Cut-Elimination by Resolution and Skolemization in Second-Order Logic *

Stefan Hetzl Alexander Leitsch Daniel Weller
Bruno Woltzenlogel Paleo

Institute of Computer Languages (E185),
Vienna University of Technology,
Favoritenstraße 9, 1040 Vienna, Austria

October 17, 2008

The first-order cut-elimination method CERES (cut-elimination by resolution) has several advantages over the traditional reductive cut-elimination methods: firstly, the reductive methods are subsumed by CERES (i.e. every proof obtained by a reductive method can also be obtained by CERES, see [2]), and secondly, a non-elementary speed-up over Gentzen's method by the use of CERES is possible (see [1]). The CERES method has been implemented in the `CERES` system[1].

CERES in first-order logic is based on the observation that, given an **LK** proof π of S one can construct a proof ψ of the empty sequent $\vdash$ by removing (1) those atoms in the axioms which are not ancestors of a cut and (2) those rule applications in π that operate on ancestors of the end-sequent (performing a merge operation when removing binary rules), leaving just the rule applications operating on ancestors of cuts. The sequents on the leaves of ψ then form the *characteristic clause set* $\mathrm{CL}(\pi)$, which is unsatisfiable and can therefore be refuted using a resolution theorem prover. Let δ be such a refutation, and note that by applying the global substitution of δ we obtain a resolution refutation without unification from instances of clauses in $\mathrm{CL}(\pi)$. The only rules in this ground refutation are resolution with the empty substitution, which is essentially atomic cut, and factoring with the empty substitution, which corresponds to contraction, so we can construct an **LK** refutation δ^* with atomic cuts from instances of clauses in $\mathrm{CL}(\pi)$. For every such instance c, we can construct a *projection*, which is an **LK** proof of $S \circ c$ (where $\circ$ is the merge operation on sequents), from π by leaving out the rule applications operating on cut ancestors. By appending these projections to the leaves of δ^*, we obtain a proof π' of S with at most atomic cuts which is basically analytic. Note that it is important for the construction of the projections that no strong quantifier rules operate on

*Supported by the Austrian Science Fund (project P19875)
[1] `http://www.logic.at/ceres`

ancestors of the end-sequent: otherwise, eigenvariable violations may occur and the constructed trees fail to be proofs. Luckily, in first-order logic it is possible to skolemize proofs, removing those strong quantifier rules, and to deskolemize π' to obtain a proof of the original end-sequent.

The method cannot be extended in a straightforward way to second-order logic: first, the resolution calculi differ from those in first-order logic because clauses are not closed under substitution, so transformation to clause form has to be included in the calculus.

Secondly, skolemization in second-order logic is more problematic than in first-order logic. In a first-order logic proof, all quantifiers introduced by quantifier rules that operate on ancestors of the end-sequent have an occurrence in the end-sequent — for the removal of strong quantifier rules, it suffices to skolemize the end-sequent and propagate the changes upwards. This is not the case in second-order logic: there, a strong quantifier may be eliminated by a weak second-order quantifier introduction and will therefore not occur in the end-sequent.

Concerning proof skolemization, a first approach would be to skolemize the formulas substituted by weak second-order quantifier rules in addition to the end-sequent. Unfortunately, the class of proofs skolemizable in this way is rather small. It contains the class of **QFC**-proofs (the class of $\mathbf{LK^2}$-proofs using quantifier-free comprehension). For **QFC**-proofs π, the construction of a **QFC**-refutation from a resolution refutation can be obtained by simulating the clause form transformation in $\mathbf{LK^2}$ using quantifier-free cuts.

By showing that the characteristic clause sets of **QFC**-proofs admit quantifier-free resolution refutations, we obtain the CERES^2 method for **QFC**-proofs. In addition, we obtain a syntactic characterization of the skolemizable $\mathbf{LK^2}$-proofs. To provide evidence that the class of **QFC**-proofs contains interesting proofs, we use CERES^2 to analyze an example involving induction and the least number principle.

It is future work to extend the method to cover a larger class of $\mathbf{LK^2}$-proofs. The primary obstacle here is that the analogue to proof skolemization in **LK** fails in $\mathbf{LK^2}$. To overcome this, we are developing a variant of $\mathbf{LK^2}$ with built-in skolemization/deskolemization.

References

[1] Matthias Baaz and Alexander Leitsch. Cut-elimination and Redundancy-elimination by Resolution. *Journal of Symbolic Computation*, 29(2):149–176, 2000.

[2] Matthias Baaz and Alexander Leitsch. Towards a clausal analysis of cut-elimination. *Journal of Symbolic Computation*, 41:381–410, 2006.