Gödel's Incompleteness Theorem and the 'Theory' of Everything

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- Basic Consepts of FOL
    - Language of FOL
    - Semantics & Proof Theory
    - Theory
- Arithmetization
    - Language of Arithmetic
    - Gdel Numbers
    - Encoding

Language of First Order Logic

- **Logical Connectives**
  $\neg$ (not), $\wedge$ (and), $\vee$ (or) and $\supset$ (implies)
- **Variables**
  x, y, z...
- **Function Symbols**
  f, g, h..
- **Predicate Symbols**
  A, B...
- **Quantifiers**
  $\forall$ (universal quantifier), $\exists$ (existential quantifier)
- **Auxiliary Symbols**
  (,) (paranthesis) and , (comma)

Language of First Order Logic

- Each function and predicate symbol has an arity $n \in \mathbb{N}$
- Nullary function symbols are called constant symbols and denoted by a, b, c..
- *Terms* are built from the variables and the function symbols.

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    - Examples:
      x, c, $f(a)$, $g(x)$, $f(b, x, z)$ are terms.

Language of First Order Logic

Definition of (FOL)Formula

- If A is an n-ary predicate symbol and $t_1 ... t_n$ are terms where $n \in \mathbb{N}$ then $A(t_1, ... t_n)$ is an atomic formula.
- Every atomic formula is a formula
- If A and B is formulas, then $\neg A$, $A \wedge B$, $A \vee B$ and $A \supset B$ are formulas.
- If A is a formula, x is a variable then $\forall x \; A$ and $\exists x \; A$ are formulas.

Sentence

- Let Q $\in \{\forall, \exists\}$ and QxA be an arbitrary formula, then
    - A is called *scope* of the quantifier Q.
    - each occurence of variable x in A, is *bound*
    - each occurence of variable x in A, is *free* iff it is not *bound* by any quantifier.
- A formula is a *closed formula (sentence)* iff it has no free variable.

Semantics and Proof Theory

Each ‘proper’ logic is equipped with basic two notion:

- formal semantics
  - which assigns a *precise meaning* to each logical expression, by defining *a truth value* for each particular formula
- proof theory
  - which deals to define the notion of a *formal proof* which is defined in terms of a calculus involving *axioms* and *inference rules*

Semantics

Semantics is defined in terms of *interpretations* and *semantic consequence* $\models$.

An interpretation M consists of

- a non empty set $|M|$ is called domain, and
- denotation (meaning) of non-logical symbols:

mapping $.^M$,assigns elements of $|M|$ to constantsfunctions
$|M|^n \longmapsto |M|$ to n-ary function symbols,
relations $\subseteq |M|^n$ to n-ary predicate symbols

Semantics

- A *model* of a formula A is an interpretation which makes the formula A true.
- Example: $|M_1| = \mathbb{N}$ with successor function s and addition $+$ and unary relation Even_Number
  $|M_2| =$ set of strings over $\sum = \{$m,u$\}$, conc and add_u and unary relation StringLength_is_3

	a	f	h	h(f(a,a))	P(h(f(a,a)))
M_1	1	$+$	s	3	false
M_2	m	conc	add_u	mmu	true

- It is denoted as $M_2 \models P(h(f(a,a)))$ but $M_1 P(h(f(a,a)))$

Theory

- The relation $T \models A$ holds if all models of T are also models of A.
- A theory T is a set of sentences which is closed under logical consequence $\models$
- M is a *model* of a theory T iff M is model of all elements of T.
- Particularly, for $T = \emptyset$, $T \models A$ means that A is *valid* which means A is true under every interpretation, and it is written as $\models A$ instead of $\emptyset \models A$.

Proof Theory

- An *Axiomatic System* consists of
  - a set of formulas as *axioms*
  - a set of relation between the formulas, called *inference rules*
    - if I is an inference rule and if $\langle A_1, .., A_n, B\rangle \in I$, then I is a *direct consequence* of $A_1, .., A_n$ under I
- A *proof* is a finite sequence $A_1, .., A_k$ such that for each A_i where $1 < i < k$, either A_i is an axiom or a direct consequence of earlier elements under some inference rule.
- A formula A is provable (denoted as $\vdash A$) iff there exists a derivation sequence $\langle A_1, ..., A_n\rangle$ such that $A_n{=}A$ holds.

Proof Theory

- A *derivation* from a theory T is a finite sequence $A_1, .., A_k$ such that for each A_i where $1 < i < k$, either
  - A_i is an axiom or
  - $A_i \in T$ or
  - A_i a direct consequence of earlier elements under some inference rule.
- A closed formula (sentence) A is derivable from a theory T (denoted as $T \vdash A$) iff there exists a derivation sequence $\langle A_1, ..., A_n \rangle$ of T such that $A_n{=}A$ holds.

Completeness

- Completeness of a Logical Calculus
  if $T \models A$, then $T \vdash A$
- A Complete Theory
  A theory T is complete if for every sentence B of its language, either B or $\neg B$ is in T.

Axiomatizablity & Consistency

- If there exists a set Γ which is recursive such that a theory T consists of all and only the sentences provable from Γ, then T is a *axiomatizable*
- A theory T is consistent if not every sentence of its language is in T.

Language of Arithmetic

Non-logical symbols are

- constant: $\mathbf{0}$
- predicate symbol: ¡/2
- function symbol: successor '/1 , +/2, ./2

Idea

If we can reason over numbers, we might reason over arbitrary statements, once they are mapped to numbers

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- Map variables to numbers
- Map formulas to numbers
- Map proofs to numbers

Map symbols to numbers

Symbol	Gödel number
$\neg$	1
$\vee$	2
$\supset$	3
$\exists$	4
$=$	5
0	6
s	7
(	8
)	9
,	10
$+$	11
.	12

Map variables to numbers

There are three kinds of variables:

- numerical variables (x, y, z): ranging over numerals 0, s0...or numerical expressions like x+y
- sentential variables (p, q, r): ranging over sentences (closed formulas)
- predicate variables (A, B, C): ranging over predicates

type	examples	gdel numbers
numerical variables	x,y,z..	13, 17, 19..
sentential varaibles	p,q,r..	$13^2, 17^2, 19^2$..
predicate variables	P,Q,R..	$13^3, 17^3, 19^3$..

Map formulas to numbers

$$\begin{array}{lcccccccccc}
 & (& \exists & \mathrm{x} &) & (& \mathrm{x} & = & \mathrm{s} & \mathrm{y} &) \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & 8 & 4 & 13 & 9 & 8 & 13 & 5 & 7 & 17 & 9 \\
 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
\mathrm{m}= & 2^{8}\mathrm{x} & 3^{4}\mathrm{x} & 5^{13}\mathrm{x} & 7^{9}\mathrm{x} & 11^{8}\mathrm{x} & 13^{13}\mathrm{x} & 17^{5}\mathrm{x} & 19^{7}\mathrm{x} & 23^{17}\mathrm{x} & 29^{9}
\end{array}$$

formulas to numbers

$$\begin{tikzcd}
 & 243{,}000{,}000 \arrow[d] & \\
2^6\mathrm{x} \arrow[d] & 3^5\mathrm{x} \arrow[d] & 5^6 \arrow[d] \\
6 \arrow[d] & 5 \arrow[d] & 6 \arrow[d] \\
0 & = & 0
\end{tikzcd}$$

result

- For every formula there is a unique Gödel number
- For every proof there exists a unique Gödel number
- Gödel number function and it's inverse is computable