

Gödel's Incompleteness Theorem and the 'Theory' of Everything

Bruno Woltzenlogel Paleo, Erman Acar

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- ▶ Basic Concepts of FOL
 - ▶ Language of FOL
 - ▶ Semantics & Proof Theory
 - ▶ Theory
- ▶ Arithmetization
 - ▶ Language of Arithmetic
 - ▶ Gdel Numbers
 - ▶ Encoding

Language of First Order Logic

- ▶ **Logical Connectives**

\neg (not), \wedge (and), \vee (or) and \supset (implies)

- ▶ **Variables**

$x, y, z \dots$

- ▶ **Function Symbols**

$f, g, h \dots$

- ▶ **Predicate Symbols**

$A, B \dots$

- ▶ **Quantifiers**

\forall (universal quantifier), \exists (existential quantifier)

- ▶ **Auxiliary Symbols**

$(,)$ (paranthesis) and $,$ (comma)

Language of First Order Logic

- ▶ Each function and predicate symbol has an arity $n \in \mathbb{N}$
- ▶ Nullary function symbols are called constant symbols and denoted by $a, b, c..$
- ▶ *Terms* are built from the variables and the function symbols.

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- ▶
 - ▶ Examples:
 $x, c, f(a), g(x), f(b, x, z)$ are terms.

Language of First Order Logic

Definition of (FOL)Formula

- ▶ If A is an n -ary predicate symbol and $t_1 \dots t_n$ are terms where $n \in \mathbb{N}$ then $A(t_1, \dots t_n)$ is an atomic formula.
- ▶ Every atomic formula is a formula
- ▶ If A and B is formulas, then $\neg A$, $A \wedge B$, $A \vee B$ and $A \supset B$ are formulas.
- ▶ If A is a formula, x is a variable then $\forall x A$ and $\exists x A$ are formulas.

Sentence

- ▶ Let $Q \in \{\forall, \exists\}$ and QxA be an arbitrary formula, then
 - ▶ A is called *scope* of the quantifier Q .
 - ▶ each occurrence of variable x in A , is *bound*
 - ▶ each occurrence of variable x in A , is *free* iff it is not *bound* by any quantifier.
- ▶ A formula is a *closed formula (sentence)* iff it has no free variable.

Semantics and Proof Theory

Each 'proper' logic is equipped with basic two notion:

- ▶ formal semantics
 - ▶ which assigns a *precise meaning* to each logical expression, by defining a *truth value* for each particular formula
- ▶ proof theory
 - ▶ which deals to define the notion of a *formal proof* which is defined in terms of a calculus involving *axioms* and *inference rules*

Semantics

Semantics is defined in terms of *interpretations* and *semantic consequence* \models .

An interpretation M consists of

- ▶ a non empty set $|M|$ is called domain, and
- ▶ denotation (meaning) of non-logical symbols:

mapping \cdot^M , assigns elements of $|M|$ to constants
 functions $|M|^n \rightarrow |M|$ to n-ary function symbols,
 relations $\subseteq |M|^n$ to n-ary predicate symbols

Semantics

- ▶ A *model* of a formula A is an interpretation which makes the formula A true.
- ▶ Example: $|M_1| = \mathbb{N}$ with successor function s and addition $+$ and unary relation `Even_Number`
 $|M_2| =$ set of strings over $\Sigma = \{m, u\}$, `conc` and `add_u` and unary relation `StringLength_is_3`

	a	f	h	$h(f(a,a))$	$P(h(f(a,a)))$
M_1	1	+	s	3	false
M_2	m	conc	<code>add_u</code>	mmu	true

- ▶ It is denoted as $M_2 \models P(h(f(a, a)))$ but $M_1 \not\models P(h(f(a, a)))$

Theory

- ▶ The relation $T \models A$ holds if all models of T are also models of A .
- ▶ A theory T is a set of sentences which is closed under logical consequence \models
- ▶ M is a *model* of a theory T iff M is model of all elements of T .
- ▶ Particularly, for $T = \emptyset$, $T \models A$ means that A is *valid* which means A is true under every interpretation, and it is written as $\models A$ instead of $\emptyset \models A$.

Proof Theory

- ▶ An *Axiomatic System* consists of
 - ▶ a set of formulas as *axioms*
 - ▶ a set of relation between the formulas, called *inference rules*
 - ▶ if I is an inference rule and if $\langle A_1, \dots, A_n, B \rangle \in I$, then I is a *direct consequence* of A_1, \dots, A_n under I
- ▶ A *proof* is a finite sequence A_1, \dots, A_k such that for each A_i where $1 < i < k$, either A_i is an axiom or a direct consequence of earlier elements under some inference rule.
- ▶ A formula A is provable (denoted as $\vdash A$) iff there exists a derivation sequence $\langle A_1, \dots, A_n \rangle$ such that $A_n = A$ holds.

Proof Theory

- ▶ A *derivation* from a theory T is a finite sequence A_1, \dots, A_k such that for each A_i where $1 < i < k$, either
 - ▶ A_i is an axiom or
 - ▶ $A_i \in T$ or
 - ▶ A_i a direct consequence of earlier elements under some inference rule.

- ▶ A closed formula (sentence) A is derivable from a theory T (denoted as $T \vdash A$) iff there exists a derivation sequence $\langle A_1, \dots, A_n \rangle$ of T such that $A_n = A$ holds.

Completeness

- ▶ Completeness of a Logical Calculus
if $T \models A$, then $T \vdash A$
- ▶ A Complete Theory
A theory T is complete if for every sentence B of its language, either B or $\neg B$ is in T .

Axiomatizability & Consistency

- ▶ If there exists a set Γ which is recursive such that a theory T consists of all and only the sentences provable from Γ , then T is a *axiomatizable*
- ▶ A theory T is consistent if not every sentence of its language is in T .

Language of Arithmetic

Non-logical symbols are

- ▶ constant: **0**
- ▶ predicate symbol: $i/2$
- ▶ function symbol: successor $'/1$, $+/2$, $./2$

Idea

If we can reason over numbers, we might reason over arbitrary statements, once they are mapped to numbers

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- ▶ Map symbols to numbers
- ▶ Map variables to numbers
- ▶ Map formulas to numbers
- ▶ Map proofs to numbers

Map symbols to numbers

Symbol	Gödel number
\neg	1
\forall	2
\supset	3
\exists	4
$=$	5
0	6
s	7
$($	8
$)$	9
$,$	10
$+$	11
$.$	12

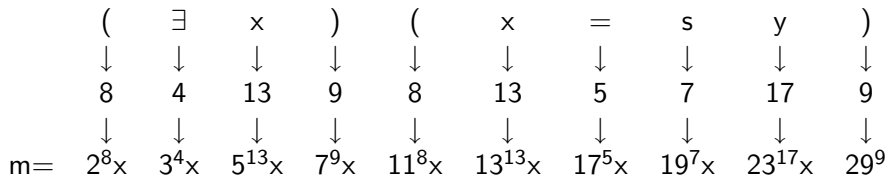
Map variables to numbers

There are three kinds of variables:

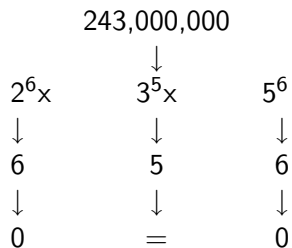
- ▶ numerical variables (x, y, z): ranging over numerals $0, s0\dots$ or numerical expressions like $x+y$
- ▶ sentential variables (p, q, r): ranging over sentences (closed formulas)
- ▶ predicate variables (A, B, C): ranging over predicates

type	examples	gdel numbers
numerical variables	$x, y, z..$	$13, 17, 19..$
sentential variables	$p, q, r..$	$13^2, 17^2, 19^2..$
predicate variables	$P, Q, R..$	$13^3, 17^3, 19^3..$

Map formulas to numbers



formulas to numbers



result

- ▶ For every formula there is a unique Gödel number
- ▶ For every proof there exists a unique Gödel number
- ▶ Gödel number function and its inverse is computable