Gödel's Incompleteness Theorem and the "Theory" of Everything

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- To introduce Gödel's Incompleteness Theorem to physicists...
- ... to allow a proper investigation of the relation between this theorem and physicists' dream "Theory" of Everything ('T'OE).
    - Does the theorem imply that it is impossible to obtain a 'T'OE?

- Nomenclature clashes
- Informal Discussion of Goedel's Incompleteness Theorem
- Basic Concepts of Logic
- Arithmetic
    - Arithmetization
    - Arithmetic Representability
- The Diagonal Lemma
- Proof of Goedel's Theorem

Nomenclature Clashes

- Physicists and Logicians (sometimes) have different meanings for:
  - Theory
  - Model
  - Solution
  - Proof
  - Reality
  - ...

Nomenclature Clashes
A logician's view

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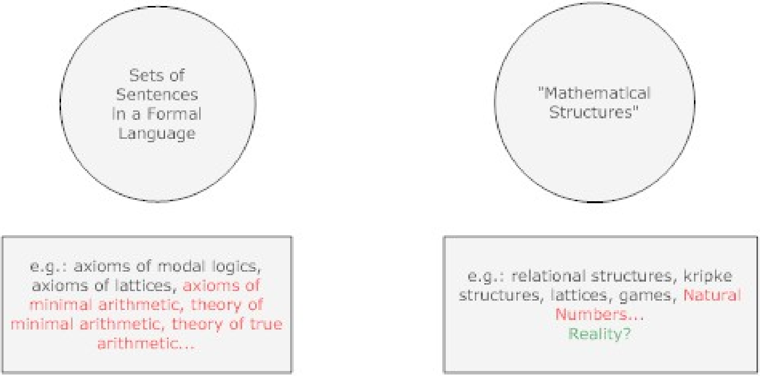
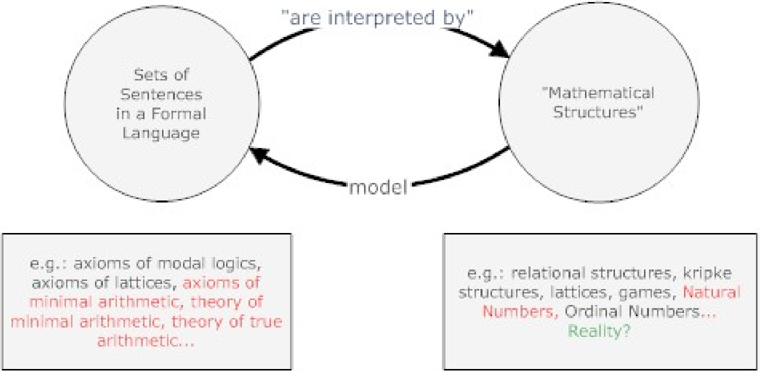


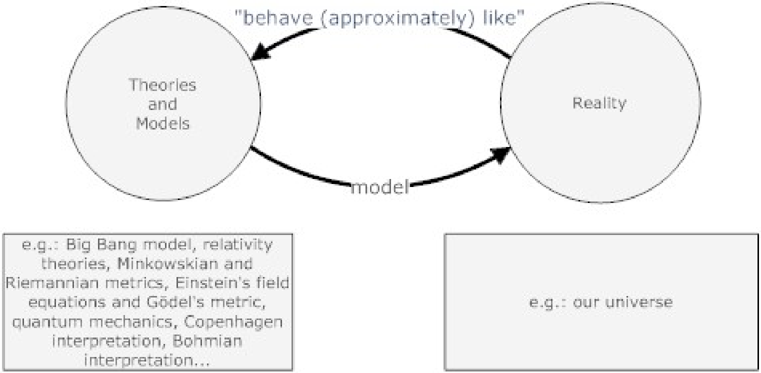
Nomenclature Clashes

A logician's view



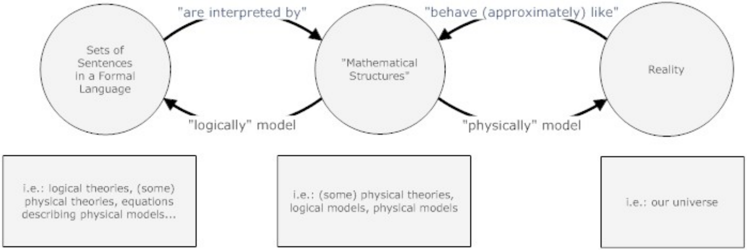
Nomenclature Clashes

A logician's rough idea of what a physicist's view seems to be



Nomenclature Clashes

A logician's rough unification proposal



Goedel's First Incompleteness Theorem

There is no consistent, complete, axiomatizable extension of Q (the theory of minimal arithmetic).

Goedel's First Incompleteness Theorem: More Informally

For any consistent, axiomatizable extension of Q, there are sentences that are neither provable nor disprovable.

"Undecidability" of Arithmetic

The set of (Goedel numbers of) sentences of the language of arithmetic that are true in the standard interpretation is not recursive.

"Undecidability" of Arithmetic: Informally

There are sentences that are true (in the standard model) but not provable (in any computable "proof system").

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"Undecidability" of Arithmetic: Informally

There are sentences that are true (in the standard model) but not provable (in any computable "proof system").

- Construct a sentence G: "this sentence is not provable".
- Show that G cannot be proved nor disproved.

1. $G \doteq$ " this sentence is not provable".
2. $G \doteq$ " $\ulcorner G \urcorner$ is not provable".
3. $G \doteq$ " $\neg Pr(\ulcorner G \urcorner)$".
4. $G \leftrightarrow \neg Pr(\ulcorner G \urcorner)$

- How can a formula G be encoded as a numeral $\ulcorner G \urcorner$??
  - By Arithmetization ...
- How can the notion of proof be expressed as a formula $Pr(.)$ in the language of arithmetic?
  - By Arithmetic Representation ...
- Does G exist?
  - G is analogous to a "fix-point" for $\neg Pr(.)$ ...
  - The Diagonal Lemma proves that G exists.

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Basic Concepts of Logic

Shown in Erman's Slides

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Goal

To represent provability in a theory T as a formula ($Pr_T(.)$) in the language of arithmetic.

Remarks

- If T is axiomatizable, then $Pr_T(x) = \exists y Prw_T(x, y)$ such that:
  - $Prw_T(x, y)$ can be read as “y is the goedel numeral of a proof in T of a sentence with goedel numeral x”.
  - $Prw_T(x, y)$ is recursive (because the axioms and rules of inference are recursive sets)

Sub-Goals

- to define definability and representability in T
- to find a suitable (as weak as possible, but strong enough) T
- to prove representability of recursive functions in T.

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- to define definability and representability in T
- to find a suitable (as weak as possible, but strong enough) T
- to prove representability of recursive functions in T.

- $F(x)$ *defines* in T a set S iff:
  $n \in S$ iff $F(\mathbf{n})$ is a theorem of T.
- $F(x, y)$ *defines* in T a relation R iff:
  $R(a, b)$ iff $F(\mathbf{a}, \mathbf{b})$ is a theorem of T.
- $F(x, y)$ *defines* in T a function f iff:
  $f(a) = b$ iff $F(\mathbf{a}, \mathbf{b})$ is a theorem of T.
- $F(x, y)$ *represents* in T a function f iff:
  $f(a) = b$ iff $\forall y(F(\mathbf{a}, y) \leftrightarrow y = \mathbf{b}))$ is a theorem of T.
- *arithmetically defines* $\equiv$
  defines in the theory of true arithmetic.
- *arithmetical* $\equiv$ *arithmetically definable*.

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Theorem

Every recursive function is arithmetically definable by a Σ_1^0-formula.

Proof Idea

- Proof by induction on the definition of recursive functions:
  - Show that the basic functions are definable.
  - Show that if some recursive functions are definable, then their composition, primitive recursion or minimization is definable.
- Trickiest part: to show definability for the "primitive recursion" case in a way that does not need exponentiation (β-function lemma).

The axioms of **Q**:

- (Q1) $\forall x \quad \mathbf{0} \neq x'$
- (Q2) $\forall x \forall y \ (x' = y' \rightarrow x = y)$
- (Q3) $\forall x \quad x + \mathbf{0} = x$
- (Q4) $\forall x \forall y \ x + y' = (x + y)'$
- (Q5) $\forall x \quad x \cdot \mathbf{0} = \mathbf{0}$
- (Q6) $\forall x \forall y \ x \cdot y' = (x \cdot y) + x$
- (Q7) $\forall x \quad \neg x < \mathbf{0}$
- (Q8) $\forall x \forall y \ (x < y' \leftrightarrow (x < y \vee x = y))$
- (Q9) $\forall x \forall y \ (x < y \vee x = y \vee x > y)$

Theorem

A Σ^0_1-sentence is true in the standard interpretation iff it is a theorem of **Q**.

Proof Idea

A proof can be given by patiently and methodically listing all theorems provable from the axioms of **Q** roughly in order of increasing formula and term complexity . . .

Theorem

There are Π^0_1-sentences that are true in the standard interpretation but are not theorems of Q.

Examples

- $\forall x((1)+x = x+(1))$
- $\forall x((2)\cdot x = x\cdot(2))$

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Theorem

Every recursive function is representable in **Q** (by a Σ^0_1-formula).

Theorem

Every recursive relation is definable in **Q** (by a Σ^0_1-formula).

Diagonal Lemma

Diagonal Lemma

Let T be a theory extending **Q**. Then, for any formula $B(y)$, there is a sentence G such that $\vdash_T G \leftrightarrow B(\ulcorner G \urcorner)$.

Proof

To be shown in the blackboard.

Proof of Goedel's Incompleteness Theorem

- Suppose G_T (Goedel sentence for T, i.e. $\vdash_T G_T \leftrightarrow \neg\exists y Prw(\ulcorner G_T \urcorner, y)$) is provable in T (i.e. $\vdash_T G_T$).
  - Then $\exists y Prw(\ulcorner G_T \urcorner, y)$ is true (in the standard interpretation). Moreover, $\exists y Prw(\ulcorner G_T \urcorner, y)$ is Σ^0_1, and hence $\vdash_T \exists y Prw(\ulcorner G_T \urcorner, y)$.
  - But also $\vdash_T G_T \leftrightarrow \neg\exists y Prw(\ulcorner G_T \urcorner, y)$, and hence $\vdash_T \neg G_T$.
  - Therefore, T is inconsistent. (Contradiction!)
- Suppose G_T (Goedel sentence for T, i.e. $\vdash_T G_T \leftrightarrow \neg\exists y Prw(\ulcorner G_T \urcorner, y)$) is disprovable in T (i.e. $\vdash_T \neg G_T$).
  - Then, by $\vdash_T G_T \leftrightarrow \neg\exists y Prw(\ulcorner G_T \urcorner, y)$ and pure logic, $\vdash_T \neg\neg\exists y Prw(\ulcorner G_T \urcorner, y)$ or, equivalently, $\vdash_T \exists y Prw(\ulcorner G_T \urcorner, y)$.
  - But, by consistency of T, for any n $\neg Prw(\ulcorner G_T \urcorner, \mathbf{n})$ is true (in the standard interpretation). Moreover, $\neg Prw(\ulcorner G_T \urcorner, \mathbf{n})$ is Σ^0_1, and hence $\vdash_T \neg Prw(\ulcorner G_T \urcorner, \mathbf{n})$.
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- Goedel's original theorem required ω-consistency of T.
- Rosser modified Goedel's sentence in such a way that only consistency of T is required.
- It is also possible to prove Goedel's theorem more directly (but less constructively) from the diagonal lemma, without showing G_T explicitly. This is a more modern approach.
- In this more modern approach, it is easier to see why the diagonal lemma is called diagonal lemma (it resembles Cantor's diagonal argument).
- The proof presented here is closer in style to the original proof. It gives a more intuitive motivation for the arithmetization of formulas and arithmetical representation of functions.
- In the approach presented here, the diagonal lemma is more like a fix-point lemma.

The End

This is an incomplete presentation . . .

. . . to be completed by the next talk, on physics and the ‘T’OE.