, **in terms of the normalized proofs that they produce.**
- Define methods that are “intermediary” between CERes and reductive methods.

- Reductive Cut-Elimination Methods (\triangleright)
- Cut-Elimination by Resolution (CERes)
- CR-Simulation of \triangleright by CERes
- The Problem of Resolution Proof Search in CERes
- Some Properties of \triangleright
- Resolution Refinements (Proof Search Restrictions) for CERes based on the properties of \triangleright

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Reductive Cut-Elimination (\triangleright)

Example

$$\frac{\frac{\frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_i \quad \frac{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))} \exists_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))} \forall_i \quad \frac{\frac{\frac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a), P(a) \rightarrow Q(v) \vdash Q(v)} \rightarrow_i \quad \frac{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)}{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_i}{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_i \quad \text{cut}}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_r$$

Reductive Cut-Elimination (\triangleright)

Example

$$\frac{\frac{\frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_i \quad \frac{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))} \rightarrow_r \quad \exists_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))} \forall_i \quad \forall_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))} \forall_i \quad \exists_i \quad \text{cut}}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_i \quad \exists_i \quad \text{cut}$$

Reductive Cut-Elimination (\triangleright)

Example

$$\frac{\frac{\frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r}{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l \quad \frac{\frac{\frac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a), P(a) \rightarrow Q(v) \vdash Q(v)} \rightarrow_l}{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)} \rightarrow_r}{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_l}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \text{cut}$$

Reductive Cut-Elimination (\triangleright)

Example

$$\frac{\frac{\frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{\rightarrow_l} \rightarrow_r}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r}{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r \quad \frac{\frac{\frac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{\rightarrow_l} \rightarrow_r}{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)} \rightarrow_r}{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_l \text{ cut}$$
$$\frac{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l$$

Reductive Cut-Elimination (\triangleright)

Example

$$\frac{\frac{\frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l \quad \frac{\frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_r}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r}{\frac{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))}{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r \text{ cut}}{\frac{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l}$$

Reductive Cut-Elimination (\triangleright)

Example

$$\frac{\frac{\frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r \quad \frac{\frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r}{\frac{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)}{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l} \text{cut}$$

Reductive Cut-Elimination (\triangleright)

Example

$$\frac{\frac{\frac{P(a) \vdash P(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_r \quad \frac{Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r \quad \frac{\frac{P(a) \vdash P(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l \quad \frac{Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_l}{\frac{\frac{P(a) \rightarrow Q(a), P(a) \vdash Q(a)}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r}{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l} \text{cut}$$

Reductive Cut-Elimination (\triangleright)

Example

$$\frac{\frac{P(a) \vdash P(a) \quad Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \rightarrow_i \quad Q(a) \vdash Q(a)}{\frac{P(a) \vdash P(a) \quad P(a), P(a) \rightarrow Q(a) \vdash Q(a)}{P(a), P(a) \rightarrow Q(a) \vdash Q(a)} \text{ cut}} \text{ cut}$$
$$\frac{\frac{P(a) \rightarrow Q(a), P(a) \vdash Q(a)}{P(a) \rightarrow Q(a) \vdash P(a) \rightarrow Q(a)} \rightarrow_r \quad \frac{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))}{P(a) \rightarrow Q(a) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l$$

- Reductive Cut-Elimination Methods (\triangleright)
- **Cut-Elimination by Resolution (CERes)**
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Cut-Elimination by Resolution

An Example

$$\frac{\frac{\frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_l}{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)} \rightarrow_r}{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))} \exists_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))} \forall_l}{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))} \forall_r}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \text{cut}$$
$$\frac{\frac{\frac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a), P(a) \rightarrow Q(v) \vdash Q(v)} \rightarrow_l}{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)} \rightarrow_r}{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r}{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_l}{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \text{cut}$$

Cut-Elimination by Resolution

An Example

$$\frac{
 \frac{
 \frac{
 \frac{P(u) \vdash P(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)}{\rightarrow_l}
 \quad
 \frac{Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)}{\rightarrow_r}
 }{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}{\rightarrow_r}
 \quad
 \frac{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))}{\exists_r}
 }{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall_l}
 \quad
 \frac{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall_r}
 }{
 \frac{
 \frac{
 \frac{
 \frac{P(a) \vdash P(a)}{P(a), P(a) \rightarrow Q(v) \vdash Q(v)}{\rightarrow_l}
 \quad
 \frac{Q(v) \vdash Q(v)}{P(a), P(a) \rightarrow Q(v) \vdash Q(v)}{\rightarrow_r}
 }{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)}{\rightarrow_r}
 \quad
 \frac{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)}{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))}{\exists_r}
 }{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))}{\exists_l}
 \quad
 \frac{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))}{\forall_l}
 }{\forall_r}
 }{
 \frac{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))}{\forall_r}
 \quad
 \frac{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))}{\forall_l}
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))
 }
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y)) \quad \forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))
 }
 }$$

$$S_\varphi \equiv (\neg P(u) \otimes Q(u)) \oplus (P(a) \oplus \neg Q(v))$$

$$C_\varphi^W \equiv \{P(u) \vdash Q(u) ; \vdash P(a) ; Q(v) \vdash\}$$

$$\frac{
 \frac{\vdash P(a) \quad P(u) \vdash Q(u)}{\vdash Q(a)} \quad R
 \quad
 Q(v) \vdash R
 }{\vdash}$$

Cut-Elimination by Resolution

An Example

$$\begin{array}{c}
 \frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_l \\
 \frac{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))} \rightarrow_r \\
 \frac{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))} \exists_r \\
 \frac{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))} \forall_l \\
 \frac{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))}{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_r \\
 \frac{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \text{cut}
 \end{array}$$

$$\frac{\frac{\vdash P(a) \quad P(u) \vdash Q(u)}{\vdash Q(a)} R \quad Q(v) \vdash R}{\vdash} R$$

$\llbracket \varphi \rrbracket_{\vdash P(a)}^O :$

$$\frac{\frac{\frac{P(a) \vdash P(a)}{P(a) \vdash P(a), Q(v)} w_r}{\vdash P(a), P(a) \rightarrow Q(v)} \rightarrow_r}{\vdash P(a), \exists y(P(a) \rightarrow Q(y))} \exists_r$$

Cut-Elimination by Resolution

An Example

$$\begin{array}{c}
 \frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_l \\
 \frac{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))} \rightarrow_r \\
 \frac{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))} \exists_r \\
 \frac{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))} \forall_l \\
 \frac{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_r \\
 \frac{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \text{cut}
 \end{array}$$

$$\frac{\frac{\vdash P(a) \quad P(u) \vdash Q(u)}{\vdash Q(a)} R \quad Q(v) \vdash R}{\vdash} R$$

$\llbracket \varphi \rrbracket_{Q(v) \vdash}^{\circ}$:

$$\frac{\frac{\frac{Q(v) \vdash Q(v)}{Q(v), P(a) \vdash Q(v)} w_l}{Q(v) \vdash P(a) \rightarrow Q(v)} \rightarrow_r}{Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r$$

Cut-Elimination by Resolution

An Example

$$\begin{array}{c}
 \frac{\frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_l \quad \frac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a), P(a) \rightarrow Q(v) \vdash Q(v)} \rightarrow_l \\
 \frac{P(u), P(u) \rightarrow Q(u) \vdash Q(u)}{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)} \rightarrow_r \quad \frac{P(a), P(a) \rightarrow Q(v) \vdash Q(v)}{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)} \rightarrow_r \\
 \frac{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))} \exists_r \quad \frac{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)}{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r \\
 \frac{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))} \forall_l \quad \frac{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))}{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_l \\
 \frac{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))} \forall_r \quad \frac{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))}{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l \\
 \hline
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y)) \quad \text{cut}
 \end{array}$$

$$\frac{\frac{\vdash P(a) \quad P(u) \vdash Q(u)}{\vdash Q(a)} R \quad Q(v) \vdash R}{\vdash R} R$$

$\llbracket \varphi \rrbracket_{P(u) \vdash Q(u)}^0$:

$$\frac{\frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_l}{P(u), \forall x(P(x) \rightarrow Q(x)) \vdash Q(u)} \forall_l$$

Cut-Elimination by Resolution

An Example

$$\begin{array}{c}
 \frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_I \\
 \frac{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)}{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))} \rightarrow_r \\
 \frac{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))} \exists_r \\
 \frac{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))} \forall_I \\
 \frac{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_r \\
 \hline
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(a) \vdash P(a) \quad Q(v) \vdash Q(v)}{P(a), P(a) \rightarrow Q(v) \vdash Q(v)} \rightarrow_I \\
 \frac{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)}{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \rightarrow_r \\
 \frac{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))}{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r \\
 \frac{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))}{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_I \\
 \frac{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))}{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_I \\
 \hline
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y)) \quad \text{cut}
 \end{array}$$

$$\begin{array}{c}
 \frac{P(a) \vdash P(a)}{P(a) \vdash P(a), Q(v)} w_r \\
 \frac{P(a) \vdash P(a), Q(v)}{\vdash P(a), P(a) \rightarrow Q(v)} \rightarrow_r \\
 \frac{\vdash P(a), P(a) \rightarrow Q(v)}{\vdash P(a), \exists y(P(a) \rightarrow Q(y))} \exists_r \\
 \hline
 \vdash Q(a)
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(u) \vdash P(u) \quad Q(u) \vdash Q(u)}{P(u), P(u) \rightarrow Q(u) \vdash Q(u)} \rightarrow_I \\
 \frac{P(u), P(u) \rightarrow Q(u) \vdash Q(u)}{P(u), \forall x(P(x) \rightarrow Q(x)) \vdash Q(u)} \forall_I \\
 \hline
 \vdash Q(a) \quad R
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{Q(v) \vdash Q(v)}{Q(v), P(a) \vdash Q(v)} w_l \\
 \frac{Q(v), P(a) \vdash Q(v)}{Q(v) \vdash P(a) \rightarrow Q(v)} \rightarrow_r \\
 \frac{Q(v) \vdash P(a) \rightarrow Q(v)}{Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r \\
 \hline
 \vdash Q(a) \quad R
 \end{array}$$

Cut-Elimination by Resolution

An Example

$$\frac{
 \frac{
 \frac{
 \frac{P(u) \vdash P(u)}{\vdash P(u), P(u) \rightarrow Q(u)} \rightarrow_l
 \quad
 \frac{Q(u) \vdash Q(u)}{\vdash P(u) \rightarrow Q(u)} \rightarrow_r
 }{P(u) \rightarrow Q(u) \vdash P(u) \rightarrow Q(u)} \rightarrow_r
 \quad
 \frac{P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))}{\exists_r}
 }{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))} \forall_l
 }{\forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))} \forall_r
 \quad
 \frac{
 \frac{
 \frac{
 \frac{P(a) \vdash P(a)}{\vdash P(a), P(a) \rightarrow Q(v)} \rightarrow_l
 \quad
 \frac{Q(v) \vdash Q(v)}{\vdash P(a) \rightarrow Q(v)} \rightarrow_r
 }{P(a) \rightarrow Q(v) \vdash P(a) \rightarrow Q(v)} \rightarrow_r
 \quad
 \frac{P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))}{\exists_r}
 }{\exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_l
 }{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \forall_l
 }{\forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))} \text{cut}
 }{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))}$$

$$\frac{
 \frac{
 \frac{P(a) \vdash P(a)}{\vdash P(a), P(a), Q(v)} \text{wr}
 \quad
 \frac{
 \frac{
 \frac{P(u) \vdash P(u)}{\vdash P(u), P(u) \rightarrow Q(u)} \rightarrow_l
 \quad
 \frac{Q(u) \vdash Q(u)}{\vdash P(u), \forall x(P(x) \rightarrow Q(x)) \vdash Q(u)} \forall_l
 }{P(u), \forall x(P(x) \rightarrow Q(x)) \vdash Q(u)} \forall_l
 }{\forall x(P(x) \rightarrow Q(x)) \vdash Q(a), \exists y(P(a) \rightarrow Q(y))} \forall_l
 }{\vdash P(a), \exists y(P(a) \rightarrow Q(y))} \exists_r
 }{\forall x(P(x) \rightarrow Q(x)) \vdash Q(a), \exists y(P(a) \rightarrow Q(y))} \forall_l
 \quad
 \frac{
 \frac{
 \frac{Q(v) \vdash Q(v)}{\vdash Q(v), P(a) \rightarrow Q(v)} \text{wl}
 \quad
 \frac{Q(v) \vdash P(a) \rightarrow Q(v)}{\vdash Q(v) \vdash P(a) \rightarrow Q(v)} \rightarrow_r
 }{Q(v) \vdash \exists y(P(a) \rightarrow Q(y))} \exists_r
 }{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y)), \exists y(P(a) \rightarrow Q(y))} \forall_l
 }{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))} \text{cr}
 }{\forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))}$$

Cut-Elimination by Resolution

An Example

$$\frac{
 \frac{
 \frac{
 \frac{P(u) \vdash P(u)}{\vdash P(u), P(u) \rightarrow Q(u)} \rightarrow_r
 \quad
 \frac{Q(u) \vdash Q(u)}{\vdash P(u) \rightarrow Q(u)} \rightarrow_l
 }{
 \vdash P(u) \rightarrow Q(u) \vdash \exists y(P(u) \rightarrow Q(y))
 } \exists_r
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(u) \rightarrow Q(y))
 } \forall_l
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash \forall x \exists y(P(x) \rightarrow Q(y))
 } \forall_r
 \quad
 \frac{
 \frac{
 \frac{
 \frac{P(a) \vdash P(a)}{\vdash P(a), P(a) \rightarrow Q(v)} \rightarrow_r
 \quad
 \frac{Q(v) \vdash Q(v)}{\vdash P(a) \rightarrow Q(v)} \rightarrow_l
 }{
 \vdash P(a) \rightarrow Q(v) \vdash \exists y(P(a) \rightarrow Q(y))
 } \exists_r
 }{
 \exists y(P(a) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))
 } \exists_l
 }{
 \forall x \exists y(P(x) \rightarrow Q(y)) \vdash \exists y(P(a) \rightarrow Q(y))
 } \forall_l
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))
 } \text{cut}$$

$$\frac{
 \frac{
 \frac{P(a) \vdash P(a)}{\vdash P(a) \vdash P(a), Q(a)} w_r
 }{
 \vdash P(a), P(a) \rightarrow Q(a)
 } \rightarrow_r
 }{
 \vdash P(a), \exists y(P(a) \rightarrow Q(y))
 } \exists_r
 \quad
 \frac{
 \frac{
 \frac{P(a) \vdash P(a)}{\vdash P(a), P(a) \rightarrow Q(a)} \rightarrow_l
 \quad
 \frac{Q(a) \vdash Q(a)}{\vdash P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)} \forall_l
 }{
 \vdash P(a), \forall x(P(x) \rightarrow Q(x)) \vdash Q(a)
 } \text{cut}
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash Q(a), \exists y(P(a) \rightarrow Q(y))
 } \text{cut}
 \quad
 \frac{
 \frac{
 \frac{Q(a) \vdash Q(a)}{\vdash Q(a), P(a) \vdash Q(a)} w_l
 }{
 \vdash Q(a) \vdash P(a) \rightarrow Q(a)
 } \rightarrow_r
 }{
 \vdash Q(a) \vdash \exists y(P(a) \rightarrow Q(y))
 } \exists_r
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y)), \exists y(P(a) \rightarrow Q(y))
 } \text{cut}
 }{
 \forall x(P(x) \rightarrow Q(x)) \vdash \exists y(P(a) \rightarrow Q(y))
 } c_r$$

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Let φ be the proof below:

$$\frac{\frac{\frac{A \vdash A}{A, C \vdash A} w_l}{A \wedge C \vdash A} \wedge_l \quad \frac{\frac{A \vdash A}{A \vdash B, A \wedge \neg B} \wedge_r \quad \frac{B \vdash B}{\vdash B, \neg B} \neg_r}{A \wedge C \vdash B, A \wedge \neg B} cut}{\frac{A \wedge C \vdash B, A \wedge \neg B}{A \wedge C \vdash B, A \wedge \neg B} cut} cut$$

A canonic refutation $CR(\varphi)$ can be easily extracted from φ :

$$\frac{\frac{\vdash A \quad A \vdash B}{\vdash B} R \quad B \vdash}{\vdash} R$$

Theorem (Correctness of Canonic Refutations)

$CR(\varphi)$ is a refutation of C_φ^W .

Let φ be the proof below:

$$\frac{\frac{\frac{A \vdash A}{A \vdash A} \text{ cut} \quad \frac{A \vdash A}{A \vdash A, C} w_r}{A, B \vdash A, B, C \wedge D} \quad \frac{\frac{\frac{B \vdash B}{B \vdash B} \text{ cut} \quad \frac{B \vdash B}{B \vdash B, D} w_r}{A, B \vdash A, B, C \wedge D} \wedge_r}$$

φ can be reduced ($\gg_{\oplus \otimes}$) to the following two proofs φ_1 and φ_2 :

$$\frac{\frac{\frac{A \vdash A}{A \vdash A} \text{ cut} \quad \frac{A \vdash A}{A \vdash A} w^*}{A, B \vdash A, B, C \wedge D} \quad \frac{\frac{\frac{B \vdash B}{B \vdash B} \text{ cut} \quad \frac{B \vdash B}{B \vdash B} w^*}{A, B \vdash A, B, C \wedge D}}$$

Canonic refutations $CR(\varphi_1)$ and $CR(\varphi_2)$ can be extracted:

$$\frac{\frac{\vdash A}{\vdash} \quad \frac{A \vdash}{\vdash} R}{\vdash} \quad \frac{\frac{\vdash B}{\vdash} \quad \frac{B \vdash}{\vdash} R}{\vdash}$$

And then: $S_{CR}(\varphi) = \{CR(\varphi_1), CR(\varphi_2)\}$

Definition (CR-Equivalence)

Two proofs φ_1 and φ_2 (with equal end-sequents) with atomic cuts only are *strongly CR-equivalent*, denoted $\varphi_1 =_{sCR} \varphi_2$, if and only if $S_{CR}(\varphi_1) = S_{CR}(\varphi_2)$. They are *weakly CR-equivalent*, denoted $\varphi_1 =_{wCR} \varphi_2$, if and only if $S_{CR}(\varphi_1) \cap S_{CR}(\varphi_2) \neq \emptyset$.

Definition (CR-Simulation)

Let \triangleright_1 and \triangleright_2 be two cut-elimination methods. \triangleright_1 *CR-simulates* \triangleright_2 if and only if, for any proofs φ and φ_2 with $\varphi \triangleright_2 \varphi_2$, there exists φ_1 such that $\varphi \triangleright_1 \varphi_1$ and $\varphi_1 =_{wCR} \varphi_2$.

Theorem (Precedence under \triangleright)

If $\varphi \triangleright \varphi'$ then for any $\sim_{\oplus \otimes W}$ -normal-form S' of $S_{\varphi'}$ there exists a $\sim_{\oplus \otimes W}$ -normal-form S of S_{φ} such that $S \preceq S'$.

$$\frac{
 \frac{
 \frac{S_{\varphi_1}}{\varphi_1} \quad \Gamma_1 \vdash \Delta_1, B \quad \frac{S_{\varphi_2}}{\varphi_2} \quad \Gamma_2 \vdash \Delta_2, C}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, B \wedge C} \wedge_r
 \quad \frac{S_{\varphi_r}}{\varphi_r} \quad B, C, \Pi \vdash \Lambda}{B \wedge C, \Pi \vdash \Lambda} \wedge_l
 }{
 \Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda
 } cut$$

$(S_{\varphi_1} \oplus S_{\varphi_2}) \oplus S_{\varphi_r}$

\Downarrow

$$\frac{
 \frac{S_{\varphi_2}}{\varphi_2} \quad \Gamma_2 \vdash \Delta_2, C \quad \frac{
 \frac{S_{\varphi_1}}{\varphi_1} \quad \Gamma_1 \vdash \Delta_1, B \quad \frac{S_{\varphi_r}}{\varphi_r} \quad B, C, \Pi \vdash \Lambda}{C, \Gamma_1, \Pi \vdash \Delta_1, \Lambda} cut
 }{
 \Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda
 } cut
 }{
 \Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda
 } cut$$

$S_{\varphi_2} \oplus (S_{\varphi_1} \oplus S_{\varphi_r})$

Theorem (Precedence under \triangleright)

If $\varphi \triangleright \varphi'$ then for any $\sim_{\oplus \otimes W}$ -normal-form S' of $S_{\varphi'}$ there exists a $\sim_{\oplus \otimes W}$ -normal-form S of S_{φ} such that $S \leq S'$.

Theorem (Refutations of Preceding Clause Sets)

Let S_1 and S_2 be structs in $\sim_{\oplus \otimes W}$ -normal-form such that $S_1 \leq^ S_2$. Then, any resolution refutation δ of $\text{cl}(S_2)$ is also a refutation of $\text{cl}(S_1)$.*

Theorem

CERes CR-simulates $\triangleright \downarrow$.

Theorem

No cut-elimination method based on \triangleright CR-simulates CERes.

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Let φ be the proof below:

$$\frac{\frac{\frac{A \vdash A}{\vdash \neg A, A} \neg_r \quad B \vdash B}{A \rightarrow B \vdash \neg A, B} \rightarrow_l \quad \frac{\frac{\frac{A \vdash A}{\neg A \vdash \neg A} \neg^* \quad \frac{B \vdash B}{\neg B, B \vdash} \neg_l}{\neg B, \neg A \vee B \vdash \neg A} \vee_l}{\neg A \vee B \vdash \neg B \rightarrow \neg A} \rightarrow_r}{A \rightarrow B \vdash \neg B \rightarrow \neg A} \text{cut}$$

Then, its cut-pertinent struct is:

$$S_\varphi \equiv (\neg A \otimes B) \oplus (A \oplus \neg B)$$

And its clause set is:

$$C_\varphi^W \equiv \{ A \vdash B ; \vdash A ; B \vdash \}$$

Let φ be the proof below:

$$\frac{\frac{\frac{A \vdash A}{\vdash \neg A, A} \neg_r \quad B \vdash B}{A \rightarrow B \vdash \neg A, B} \rightarrow_l \quad \frac{\frac{\frac{A \vdash A}{\neg A \vdash \neg A} \neg^* \quad \frac{B \vdash B}{\neg B, B \vdash} \neg_l}{\neg B, \neg A \vee B \vdash \neg A} \vee_l}{\neg A \vee B \vdash \neg B \rightarrow \neg A} \rightarrow_r}{A \rightarrow B \vdash \neg B \rightarrow \neg A} \text{cut}$$

The formula corresponding to end-sequent of φ is:

$$F_\varphi \doteq \mathcal{F}(A \rightarrow B \vdash \neg B \rightarrow \neg A) \equiv ((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A))$$

The clause form of $\neg F_\varphi$ is:

$$C \doteq \{ A \vdash B ; \vdash A ; B \vdash \}$$

Let φ be the proof below:

$$\frac{\frac{\frac{A \vdash A}{\vdash \neg A, A} \neg_r \quad B \vdash B}{A \rightarrow B \vdash \neg A, B} \rightarrow_l \quad \frac{\frac{\frac{A \vdash A}{\neg A \vdash \neg A} \neg^* \quad \frac{B \vdash B}{\neg B, B \vdash} \neg_l}{\neg B, \neg A \vee B \vdash \neg A} \vee_l}{\neg A \vee B \vdash \neg B \rightarrow \neg A} \rightarrow_r}{A \rightarrow B \vdash \neg B \rightarrow \neg A} \text{cut}$$

The clause set is:

$$C_\varphi^W \equiv \{ A \vdash B ; \vdash A ; B \vdash \}$$

The clause form of $\neg F_\varphi$ is:

$$C \doteq \{ A \vdash B ; \vdash A ; B \vdash \}$$

Both are equally hard to refute!!

- Consider a formalization of Fuerstenberg's proof of the infinitude of primes.
- By skolemization, we get a schematic proof φ_k proving that there are more than k primes.
- Current theorem provers like Otter and Prover9 cannot refute the clause set of φ_k for $k \geq 3$ (in a reasonable time).

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In the proof φ below, occurrences highlighted with the same color are cut-linked to each other.

$$\frac{
 \frac{
 \frac{
 \frac{P(x) \vdash P(x)}{P(x), \neg P(x) \vdash P(x)} w_r
 }{P(x), \vdash \neg P(x) \rightarrow P(x)} \rightarrow_r
 }{(\forall x)P(x), \vdash \neg P(x) \rightarrow P(x)} \forall_I
 }{(\forall x)P(x), \vdash (\forall x)(\neg P(x) \rightarrow P(x))} \forall_r
 }{
 \frac{
 \frac{
 \frac{
 \frac{P(s) \vdash P(s)}{\vdash \neg P(s), P(s)} \neg_r
 }{\neg P(s) \rightarrow P(s) \vdash P(s), P(s)} \rightarrow_I
 }{\neg P(s) \rightarrow P(s) \vdash P(s)} c_r
 }{(\forall x)(\neg P(x) \rightarrow P(x)) \vdash P(s)} \forall_I
 }{(\forall x)(\neg P(x) \rightarrow P(x)) \vdash (\exists y)P(y)} \exists_r
 }{(\forall x)P(x) \vdash (\exists y)P(y)} cut
 }{(\forall x)P(x) \vdash (\exists y)P(y)} cut$$

In the proof φ below, occurrences highlighted with the same color are atomically cut-linked to each other.

$$\frac{
 \frac{
 \frac{
 \frac{
 \frac{
 P(\alpha) \vdash P(\alpha)
 }{P(\alpha), \neg P(\alpha) \vdash P(\alpha)}
 w_r
 }{P(\alpha), \vdash \neg P(\alpha) \rightarrow P(\alpha)}
 \rightarrow_r
 }{(\forall x)P(x), \vdash \neg P(\alpha) \rightarrow P(\alpha)}
 \forall_I
 }{(\forall x)P(x), \vdash (\forall x)\neg P(x) \rightarrow P(x)}
 \forall_r
 }{(\forall x)P(x) \vdash (\exists y)P(y)}
 }{
 \frac{
 \frac{
 \frac{
 \frac{
 \frac{
 P(s) \vdash P(s)
 }{\vdash \neg P(s), P(s)}
 \neg_r
 }{\neg P(s) \rightarrow P(s) \vdash P(s), P(s)}
 \rightarrow_I
 }{\neg P(s) \rightarrow P(s) \vdash P(s)}
 c_r
 }{(\forall x)\neg P(x) \rightarrow P(x) \vdash P(s)}
 \forall_I
 }{(\forall x)\neg P(x) \rightarrow P(x) \vdash (\exists y)P(y)}
 \exists_r
 }{(\forall x)\neg P(x) \rightarrow P(x) \vdash (\exists y)P(y)}
 cut
 }{(\forall x)P(x) \vdash (\exists y)P(y)}
 }
 }
 }
 }
 }
 }$$

Theorem (Atomic Cut-Linkage Preservation under Reductive Cut-Elimination)

Let φ be a proof with cuts and φ^ a proof such that $\varphi \triangleright^* \varphi^*$. Let ρ be a cut in φ^* with auxiliary occurrences v_l and v_r . Then, for any position π , $\text{cutlink}_a(\Gamma v_l \neg^\pi, \varphi) = \text{cutlink}_a(\Gamma v_r \neg^\pi, \varphi)$.*

$$\frac{\frac{\frac{\varphi_1}{\Gamma_1 \vdash \Delta_1, F_1[B]_{\pi_1}}}{\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2, F_1[B]_{\pi_1} \wedge F_2[C]_{\pi_2}} \wedge_r \quad \frac{\frac{\varphi_2}{\Gamma_2 \vdash \Delta_2, F_2[C]_{\pi_2}}}{F_1[B]_{\pi_1}, F_2[C]_{\pi_2}, \Pi \vdash \Lambda} \wedge_l \quad \frac{\varphi_r}{\Pi \vdash \Lambda}}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda} \text{cut}}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda} \text{cut}$$

\Downarrow

$$\frac{\frac{\varphi_2}{\Gamma_2 \vdash \Delta_2, F_2[C]_{\pi_2}} \quad \frac{\frac{\varphi_1}{\Gamma_1 \vdash \Delta_1, F_1[B]_{\pi_1}}}{F_2[C]_{\pi_2}, \Gamma_1, \Pi \vdash \Delta_1, \Lambda} \text{cut} \quad \frac{F_1[B]_{\pi_1}, F_2[C]_{\pi_2}, \Pi \vdash \Lambda}{\Pi \vdash \Lambda} \text{cut}}{\Gamma_1, \Gamma_2, \Pi \vdash \Delta_1, \Delta_2, \Lambda} \text{cut}$$

Theorem (Cut-Linkage Preservation under Reductive Cut-Elimination)

Let φ be a proof with cuts and φ^ a proof such that $\varphi \triangleright^* \varphi^*$. Let ρ be a cut in φ^* with auxiliary occurrences v_l and v_r . Then, for any position π , $\text{cutlink}(\Gamma v_l \neg^\pi, \varphi) = \text{cutlink}(\Gamma v_r \neg^\pi, \varphi)$.*

Some Properties of \triangleright

Cut-Side

$$\begin{array}{c}
 \frac{P(\alpha) \vdash P(\alpha)^I}{(\forall z)P(z) \vdash P(\alpha)^I} \forall_I \\
 \frac{(\forall z)P(z) \vdash P(\alpha)^I}{(\forall z)P(z) \vdash (\forall x)P(x)^I} \forall_r \\
 \frac{(\forall z)P(z) \vdash (\forall x)P(x)^I, (\forall y)P(y)^I}{(\forall z)P(z) \vdash (\forall x)P(x)^I \vee (\forall y)P(y)^I} \vee_r \\
 \frac{(\forall z)P(z) \vdash (\forall x)P(x)^I \vee (\forall y)P(y)^I}{(\forall z)P(z) \vdash (\exists w)P(w)} \text{cut}
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(t)^r \vdash P(t)}{P(t)^r \vdash (\exists w)P(w)} \exists_r \\
 \frac{P(t)^r \vdash (\exists w)P(w)}{(\forall x)P(x)^r \vdash (\exists w)P(w)} \forall_I \\
 \frac{(\forall x)P(x)^r \vee (\forall y)P(y)^r \vdash (\exists w)P(w), (\exists w)P(w)}{(\forall x)P(x)^r \vee (\forall y)P(y)^r \vdash (\exists w)P(w)} c_r
 \end{array}
 \qquad
 \begin{array}{c}
 \frac{P(s)^r \vdash P(s)}{P(s)^r \vdash (\exists w)P(w)} \exists_r \\
 \frac{P(s)^r \vdash (\exists w)P(w)}{(\forall y)P(y)^r \vdash (\exists w)P(w)} \forall_I \\
 \frac{(\forall y)P(y)^r \vdash (\exists w)P(w)}{(\forall x)P(x)^r \vee (\forall y)P(y)^r \vdash (\exists w)P(w)} \forall_I
 \end{array}$$

Theorem (Cut-Side Preservation under Reductive Cut-Elimination)

Let φ be a proof with cuts and φ^ a proof such that $\varphi \triangleright^* \varphi^*$. Let ρ be a cut in φ^* with auxiliary occurrences v_l and v_r . Then, for any position π , $\text{outside}(\Gamma v_l \neg^\pi, \varphi) \neq \text{outside}(\Gamma v_r \neg^\pi, \varphi)$.*

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Let φ be the proof below.

$$\frac{
 \frac{
 \frac{
 \frac{P(\alpha) \vdash P(\alpha)}{P(\alpha), \neg P(\alpha) \vdash P(\alpha)} w_r
 }{\rightarrow_r}
 }{\forall_I}
 }{\forall_r}
 }{(\forall x)P(x), \vdash (\forall x)(\neg P(x) \rightarrow P(x))}
 \quad
 \frac{
 \frac{
 \frac{
 \frac{
 \frac{P(s) \vdash P(s)}{\vdash \neg P(s), P(s)} \neg_r
 }{\rightarrow_I}
 }{c_r}
 }{\forall_I}
 }{(\forall x)(\neg P(x) \rightarrow P(x)) \vdash P(s)}
 }{\exists_r}
 }{P(s) \vdash (\exists y)P(y)}
 }{cut}
 }{(\forall x)(\neg P(x) \rightarrow P(x)) \vdash (\exists y)P(y)}$$

Then its clause set is:

$$C_\varphi^W \equiv \{ \underbrace{\vdash P(\alpha)}_{c_1}; \underbrace{P(s) \vdash P(s)}_{c_2}; \underbrace{P(s) \vdash P(s)}_{c_3}; \underbrace{P(s) \vdash}_{c_4} \}$$

Let δ be the following **R**-refutation of C_φ^W :

$$\frac{C_1 \quad C_4}{\vdash} r$$

Resolution Refinements (Proof Search Restrictions) for CERes

Unrestricted CERes does not preserve Cut-Linkage

Let φ be the proof below.

$$\begin{array}{c}
 \frac{P(\alpha) \vdash P(\alpha)}{P(\alpha), \neg P(\alpha) \vdash P(\alpha)} w_r \\
 \frac{P(\alpha), \neg P(\alpha) \vdash P(\alpha)}{P(\alpha), \vdash \neg P(\alpha) \rightarrow P(\alpha)} \rightarrow_r \\
 \frac{P(\alpha), \vdash \neg P(\alpha) \rightarrow P(\alpha)}{(\forall x)P(x), \vdash \neg P(\alpha) \rightarrow P(\alpha)} \forall_I \\
 \frac{(\forall x)P(x), \vdash \neg P(\alpha) \rightarrow P(\alpha)}{(\forall x)P(x), \vdash (\forall x)(\neg P(x) \rightarrow P(x))} \forall_r \\
 \\
 \frac{P(s) \vdash P(s)}{\vdash \neg P(s), P(s)} \neg_r \\
 \frac{\vdash \neg P(s), P(s)}{\neg P(s) \rightarrow P(s) \vdash P(s), P(s)} \rightarrow_I \\
 \frac{\neg P(s) \rightarrow P(s) \vdash P(s), P(s)}{\neg P(s) \rightarrow P(s) \vdash P(s)} c_r \\
 \frac{\neg P(s) \rightarrow P(s) \vdash P(s)}{(\forall x)(\neg P(x) \rightarrow P(x)) \vdash P(s)} \forall_I \\
 \frac{(\forall x)(\neg P(x) \rightarrow P(x)) \vdash P(s)}{(\forall x)(\neg P(x) \rightarrow P(x)) \vdash (\exists y)P(y)} \exists_r \\
 \frac{(\forall x)P(x), \vdash (\forall x)(\neg P(x) \rightarrow P(x)) \quad (\forall x)(\neg P(x) \rightarrow P(x)) \vdash (\exists y)P(y)}{(\forall x)P(x) \vdash (\exists y)P(y)} cut
 \end{array}$$

Then its clause set is:

$$C_\varphi^W \equiv \{ \underbrace{\vdash P(\alpha)}_{c_1}; \underbrace{P(s) \vdash P(s)}_{c_2}; \underbrace{P(s) \vdash P(s)}_{c_3}; \underbrace{P(s) \vdash}_{c_4} \}$$

Let δ be the following **R**-refutation of C_φ^W :

$$\frac{C_1 \quad C_4}{\vdash} r$$

- **Resolution Rule** (using cut-linkage):

$$\frac{\Gamma_1 \vdash \Delta_1, A_1 \quad A_2, \Gamma_2 \vdash \Delta_2}{\Gamma_1 \sigma, \Gamma_2 \sigma \vdash \Delta_1 \sigma, \Delta_2 \sigma} r(\sigma)$$

only if $cutlink(A_1) = cutlink(A_2)$.

- **Factoring Rules:**

$$\frac{A, A', \Gamma \vdash \Delta}{A \sigma, \Gamma \sigma \vdash \Delta \sigma} f_l(\sigma)$$

$$\frac{\Gamma \vdash \Delta, A, A'}{\Gamma \sigma \vdash \Delta \sigma, A \sigma} f_r(\sigma)$$

only if $cutlink(A) = cutlink(A')$.

Resolution Refinements (Proof Search Restrictions) for CERes

Unrestricted CERes does not preserve Cut-Side

Let φ be the following proof:

$$\begin{array}{c}
 \frac{P(\alpha) \vdash P(\alpha)}{P(\alpha) \vdash \neg P(\alpha), P(\alpha)} w_r \\
 \frac{P(\alpha) \vdash \neg P(\alpha), P(\alpha)}{P(\alpha) \vdash \neg P(\alpha) \vee P(\alpha)} \vee_r \\
 \frac{P(\alpha) \vdash \neg P(\alpha) \vee P(\alpha)}{(\forall x)P(x) \vdash \neg P(\alpha) \vee P(\alpha)} \forall_l \\
 \frac{(\forall x)P(x) \vdash \neg P(\alpha) \vee P(\alpha)}{(\forall x)P(x) \vdash (\forall x)\neg P(x) \vee P(x)} \forall_r \\
 \frac{P(t) \vdash P(t)}{\neg P(t), P(t) \vdash} \neg_l \\
 \frac{\neg P(t), P(t) \vdash}{\neg P(t) \vdash \neg P(t)} \neg_r \\
 \frac{P(t) \vdash P(t)}{\neg P(t) \vee P(t) \vdash P(t), \neg P(t)} \vee_l \\
 \frac{\neg P(t) \vee P(t) \vdash P(t), \neg P(t)}{\neg P(t) \vee P(t) \vdash P(t) \vee \neg P(t)} \vee_r \\
 \frac{(\forall x)(\neg P(x) \vee P(x)) \vdash P(t) \vee \neg P(t)}{(\forall x)P(x) \vdash P(t) \vee \neg P(t)} \forall_l \\
 \text{cut}
 \end{array}$$

Its clause set is:

$$C_\varphi^W \equiv \{ \underbrace{\vdash P(\alpha)}_{c_1}; \underbrace{\vdash P(t)}_{c_2}; \underbrace{P(t) \vdash}_{c_3} \}$$

A possible \mathbf{R}_{cls} -refutation of C_φ^W is shown below:

$$\frac{c_2 \quad c_3}{\vdash} r$$

Resolution Refinements (Proof Search Restrictions) for CERes

Unrestricted CERes does not preserve Cut-Side

Let φ be the following proof:

$$\frac{
 \frac{
 \frac{
 \frac{
 P(\alpha) \vdash P(\alpha)^l
 }{
 P(\alpha) \vdash \neg P(\alpha)^l, P(\alpha)^l
 } w_r
 }{
 P(\alpha) \vdash \neg P(\alpha)^l \vee P(\alpha)^l
 } \vee_r
 }{
 (\forall x)P(x) \vdash \neg P(\alpha)^l \vee P(\alpha)^l
 } \forall_l
 }{
 (\forall x)P(x) \vdash (\forall x)\neg P(x)^l \vee P(x)^l
 } \forall_r
 }{
 (\forall x)P(x) \vdash P(t) \vee \neg P(t)
 } \text{cut}
 }{
 \frac{
 \frac{
 \frac{
 P(t) \vdash P(t)^r
 }{
 \neg P(t)^r, P(t) \vdash
 } \neg_l
 }{
 \neg P(t)^r \vdash \neg P(t)
 } \neg_r
 }{
 \neg P(t)^r \vee P(t)^r \vdash P(t), \neg P(t)
 } \vee_r
 }{
 \neg P(t)^r \vee P(t)^r \vdash P(t) \vee \neg P(t)
 } \forall_l
 }{
 (\forall x)(\neg P(x)^r \vee P(x)^r) \vdash P(t) \vee \neg P(t)
 } \text{cut}
 }$$

Its clause set is:

$$C_\varphi^W \equiv \{ \underbrace{\vdash P(\alpha)^l}_{c_1}; \underbrace{\vdash P(t)^r}_{c_2}; \underbrace{P(t)^r \vdash}_{c_3} \}$$

A possible \mathbf{R}_{cls} -refutation of C_φ^W is shown below:

$$\frac{c_2 \quad c_3}{\vdash} r$$

- **Resolution Rule** (using cut-side and cut-linkage):

$$\frac{\Gamma_1 \vdash \Delta_1, A_1 \quad A_2, \Gamma_2 \vdash \Delta_2}{\Gamma_1 \sigma, \Gamma_2 \sigma \vdash \Delta_1 \sigma, \Delta_2 \sigma} r(\sigma)$$

only if $\text{cutlink}(A_1) = \text{cutlink}(A_2)$ and $\text{outside}(A_1) \neq \text{outside}(A_2)$.

- **Factoring Rules** (using cut-side and cut-linkage):

$$\frac{A, A', \Gamma \vdash \Delta}{A \sigma, \Gamma \sigma \vdash \Delta \sigma} f_l(\sigma) \quad \frac{\Gamma \vdash \Delta, A, A'}{\Gamma \sigma \vdash \Delta \sigma, A \sigma} f_r(\sigma)$$

only if $\text{cutlink}(A) = \text{cutlink}(A')$ and $\text{outside}(A) = \text{outside}(A')$.

Resolution Refinements (Proof Search Restrictions) for CERes

Unrestricted CERes does not preserve Atomic Cut-Linkage

Let φ be the proof below:

$$\frac{
 \frac{
 \frac{
 \frac{P(\alpha) \vdash P(\alpha)^I}{(\forall z)P(z) \vdash P(\alpha)^I} \forall_I
 }{(\forall z)P(z) \vdash (\forall x)P(x)^I} \forall_r
 }{(\forall z)P(z) \vdash (\forall x)P(x)^I, (\forall y)P(y)^I} w_r
 }{(\forall z)P(z) \vdash (\forall x)P(x)^I \vee (\forall y)P(y)^I} \forall_r
 }{(\forall z)P(z) \vdash (\exists w)P(w)} \text{cut}
 }{
 \frac{
 \frac{
 \frac{P(t)^r \vdash P(t)}{P(t)^r \vdash (\exists w)P(w)} \exists_r
 }{(\forall x)P(x)^r \vdash (\exists w)P(w)} \forall_I
 }{(\forall x)P(x)^r \vee (\forall y)P(y)^r \vdash (\exists w)P(w), (\exists w)P(w)} c_r
 }{(\forall x)P(x)^r \vee (\forall y)P(y)^r \vdash (\exists w)P(w)} \text{cut}
 }{(\forall z)P(z) \vdash (\exists w)P(w)} \text{cut}
 }$$

Its clause set is:

$$C_\varphi^W \equiv \{ \underbrace{\vdash P(\alpha)^I}_{c_1}; \underbrace{P(t)^r \vdash}_{c_2}; \underbrace{P(s)^r \vdash}_{c_3} \}$$

Let δ be the following \mathbf{R}_{scl} -refutation of C_φ^W :

$$\frac{c_1 \quad c_3}{\vdash} r$$

Resolution Refinements (Proof Search Restrictions) for CERes

Unrestricted CERes does not preserve Atomic Cut-Linkage

Let φ be the proof below:

$$\frac{
 \frac{
 \frac{
 \frac{
 P(\alpha) \vdash P(\alpha)
 }{(\forall z)P(z) \vdash P(\alpha)}
 \forall_I
 }{(\forall z)P(z) \vdash (\forall x)P(x)}
 \forall_r
 }{(\forall z)P(z) \vdash (\forall x)P(x), (\forall y)P(y)}
 w_r
 }{(\forall z)P(z) \vdash (\forall x)P(x) \vee (\forall y)P(y)}
 \forall_r
 }{(\forall z)P(z) \vdash (\exists w)P(w)}
 cut
 }{
 \frac{
 \frac{
 \frac{
 P(t) \vdash P(t)
 }{P(t) \vdash (\exists w)P(w)}
 \exists_r
 }{(\forall x)P(x) \vdash (\exists w)P(w)}
 \forall_I
 }{(\forall x)P(x) \vee (\forall y)P(y) \vdash (\exists w)P(w), (\exists w)P(w)}
 c_r
 }{(\forall x)P(x) \vee (\forall y)P(y) \vdash (\exists w)P(w)}
 c_r
 }{(\forall z)P(z) \vdash (\exists w)P(w)}
 cut
 }$$

Its clause set is:

$$C_\varphi^W \equiv \{ \underbrace{\vdash P(\alpha)}_{c_1}; \underbrace{P(t) \vdash}_{c_2}; \underbrace{P(s) \vdash}_{c_3} \}$$

Let δ be the following \mathbf{R}_{scl} -refutation of C_φ^W :

$$\frac{c_1 \quad c_3}{\vdash} r$$

- **Resolution Rule** (using atomic cut-linkage):

$$\frac{\Gamma_1 \vdash \Delta_1, A_1 \quad A_2, \Gamma_2 \vdash \Delta_2}{\Gamma_1\sigma, \Gamma_2\sigma \vdash \Delta_1\sigma, \Delta_2\sigma} r(\sigma)$$

only if $\text{cutlink}_a(A_1) = \text{cutlink}_a(A_2)$.

- **Factoring Rules:**

$$\frac{A, A', \Gamma \vdash \Delta}{A\sigma, \Gamma\sigma \vdash \Delta\sigma} f_l(\sigma) \qquad \frac{\Gamma \vdash \Delta, A, A'}{\Gamma\sigma \vdash \Delta\sigma, A\sigma} f_r(\sigma)$$

only if $\text{cutlink}_a(A) = \text{cutlink}_a(A')$.

- Reductive Cut-Elimination Methods (\triangleright)
- Cut-Elimination by Resolution (CERes)
- CR-Simulation of \triangleright by CERes
- The Problem of Resolution Proof Search in CERes
- Some Properties of \triangleright
- Resolution Refinements (Proof Search Restrictions) for CERes based on the properties of \triangleright
- **Conclusions and Future Work**

- We have defined 3 refinements/restrictions for CERes (and we could have defined more).
- Each refined CERes is intermediary (with respect to CR-simulation) between the unrefined CERes and reductive methods.
- The design of such refinements helped to further clarify the differences between CERes and reductive methods. In particular, they clarify why reductive methods do not CR-simulate the unrestricted/unrefined CERes.
- By using the refinements, we can have a trade-off between the easiness of the refutation search and the “non-confluence” of CERes.

- Proof Procedures for Iterated Schemata:
 - Proof schemata (e.g. the skolemized Fuerstenberg's proof) lead to schemata of clause sets, which could perhaps be more easily refuted using proof procedures for iterated schemata.
- Curry-Howard Isomorphism:
 - Proofs \leftrightarrow programs.
 - Reductive cut-elimination \leftrightarrow execution of programs.
 - Cut-elimination by resolution \leftrightarrow ??
- Permutability in the Sequent Calculus:
 - Profile Clause Set \approx Inference Permutations + Standard Clause Set
 - Swapped Clause Set = Inference Permutations + Standard Clause Set
 - Profile and Swapped Clause Sets are largely invariant to inference permutations

- ETPs/Herbrand-proofs/Proof-Forests with cuts:
 - “Describing Proofs by Short Tautologies” (Stefan Hetzl) - Relates clause sets and Herbrand-sequents with cuts.
- Deep Inference:
 - Reductive cut-elimination methods are “shallow” (grade reduction operates on the shallowest connectives of cut-formulas). CERes is “deep” (it works only with the atomic components of the cut-formulas).
 - Normalization of structs is similar to the switch rule in some cases.
- Redundancy/Bureaucracy:
 - CERes is capable of eliminating certain kinds of redundancies from proofs (non-elementarily smaller “cut-free” proofs) . . .
 - . . . but it also introduces some redundancies in some cases (exponentially larger “cut-free” proofs).
 - Structs and clause sets can be seen as compact representations of (the cut-pertinent part of) a proof.
 - The relation between clause sets and proof nets has been studied in Stefan Hetzl’s PhD thesis.

- Implementation of the refinements
- Conjecture: \triangleright CR-simulates CERes with \mathbf{R}_{acl} .
- Refinements in the case of higher-order logics
- Even more restrictive refinements, based on particular reductive strategies (e.g. Gentzen's, Tait's).
- Restrictions of paramodulation.

- Thanks!
- Questions? Comments? Suggestions?