

# Sequent Calculi with Hilbert's Choice Operator

$\epsilon$

(Talk at the final seminar of Prof. Richard Zach's course on the Epsilon Calculus)

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# Motivation

- Both Gentzen's sequent calculus (and the cut-elimination method) and Hilbert's epsilon calculus (and the epsilon substitution method) were developed as approaches to Hilbert's program of consistency proofs and the reduction of "ideal" mathematics to "real" mathematics by acceptable finitistic means.
- How do these approaches relate to each other?
- Idea to answer the question: design sequent calculi that can handle the epsilon choice operator and see what happens.

# The Goal

## Design of Sequent Calculi with an $\epsilon$ operator

- Hilbert-style Calculi:  
 $\mathbf{EC}$ ,  $\mathbf{EC}_{=}$ ,  
 $\mathbf{EC}_{\forall}$ ,  $\mathbf{EC}_{=\forall}$ ,  
 $\mathbf{EC}_{\epsilon}$ ,  $\mathbf{EC}_{=\epsilon}$ ,  $\mathbf{EC}_{\forall\epsilon}$ ,  $\mathbf{EC}_{=\forall\epsilon}$ ,  
 $\mathbf{EC}_{=\epsilon_{\text{ext}}}$ ,  $\mathbf{EC}_{=\forall\epsilon_{\text{ext}}}$
- Gentzen-style Sequent Calculi:  
 $\mathbf{LK}$ ,  $\mathbf{LK}_{=}$ ,  
 $\mathbf{LK}_{\forall}$ ,  $\mathbf{LK}_{=\forall}$ ,  
 $\mathbf{LK}_{\epsilon}$ ,  $\mathbf{LK}_{=\epsilon}$ ,  $\mathbf{LK}_{\forall\epsilon}$ ,  $\mathbf{LK}_{=\forall\epsilon}$ ,  
 $\mathbf{LK}_{=\epsilon_{\text{ext}}}$ ,  $\mathbf{LK}_{=\forall\epsilon_{\text{ext}}}$

- What we want:  
 $F$  provable in  $\mathbf{EC}_\epsilon$  if and only if  $\vdash F$  derivable in  $\mathbf{LK}_\epsilon$ .
- Approach:
  - 1 Use  $\mathbf{LK}$ , adding all critical formulas to the antecedent of the sequent to be proved.
  - 2 Apply the rules of  $\mathbf{LK}$  to the relevant critical formulas.
  - 3 Think of a new rule that would have the same effect as the rules that have been applied, but would not need all the critical formulas in the antecedent. Add this rule to  $\mathbf{LK}$  to obtain  $\mathbf{LK}_\epsilon$

A proof-fragment schema in  $\mathbf{LK}$ :

(CF  $\doteq$  set of critical formulas)

( $e \doteq \epsilon_x B(x)$ )

$$\frac{\frac{CF, \Gamma \vdash \Delta, B(t) \quad B(\epsilon_x B(x)), CF, \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)}{B(t) \rightarrow B(\epsilon_x B(x)), CF, \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)} \rightarrow_I}{CF, \Gamma, \Gamma'(\epsilon_x B(x)) \vdash \Delta, \Delta'(\epsilon_x B(x))} c_i$$

An equivalent rule in  $\mathbf{LK}_\epsilon$ :

$$\frac{\Gamma, \vdash \Delta, B(t) \quad B(\epsilon_x B(x)), \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)}{\Gamma, \Gamma'(\epsilon_x B(x)) \vdash \Delta, \Delta'(\epsilon_x B(x))} \epsilon_{critical}$$

where (trick!)

$$\Gamma'(\epsilon_x B(x)) \cup \Delta'(\epsilon_x B(x)) \neq \emptyset$$

$$\frac{\Gamma \vdash \Delta, B(t) \quad B(\epsilon_x B(x)), \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)}{\Gamma, \Gamma'(\epsilon_x B(x)) \vdash \Delta, \Delta'(\epsilon_x B(x))} \epsilon_{critical}$$

- By design, (cut-free)  $\mathbf{LK}_\epsilon$  is clearly sound and complete w.r.t. (cut-free)  $\mathbf{LK}$  with all critical formulas in the antecedents of the sequents.
- (cut-free)  $\mathbf{LK}$  with all critical formulas in the antecedents of the sequents is sound and complete w.r.t.  $\mathbf{EC}_\epsilon$ .
- Hence, (cut-free)  $\mathbf{LK}_\epsilon$  is sound and complete w.r.t.  $\mathbf{EC}_\epsilon$ .
- $\mathbf{EC}_\epsilon$  is sound and complete w.r.t. semantic entailment.
- Therefore, (cut-free)  $\mathbf{LK}_\epsilon$  is sound and complete w.r.t. semantic entailment.

$$\frac{\Gamma \vdash \Delta, B(t) \quad B(\epsilon_x B(x)), \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)}{\Gamma, \Gamma'(\epsilon_x B(x)) \vdash \Delta, \Delta'(\epsilon_x B(x))} \epsilon_{critical}$$

- Yasuhara[1982] gives a direct proof of sound and completeness of cut-free  $\mathbf{LK}_{\forall\epsilon}$  w.r.t. semantic entailment.
  - Proof based on Tableaux-like proof search.
  - If the search terminates, a cut-free  $\mathbf{LK}_{\forall\epsilon}$ -proof can be easily constructed from the finite tableau.
  - If the search does not terminate, then the tableau should be infinite. By the weak koenig's lemma, there exists an infinite path. This path can be used to construct a counter model in which the end-sequent does not hold. Contradiction!
  - Construction of the counter-model is intricate, because intensional semantics for epsilon terms is used.

$$\frac{\Gamma \vdash \Delta, B(t) \quad B(\epsilon_x B(x)), \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)}{\Gamma, \Gamma'(\epsilon_x B(x)) \vdash \Delta, \Delta'(\epsilon_x B(x))} \epsilon_{critical}$$

- Gentzen's cut-elimination proof does not work for  $\mathbf{LK}_\epsilon$ .
  - Notice that the new rule,  $\epsilon_{critical}$ , can block the upward swapping of a cut.
  - $\epsilon_{critical}$ -inferences and cuts must be eliminated simultaneously.
- As  $\Gamma'(\epsilon_x B(x)) \cup \Delta'(\epsilon_x B(x)) \neq \emptyset$ , we get, as immediate corollaries:
  - Subformula property.
  - The first epsilon theorem: if  $\Gamma \vdash \Delta$  does not contain epsilon terms and is derivable in  $\mathbf{LK}_\epsilon$ , then  $\Gamma \vdash \Delta$  is derivable in (cut-free)  $\mathbf{LK}$ .
  - The first epsilon theorem: if  $\Gamma \vdash \Delta$  contains neither epsilon terms nor quantifiers and is derivable in  $\mathbf{LK}_{\forall\epsilon}$ , then  $\Gamma \vdash \Delta$  is derivable in (cut-free)  $\mathbf{LK}$ .

$$\frac{\Gamma \vdash \Delta, B(t) \quad B(\epsilon_x B(x)), \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)}{\Gamma, \Gamma'(\epsilon_x B(x)) \vdash \Delta, \Delta'(\epsilon_x B(x))} \epsilon_{critical}$$

- But the extractability of Herbrand disjunctions (mid-sequent theorem, extended first epsilon theorem) is not a corollary of cut-elimination in  $\mathbf{LK}_{\forall\epsilon}$ , as it is in  $\mathbf{LK}_\forall$ .
  - $\epsilon_{critical}$ -inferences also have to be eliminated, in order to allow the extraction of Herbrand disjunctions.
  - Conjecture:  $\epsilon_{critical}$ -elimination can be done analogously to Gentzen's cut-elimination (i.e. by swapping  $\epsilon_{critical}$ -inferences upward), because  $\epsilon_{critical}$ -inferences are essentially “cuts in disguise”, with a controlled violation of an eigenvariable condition.

$$\frac{\Gamma \vdash \Delta, B(t) \quad B(\epsilon_x B(x)), \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)}{\Gamma, \Gamma'(\epsilon_x B(x)) \vdash \Delta, \Delta'(\epsilon_x B(x))} \epsilon_{critical}$$

- $\epsilon_{critical}$ -inferences are essentially “cuts in disguise”, with a controlled violation of an eigenvariable condition.

$$\frac{\frac{\Gamma \vdash \Delta, B(t)}{\Gamma \vdash \Delta, \exists x B(x)} \exists_r \quad \frac{B(\epsilon_x B(x)), \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)}{\exists x B(x), \Gamma, \Gamma'(e) \vdash \Delta, \Delta'(e)} \exists_l}{\Gamma, \Gamma'(\epsilon_x B(x)) \vdash \Delta, \Delta'(\epsilon_x B(x))} cut$$

$$\frac{A(t) \vdash A(t) \quad A(\epsilon_x B(x)) \vdash A(\epsilon_x B(x))}{A(t) \vdash A(\epsilon_x B(x))} \epsilon_{critical} \quad \Rightarrow \quad \begin{array}{l} A(t) \vdash A(t) \\ (\varphi)\{\epsilon_x B(x) \leftarrow t\} \end{array}$$

- Problem A: what if  $\varphi$  has a contraction operating on  $\epsilon_x B(x)$ ??
  - Solution A1: Enforce contractions to occur only at a shallow level in the structure of formulas. (This is similar (but weaker) to the requirement of prenex form for the Midsequent theorem.)
  - Solution A2: Replace contractions by *array inferences*.
- Problem B: what if epsilon terms occur nested inside other epsilon terms in  $\varphi$ ?
  - Solution B: follow an appropriate  $\epsilon_{critical}$ -elimination strategy based on rank and degree of the epsilon terms. (analogously to the epsilon substitution method).

- What we want:  
 $F$  provable in  $\mathbf{EC}_{=\epsilon}$  if and only if  $\vdash F$  derivable in  $\mathbf{LK}_{=\epsilon}$ .
- Approach: Add equality rules to  $\mathbf{LK}_{\epsilon}$ .

$$\frac{}{\vdash s = s}$$

$$\frac{\Gamma \vdash \Delta, F(s)}{s = t, \Gamma \vdash \Delta, F(t)} =_{r1}$$

$$\frac{\Gamma \vdash \Delta, F(t)}{s = t, \Gamma \vdash \Delta, F(s)}$$

$$\frac{\Gamma, F(s) \vdash \Delta}{s = t, \Gamma, F(t) \vdash \Delta} =_{l1}$$

$$\frac{\Gamma, F(t) \vdash \Delta}{s = t, \Gamma, F(s) \vdash \Delta} =_{l2}$$

- What we want:

$F$  provable in  $\mathbf{EC}_{=\forall\epsilon_{\text{ext}}}$  if and only if  $\vdash F$  derivable in  $\mathbf{LK}_{=\forall\epsilon_{\text{ext}}}$ .

- Approach:

- 1 Use  $\mathbf{LK}_{=\forall\epsilon}$ , adding all extensionality formulas to the antecedent of the sequent to be proved.
- 2 Apply the rules of  $\mathbf{LK}_{=\forall\epsilon}$  to the relevant extensionality formulas.
- 3 Think of a new rule that would have the same effect as the rules that have been applied, but that would not need all the critical formulas in the antecedent. Add this rule to  $\mathbf{LK}_{=\forall\epsilon}$  to obtain  $\mathbf{LK}_{=\forall\epsilon_{\text{ext}}}$ .

A proof-fragment schema in  $\mathbf{LK}_{=\forall\epsilon}$ :  
 (EF  $\doteq$  set of extensionality formulas)  
 ( $e_A \doteq \epsilon_x A(x)$ ) ( $e_B \doteq \epsilon_x B(x)$ )

$$\frac{\frac{\frac{B(\alpha), \text{EF}, \Gamma \vdash \Delta, A(\alpha)}{\text{EF}, \Gamma \vdash \Delta, (B(\alpha) \rightarrow A(\alpha))} \rightarrow_r \quad \frac{A(\alpha), \text{EF}, \Gamma \vdash \Delta, B(\alpha)}{\text{EF}, \Gamma \vdash \Delta, (A(\alpha) \rightarrow B(\alpha))} \rightarrow_r}{\text{EF}, \Gamma \vdash \Delta, (A(\alpha) \leftrightarrow B(\alpha))} \wedge_r}{\text{EF}, \Gamma \vdash \Delta, \forall x(A(x) \rightarrow B(x))} \forall_r \quad \frac{\text{EF}, \Gamma \vdash \Delta, F(e_A)}{e_A = e_B, \text{EF}, \Gamma \vdash \Delta, F(e_B)} =_r}{\frac{\forall x(A(x) \leftrightarrow B(x)) \rightarrow e_A = e_B, \text{EF}, \Gamma \vdash \Delta, F(e_B)}{\text{EF}, \Gamma \vdash \Delta, F(e_B)} \rightarrow_l} c_l$$

An equivalent rule in  $\mathbf{LK}_{=\forall\epsilon_{\text{ext}}}$ :

$$\frac{B(\alpha), \Gamma \vdash \Delta, A(\alpha) \quad A(\alpha), \Gamma \vdash \Delta, B(\alpha) \quad \Gamma \vdash \Delta, F(e_A)}{\Gamma \vdash \Delta, F(e_B)} \epsilon_{\text{ext}}$$

(And another analogous rule for the case when  $F$  occurs in the antecedent.)

# Choice Operators and the Axiom of Choice

## Motivation

- What is the relation between the choice operator  $\epsilon$  and the axiom of choice?

# Choice Operators and the Axiom of Choice

Proving AC in  $LK_{\forall\epsilon}^{\text{high}}$

$$\frac{\frac{\frac{\exists x_i P_{oi}(x_i) \vdash P_{oi}(\epsilon_{i(oi)} P_{oi})}{\vdash \exists x_i P_{oi}(x_i) \rightarrow P_{oi}(\epsilon_{i(oi)} P_{oi})} \rightarrow_r}{\vdash \forall P_{oi}(\exists x_i P_{oi}(x_i) \rightarrow P_{oi}(\epsilon_{i(oi)} P_{oi}))} \forall_r}{\vdash \exists f_{i(oi)} \forall P_{oi}(\exists x_i P_{oi}(x_i) \rightarrow P_{oi}(f_{i(oi)} P_{oi}))} \exists_r$$





# Choice Operators and the Axiom of Choice

## Restricted Axioms of Choice

- What about restricted axioms of choice?
- The definite choice operator  $\iota$  corresponds to the axiom of choice restricted to sets having exactly one element.
- Could we define other choice operators corresponding to other restricted forms of the axiom of choice (e.g. weak koenig's lemma)?
- Could we find restricted forms of the axiom of choice, which would correspond to the other choice operators used in linguistics?

- ( $\epsilon_{critical}$  and cut)-elimination by resolution?

- Mitsuru Yasuhara: *Cut-Elimination in  $\epsilon$ -Calculi*. Zeitschrift fuer Mathematische Logik und Grundlagen der Mathematik, 28, 311–316 (1982).
- Peter B. Andrews: *An Introduction to Mathematical Logic and Type Theory: To Truth through Proof*
- Geor Moser, Richard Zach: *The Epsilon Calculus and Herbrand Complexity*
- Richard Zach: *Lecture Notes of the “Epsilon Calculus” course*, Technische Universitaet Wien, 2009.

# Thanks

- Thanks for the attention!
- Questions? Comments?