Bounded Łukasiewicz Logics

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Abstract. In this work we investigate *bounded Łukasiewicz logics*, characterised as the intersection of the k-valued Łukasiewicz logics for $k = 2, \ldots, n$ ($n \geq 2$). These logics formalise a generalisation of Ulam's game with applications in Information Theory. Here we provide an analytic proof calculus $\mathbf{GŁB_n}$ for each bounded Łukasiewicz logic, obtained by adding a single rule to **GŁ**, a hypersequent calculus for Łukasiewicz infinite-valued logic. We give a first cut-elimination proof for **GŁ** with (suitable forms of) cut rules. We then prove completeness for $\mathbf{GŁB_n}$ with cut and show that cut can also be eliminated in this case.

1 Introduction

Łukasiewicz logics were introduced for philosophical reasons by Jan Łukasiewicz in the 1920s [8] and are among the first examples of *many-valued logics*. Currently they are of great importance in several areas of research. Firstly, in *fuzzy logic* [15], where infinite-valued Łukasiewicz logic **Ł**, along with Gödel logic and Product logic, emerges as one of the fundamental "t-norm based"[1] fuzzy logics [7]. From an *algebraic* perspective, Chang's MV-algebras [2] for Łukasiewicz logics are of great interest and form the subject of a recent monograph containing many deep mathematical results [5]. Łukasiewicz logics can also be viewed from a *geometric* perspective via *McNaughton's representation theorem* [9] which establishes that formulae in **Ł** stand to particular geometric functions as formulae in classical logic stand to boolean functions. Finally, various *semantic interpretations* of Łukasiewicz logics have been provided, most importantly via Ulam's game, a variant of the game of Twenty Questions where errors/lies are allowed in the answers [11, 12]. Ulam's game models situations in the processing and sending of information that might be affected by "noise" (see e.g. [13]), and strategies for the game lead naturally to the theory of error-correcting codes.

In [10] an analytic proof calculus **GŁ** was defined for **Ł** using *hypersequents*, a natural generalisation of Gentzen sequents introduced by Avron in [1] [2]. Soundness and completeness for **GŁ** were proved *semantically* in [10] via an embedding

[1] T-norms are widely used to combine vague information in applications for approximate reasoning, knowledge representation and decision making.

[2] Note that a *single* sequent calculus for **Ł** has also been defined in [**?**]

of **Ł** into Meyer and Slaney's abelian logic **A**, the logic of abelian groups. Hence "cut" rules, which permit the introduction of lemmas or intermediary steps into proofs, were shown to be *admissible* for **GŁ** *without* proving cut-elimination, i.e. without providing an algorithm for obtaining proofs in **GŁ** from proofs in **GŁ** with cut.

Bounded Łukasiewicz logics $\mathbf{ŁB_n}$ ($n \geq 2$) arise as the *intersection* of k-valued Łukasiewicz logics, for $k = 2 \ldots n$. Informally they capture the notion of having "at most" n truth values, as expressed by the validity of the following sequent-style rule:

$$\frac{\Gamma, \overbrace{A, \ldots, A}^{\text{n times}} \vdash \Delta}{\Gamma, \underbrace{A, \ldots, A}_{\text{n-1 times}} \vdash \Delta} \ (n\text{-contraction})$$

Note that in the particular cases where $n = 2$ and $n = 3$, $\mathbf{ŁB_n}$ coincides with classical logic and 3-valued Łukasiewicz logic respectively. Other families of logics satisfying (n-contraction) were investigated in [14, 3], and in [3] also an analytic calculus for $\mathbf{ŁB_4}$ was defined.

In this work we introduce semantic interpretations and analytic proof calculi for the family of bounded Łukasiewicz logics. We start in Section 2 by introducing finite-valued, infinite-valued and bounded Łukasiewicz logics. We then show in Section 3 that bounded Łukasiewicz logics can be interpreted in terms of a *generalised* version of Ulam's game with applications in Information Theory (see e.g. [4]). In Section 4 we recall the hypersequent calculus **GŁ** presented in [10], and give first proofs of cut-elimination for **GŁ** with two forms of cut rules. Finally in Section 5 we define calculi for the bounded Łukasiewicz logics by adding a single rule to **GŁ** in each case. We then prove completeness syntactically using cut and show that the cut-elimination proofs of Section 4.1 can be extended in the presence of this extra rule.

2 Łukasiewicz Logics

We start by defining the infinite-valued Łukasiewicz logic **Ł**, noting that in this work we identify theoremhood in a logic with derivability in the corresponding Hilbert-style system.

Definition 1 (Łukasiewicz Infinite-Valued Logic, Ł). *A Hilbert-style system for* ***Ł****, using the connectives* $\supset$ *and* $\bot$*, consists of the rule:*

$$(mp)\ \frac{A \supset B, A}{B} \qquad \textit{together with the axioms:}$$

Ł1 $A \supset (B \supset A)$	*Ł3* $((A \supset B) \supset B) \supset ((B \supset A) \supset A)$
Ł2 $(A \supset B) \supset ((B \supset C) \supset (A \supset C))$	*Ł4* $((A \supset \bot) \supset (B \supset \bot)) \supset (B \supset A)$

Other connectives are defined as follows: $\neg A =^{def} A \supset \bot$, $A \oplus B =^{def} \neg A \supset B$

and $A \odot B =^{def} \neg(A \supset \neg B)$. *We also adopt the notation below:*

$$A \Leftrightarrow B = (A \supset B) \odot (B \supset A) \qquad n.A = \overbrace{A \oplus \ldots \oplus A}^{n} \qquad A^n = \overbrace{A \odot \ldots \odot A}^{n}$$

Remark 1. An alternative Hilbert-style system for **Ł** is obtained by adding axioms Ł1 and Ł3 to any axiomatization of the multiplicative additive fragment of linear logic[3] (see [14]). In particular, axiom Ł3 allows the additive connectives $\vee$ and $\wedge$ to be defined over the multiplicative ones ($\odot$ and $\oplus$ or, equivalently, $\supset$) as follows: $A \vee B =^{def} (A \supset B) \supset B$ and $A \wedge B =^{def} \neg(\neg A \vee \neg B)$.

Hilbert-style systems extending **Ł** were provided for the *finite-valued* Łukasiewicz logics by Grigolia in [6].

Definition 2 (N-Valued Łukasiewicz Logic, $\mathbf{Ł_n}$). *A Hilbert-style system for* $\mathbf{Ł_n}$ *consists of the same axioms and rules as* **Ł** *and also:*

$$Ł_n5 \;\; n.A \supset (n-1).A \qquad \text{and} \qquad Ł_n6 \;\; (p.A^{p-1}) \Leftrightarrow n.A^p$$

for every integer $p = 2, \ldots, n-2$ *that does not divide* $n-1$.

Remark 2. Axiom $Ł_n5$ corresponds to the (n-contraction) rule (see [14]).

Algebraic structures for the above logics are defined as follows, using the same notation for algebraic operations as the corresponding connectives.

Definition 3 (MV-Algebra). *An MV-algebra*[4] *is an algebra* $A_Ł = \langle A, \oplus, \neg, \perp \rangle$ *with a binary operation* $\oplus$, *a unary operation* $\neg$ *and a constant* $\perp$, *satisfying the following equations:*

$MV1 \;\; x \oplus (y \oplus z) = (x \oplus y) \oplus z$	$MV2 \;\; x \oplus y = y \oplus x$
$MV3 \;\; x \oplus \perp = x$	$MV4 \;\; \neg\neg x = x$
$MV5 \;\; x \oplus \neg\perp = \neg\perp$	$MV6 \;\; \neg(\neg x \oplus y) \oplus y = \neg(\neg y \oplus x) \oplus x$

We also define: $x \supset y =^{def} \neg x \oplus y$ *and* $\top =^{def} \neg\perp$.

Definition 4 ($\mathbf{MV_n}$-algebra). *An* MV_n*-algebra is an MV-algebra satisfying the equations:*

$$(E_{n0}) \;\; n.x = (n-1).x \qquad \text{and} \qquad (E_{np}) \;\; (p.x^{p-1})^n = n.x^p$$

for every integer $p = 2, \ldots, n-2$ *that does not divide* $n-1$.

Valuations for these algebraic structures are defined in the usual way (see, e.g., [5]). We say that a formula Φ is *valid* in an MV-algebra A (resp. in an MV$_n$-algebra A) if for all valuations v on A, $v(\Phi) = \top$.

We now introduce some important MV-algebras and MV$_n$-algebras.

[3] I.e. linear logic without exponential connectives.

[4] MV stands for many-valued.

Definition 5 ($[-1, 0]_{Ł}$, $[-1, 0]_{Ł_n}$). *Let $x \oplus y = min(0, x + y + 1)$, $\neg x = -1 - x$ and $\perp = -1$, then $[-1, 0]_{Ł} = \langle [-1, 0]_{\mathbb{R}}, \oplus, \neg, \perp \rangle$ is an MV-algebra and $[-1, 0]_{Ł_n} = \langle [-1, -(n-2)/(n-1), \ldots, -1/(n-1), 0], \oplus, \neg, \perp \rangle$ is an MV_n-algebra.*

In fact these algebras are *characteristic* for **Ł** and $\mathbf{Ł_n}$.

Theorem 1 ([2]). *The following are equivalent: (1) ϕ is a theorem of **Ł**. (2) ϕ is valid in all MV-algebras. (3) ϕ is valid in $[-1, 0]_{Ł}$.*

Theorem 2 ([6]). *The following are equivalent: (1) ϕ is a theorem of $\mathbf{Ł_n}$. (2) ϕ is valid in all MV_n-algebras. (3) ϕ is valid in $[-1, 0]_{Ł_n}$.*

We now introduce *bounded* Łukasiewicz logics.

Definition 6 (N-Bounded Łukasiewicz Logic, $\mathbf{ŁB_n}$). *A Hilbert-style system for n-bounded Łukasiewicz Logic $\mathbf{ŁB_n}$ consists of the same axioms and rules as **Ł** together with axiom $Ł_n 5$.*

Remark 3. $\mathbf{ŁB_2}$ and $\mathbf{ŁB_3}$ coincide with classical logic and $\mathbf{Ł_3}$, respectively.

$\mathbf{ŁB_n}$ is characterised by the following algebraic structures.

Definition 7 (N-Bounded MV-Algebra). *An n-bounded MV-algebra is an MV-algebra satisfying the equation (E_{n0}).*

Remark 4. In the above definition, equation (E_{n0}) can be equivalently replaced by $\neg x \vee (n-1).x$.

Theorem 3 ([5]). *The following are equivalent: (1) ϕ is a theorem of $\mathbf{ŁB_n}$. (2) ϕ is valid in all n-bounded MV-algebras. (3) ϕ is valid in all MV_k-algebras for $k = 2, \ldots, n$.*

Corollary 1. *The following are equivalent: (1) ϕ is a theorem of $\mathbf{ŁB_n}$. (2) ϕ is a theorem of $\mathbf{Ł_k}$ for $k = 2, \ldots, n$. (3) ϕ is valid in $[-1, 0]_{Ł_k}$ for $k = 2, \ldots, n$.*

Proof. Follows immediately from Theorems 2 and 3. □

3 Ulam's Game Interpretation of $\mathbf{ŁB_n}$

In [11, 12] a semantic interpretation of Łukasiewicz finite and infinite-valued logics is defined in terms of *Ulam's game* – a variant of the game of Twenty Questions, where lies, or errors, are allowed in the answers. Here we show that a useful generalisation of Ulam's game is formalised by the family of *bounded* Łukasiewicz logics $\mathbf{ŁB_n}$.

We first recall the connection between Ulam's game with $n-2$ lies and n-valued Łukasiewicz logic $\mathbf{Ł_n}$ presented in [5]. An instance of such a game proceeds as follows. An Answerer A chooses a number x from a finite subset S of natural numbers, called the *search space*. A Questioner Q then asks "questions" in the form of *subsets* of S. A, who is allowed to lie up to $n-2$ times, responds "yes" meaning that x belongs to the chosen subset, or "no" meaning that it does not. Q's objective is to identify x. It is well-known that Ulam's game models situations in the processing and sending of information that may be altered by some kind of "noise" (e.g. such as transmissions from a satellite). Here Q's aim can be interpreted as discovery of the most efficient way of recovering information in the presence of possible distortions; the *strategies* of Q then lead naturally to the theory of error-correcting codes.

Q's *state of knowledge* regarding x is uniquely determined at each point in time by a "conjunction" of the answers given by A. In general this conjunction fails to obey the rules of classical logic. For example, neither the principle of non-contradiction nor idempotency hold. If A answers "x is 2" and then "x is not 2", this is not inconsistent, it just means that A has one less lie to use. Similarly, repeated assertions that "x is 2" are more informative than one such assertion; indeed $n-1$ such assertions *guarantee* truth. One way of *describing* Q's state of knowledge is by a function $\tau : S \to \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$ that assigns to each number $y \in S$ the truth-value:

$$\tau(y) = 1 - \frac{\text{answers falsified by } y}{(n-1)}$$

Intuitively $\tau(y)$ measures, in units of $n-1$, how far y is from falsifying too many answers. Accordingly the *initial state* of the game is the constant function 1 over S. Moreover, as demonstrated in [11, 12], at each stage of the game both Q's state of knowledge and A's replies can be expressed by formulae in *n-valued Łukasiewicz logic* $\mathbf{Ł_n}$. Hence Q's ith-state of knowledge is given by the Łukasiewicz conjunction $\odot$ of the formulas expressing the jth-state of knowledge, for $j = 1, \ldots, i-1$. If we also define, for every state of knowledge τ, a "coarsest" state $\neg\tau$ that is incompatible with τ, in the sense that $\tau \odot \neg\tau = 0$ (with $\neg\tau = 1 - \tau$), then we obtain the following characterisation for $\mathbf{Ł_n}$.

Proposition 1 ([5]). *A formula Φ is a theorem of $\mathbf{Ł_n}$ if and only if Φ represents the initial state for every Ulam's game with $n-2$ lies.*

We now consider a *generalised* version of Ulam's game. In this version A and Q agree to split the search space S into $n-2$ "parts" $S_1, \ldots, S_{n-2}$, where for each S_i, A is allowed to lie i times. Equivalently, we could permit the initial state to be taken from any of Q's intermediate states of knowledge of an instance of Ulam's game with $n-2$ lies. Here the formulae representing the initial state for every generalised Ulam's game with $n-2$ parts coincide with the common tautologies of $\mathbf{Ł_k}$ for $k = 2, \ldots, n$. More formally:

Proposition 2. *A formula Φ is a theorem of $\mathbf{ŁB_n}$ if and only if Φ represents the initial state for every Ulam's game with k lies, with $k = 0, \ldots, n-2$.*

Proof. Follows by Corollary 1 and Proposition 1. □

This generalised version of Ulam's game has important applications in Information Theory, modelling search procedures for information with different probabilities of distortions; for example in transmitting "large" or "short" numbers or using different channels or frequency bands to send information (see e.g. [4]).

4 The Hypersequent Calculus GŁ

In this section we present and prove cut-elimination for the hypersequent calculus **GŁ** defined for **Ł** in [10]. Hypersequents were introduced by Avron in [1] as a natural generalisation of Gentzen sequents, consisting of a *multiset* of sequents and permitting the definition of rules that "exchange information" between different sequents. More precisely:

Definition 8 (Hypersequent). *A hypersequent is a multiset of the form*

$$\Gamma_1 \vdash \Delta_1 \mid \ldots \mid \Gamma_n \vdash \Delta_n$$

where for $i = 1, \ldots, n$, Γ_i and Δ_i are multisets[5] *of formulae, and $\Gamma_i \vdash \Delta_i$ is an ordinary sequent, called a component of the hypersequent.*

The symbol "|" is intended to denote meta-level *disjunction*. In [10] hypersequents for **Ł** are interpreted using the characteristic model $[-1, 0]_{\text{Ł}}$ as follows:

Definition 9 (Interpretation of Hypersequents for Ł). *We say that a hypersequent $\Gamma_1 \vdash \Delta_1 | \ldots | \Gamma_n \vdash \Delta_n$ is valid in **Ł**, in symbols $\models^*_{\text{Ł}} \Gamma_1 \vdash \Delta_1 | \ldots | \Gamma_n \vdash \Delta_n$, iff for all valuations v for $[-1, 0]_{\text{Ł}}$ there exists i such that $\Sigma_{A \in \Gamma_i} v(A) \leq \Sigma_{B \in \Delta_i} v(B)$, where $\Sigma_{A \in \emptyset} v(A) = 0$.*

Remark 5. We emphasize that for *formulae*, this interpretation gives the usual notion of validity for **Ł**, ie we have that a formula A is a theorem of **Ł** iff $\models^*_{\text{Ł}} \vdash A$.

Like ordinary sequent calculi, hypersequent calculi consist of axioms, logical rules and structural rules. However for hypersequent calculi the structural rules are divided into *internal* and *external* rules. The former deal with formulas within components, while the latter manipulate whole components of a hypersequent. Standard external structural rules are *external weakening* (EW) and *external contraction* (EC):

$$\frac{G}{G|\Gamma \vdash \Delta}\ (EW) \qquad \frac{G|\Gamma \vdash \Delta|\Gamma \vdash \Delta}{G|\Gamma \vdash \Delta}\ (EC)$$

where G represents a (possibly empty) side hypersequent. These rules do not really increase the expressive power of hypersequent calculi with respect to ordinary sequent calculi since they only apply to one component at a time. A rule that allows *interactions* between different components is the *splitting rule* (S):

$$\frac{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}{G|\Gamma_1 \vdash \Delta_1|\Gamma_2 \vdash \Delta_2}\ (S)$$

The hypersequent calculus **GŁ** is defined as follows:

[5] Note that by using multisets we avoid the need for exchange rules in our calculi

Definition 10 (GŁ). ***GŁ*** *has the following axioms and rules:*

Axioms

$$(ID)\ A \vdash A \qquad\qquad (\Lambda)\ \vdash \qquad\qquad (\bot)\ \bot \vdash A$$

Internal structural rules

$$\frac{G|\Gamma \vdash \Delta}{G|\Gamma, A \vdash \Delta}\ (WL) \qquad\qquad \frac{G|\Gamma_1 \vdash \Delta_1 \quad G|\Gamma_2 \vdash \Delta_2}{G|\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2}\ (M)$$

External structural rules

$$(EW),\ (EC)\ \textit{and}\ (S)$$

Logical rules[6]

$$\frac{G|\Gamma, B \vdash A, \Delta|\Gamma \vdash \Delta}{G|\Gamma, A \supset B \vdash \Delta}\ (\supset, l) \qquad\qquad \frac{G|\Gamma \vdash \Delta \quad G|\Gamma, A \vdash B, \Delta}{G|\Gamma \vdash A \supset B, \Delta}\ (\supset, r)$$

Example 1. We give a proof in **GŁ** of Ł4, the characteristic axiom for **Ł**:

$$\dfrac{\vdash \qquad \dfrac{\dfrac{\vdash}{(A \supset B) \supset B \vdash}\ (WL) \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{B \vdash B \quad A \vdash A}{B, A \vdash A, B}\ (M)}{B, A \vdash A, B|B \vdash A}\ (EW)}{B, B \supset A \vdash A}\ (\supset, l) \qquad \dfrac{\dfrac{B \vdash B \quad A \vdash A}{B, A \vdash A, B}\ (M)}{B, B \supset A, A \vdash A, B}\ (WL)}{B, B \supset A \vdash A, A \supset B}\ (\supset, r)}{B, B \supset A \vdash A, A \supset B|B \supset A \vdash A}\ (EW)}{(A \supset B) \supset B, B \supset A \vdash A}\ (\supset, l)}{(A \supset B) \supset B \vdash (B \supset A) \supset A}\ (\supset, r)}{\vdash ((A \supset B) \supset B) \supset ((B \supset A) \supset A)}\ (\supset, r)$$

Soundness and completeness of **GŁ** are proved in [10] by relating **GŁ** to a hypersequent calculus for abelian logic, and then proving the soundness and completeness for this latter calculus semantically.

Theorem 4 (Soundness and Completeness of GŁ [10]). *A hypersequent G is derivable in* ***GŁ*** *iff* $\models^*_{Ł} G$.

4.1 Cut-Elimination for GŁ

Cut-elimination is one of the most important procedures in logic. The removal of cuts corresponds to the elimination of "lemmas" from proofs. This renders a proof *analytic*, in the sense that all formuale occurring in the proof are subformulae of the formula to be proved.

In [10] it was shown that the following cut rules are *admissible* for **GŁ**.

$$\frac{G|\Gamma, A \vdash \Delta \quad G|\Pi \vdash A, \Sigma}{G|\Gamma, \Pi \vdash \Delta, \Sigma}\ (cut) \qquad\qquad \frac{G|\Gamma, A \vdash A, \Delta}{G|\Gamma \vdash \Delta}\ (gencut)$$

[6] In the logical rules, $A \supset B$ is called the *principal formula*.

Here we prove a stronger result, namely that in **GŁ** *cut-elimination* holds for these rules, i.e. that there is an algorithm for transforming a proof of a hypersequent G in **GŁ** + (*gencut*) (resp. **GŁ** + (*cut*)) into a proof of G in **GŁ**. First observe that the above cut rules are *interderivable* in **GŁ**:

$$\dfrac{\dfrac{\dfrac{\Gamma, A \vdash A, \Delta}{\Gamma \vdash \Delta | \Gamma, A \vdash A, \Delta}\,(EW)}{\Gamma, A \supset A \vdash \Delta}\,(\supset, l) \quad \dfrac{\vdash \quad A \vdash A}{\vdash A \supset A}\,(\supset, r)}{\Gamma \vdash \Delta}\,(cut) \qquad \dfrac{\dfrac{\Gamma, A \vdash \Delta \quad \Pi \vdash A, \Sigma}{\Gamma, \Pi, A \vdash A, \Delta, \Sigma}\,(M)}{\Gamma, \Pi \vdash \Delta, \Sigma}\,(gencut)$$

We therefore prove cut-elimination for **GŁ** + (*gencut*) and obtain cut-elimination for **GŁ** + (*cut*) as a corollary. Our strategy is as follows. We show that the logical rules of **GŁ** are *invertible* (i.e. that the premises of the rule are derivable if the conclusion is derivable); this allows us to reduce all applications of (*gencut*) to applications of (*gencut*) on *atomic* cut formulae. We then show that applications of (*cut*) on atomic cut formulae can be eliminated (required to deal with the rule (M)), and finally that applications of (*gencut*) on atomic cut formulae can be eliminated.

To aid exposition, we adopt the following conventions, where A is a formula and Σ is a multiset of formulae:

$$\lambda A = \{\overbrace{A, \ldots, A}^{\lambda}\} \text{ and } \lambda\Sigma = \overbrace{\Sigma \cup \ldots \cup \Sigma}^{\lambda} \text{ for } \lambda \geq 0$$

The following lemmas will be useful.

Lemma 1 ([10]). *If $G|\Gamma \vdash \bot, \Delta$ is derivable in **GŁ**, then $G|\Gamma \vdash A, \Delta$ is derivable in **GŁ**.*

Lemma 2. *If $G|\Gamma \vdash \Delta, \Sigma$ is derivable in **GŁ**, then $G|\Gamma \vdash \Delta$ is derivable in **GŁ**.*

Proof. An easy induction on the height of a derivation of $G|\Gamma \vdash \Delta, \Sigma$. □

In the next two propositions we establish the invertibility of the logical rules.

Proposition 3. *If $G|\Gamma, A \supset B \vdash \Delta$ is derivable in **GŁ**, then $G|\Gamma \vdash \Delta|\Gamma, B \vdash A, \Delta$ is derivable in **GŁ**.*

Proof. We actually prove a more general result (required to deal with the rule (EC)), namely that:

If $Q = \Gamma_1, \lambda_1(A \supset B) \vdash \Delta_1| \ldots |\Gamma_k, \lambda_k(A \supset B) \vdash \Delta_k$ is derivable in **GŁ** for $\lambda_1, \ldots, \lambda_k \geq 0$, then $Q' = \Gamma_1 \vdash \Delta_1|\Gamma_1, \lambda_1 B \vdash \Delta_1, \lambda_1 A| \ldots |\Gamma_k \vdash \Delta_k|\Gamma_k, \lambda_k B \vdash \Delta_k, \lambda_k A$ is derivable in **GŁ**.

We proceed by induction on h, the height of a proof of Q. We assume $\lambda_i > 0$ for some i, $1 \leq i \leq k$, as otherwise the proof is trivial. If $h = 0$ then $Q = A \supset B \vdash A \supset B$ and $Q' = \vdash A \supset B|B \vdash A \supset B, A$ which is derivable in **GŁ**. For $h > 0$ there are several cases for the last rule r applied.

- r is (WL) and we step to:

  $\Gamma_1, (\lambda_1 - 1)(A \supset B) \vdash \Delta_1 | \dots | \Gamma_k, \lambda_k(A \supset B) \vdash \Delta_k$

  so by the induction hypothesis:

  $\Gamma_1 \vdash \Delta_1 | \Gamma_1, (\lambda_1 - 1)B \vdash \Delta_1, (\lambda_1 - 1)A | \dots | \Gamma_k \vdash \Delta_k | \Gamma_k, \lambda_k B \vdash \Delta_k, \lambda_k A$

  is derivable in **GŁ**; hence Q' is derivable by repeatedly applying (M) and (S) $(\lambda_1 - 1)$ times together with (EC) and (EW) as necessary.

- r is (EW), (EC), (S), $(\supset, r)$ or (WL) and the weakened formula is in one of the Γ_i's. These cases involve unproblematic applications of the induction hypothesis followed by applications of the corresponding rule.

- r is (M). We step to:

  $\Gamma_1^1, \mu(A \supset B) \vdash \Delta_1^1 | \dots | \Gamma_k, \lambda_k(A \supset B) \vdash \Delta_k$ and

  $\Gamma_1^2, (\lambda_1 - \mu)(A \supset B) \vdash \Delta_1^2 | \dots | \Gamma_k, \lambda_k(A \supset B) \vdash \Delta_k$

  hence by the induction hypothesis twice we get that:

  $\Gamma_1^1 \vdash \Delta_1^1 | \Gamma_1^1, \mu B \vdash \Delta_1^1, \mu A | \dots | \Gamma_k \vdash \Delta_k | \Gamma_k, \lambda_k B \vdash \Delta_k, \lambda_k A$ and

  $\Gamma_1^2 \vdash \Delta_1^2 | \Gamma_1^2, (\lambda_1 - \mu)B \vdash \Delta_1^2, (\lambda_1 - \mu)A | \dots | \Gamma_k \vdash \Delta_k | \Gamma_k, \lambda_k B \vdash \Delta_k, \lambda_k A$

  are derivable in **GŁ**. Hence Q' is derivable in **GŁ** by suitable applications of (M) and (EW).

- r is $(\supset, l)$. There are two subcases:

  (a) If the principal formula in $(\supset, l)$ is in Γ_i, for some i, $1 \leq i \leq k$, the claim follows by the induction hypothesis and two applications of $(\supset, l)$.

  (b) Otherwise, we step to:

  $\Gamma_1, (\lambda_1 - 1)(A \supset B) \vdash \Delta_1 | \Gamma_1, (\lambda_1 - 1)(A \supset B), B \vdash A, \Delta_1 | \dots | \Gamma_k, \lambda_k(A \supset B) \vdash \Delta_k$

  By the induction hypothesis we have:

  $\Gamma_1 \vdash \Delta_1 | \Gamma_1, (\lambda_1 - 1)B \vdash (\lambda_1 - 1)A, \Delta_1 | \Gamma_1, B \vdash A, \Delta_1 | \Gamma_1, \lambda_1 B \vdash \lambda_1 A, \Delta_1 | \dots | \Gamma_k \vdash \Delta_k | \Gamma_k, \lambda_k B \vdash \Delta_k, \lambda_k A$

  By $(\lambda_1 - 1)$ applications of (M) followed by $(\lambda_1 - 1)$ applications of (S) and (EC) we have:

  $\Gamma_1 \vdash \Delta_1 | \Gamma_1, (\lambda_1 - 1)B \vdash (\lambda_1 - 1)A, \Delta_1 | \Gamma_1, \lambda_1 B \vdash A, \lambda_1 \Delta_1 | \Gamma_1, \lambda_1 B \vdash \lambda_1 A, \Delta_1 | \dots | \Gamma_k \vdash \Delta_k | \Gamma_k, \lambda_k B \vdash \Delta_k, \lambda_k A$

  Hence Q' is derivable by repeatedly applying (M) and (S) $(\lambda_1 - 1)$ times together with (EC) and (EW) as necessary. $\square$

Proposition 4. *If $G|\Gamma \vdash A \supset B, \Delta$, then $G|\Gamma \vdash \Delta$ and $G|\Gamma, A \vdash B, \Delta$ are derivable in* ***GŁ****.*

Proof. Similar to that of Proposition 3. □

We now show that applications of (cut) on *atomic*[7] formulae are eliminable.

Proposition 5. *If $G|\Gamma, \lambda q \vdash \Delta$ and $G'|\Pi \vdash \lambda q, \Sigma$ are derivable in* ***GŁ*** *for q atomic and $\lambda \geq 0$, then $G|G'|\Gamma, \Pi \vdash \Delta, \Sigma$ is derivable in* ***GŁ****.*

Proof. We prove the more general result that:

If $Q = \Gamma_1, \lambda_1 q \vdash \Delta_1 | \ldots | \Gamma_k, \lambda_k q \vdash \Delta_k$ and $Q'_i = G_i | \Pi_i \vdash \Sigma_i, \mu_i q$ for $i = 1, \ldots, k$ are derivable in **GŁ** for q atomic and $\mu_i \geq \lambda_i \geq 0$ for $i = 1, \ldots, k$, then $Q' = G_1 | \ldots | G_k | \Gamma_1, \Pi_1 \vdash \Delta_1, \Sigma_1, (\mu_1 - \lambda_1) q | \ldots | \Gamma_k, \Pi_k \vdash \Delta_k, \Sigma_k, (\mu_k - \lambda_k) q$ is derivable in **GŁ**.

We proceed by induction on h, the height of a proof of Q. We assume that $\mu_i \geq \lambda_i > 0$ for some i, $1 \leq i \leq k$ as otherwise Q is derivable easily using (EW) and (M). For $h = 0$ there are two possibilities. If $Q = q \vdash q$ then $Q' = Q'_1 = G_1 | \Pi_1 \vdash \Sigma_1, \mu_1 q$ which is derivable in **GŁ**. If $Q = \bot \vdash A$ then $Q' = G_1 | \Pi_1 \vdash \Sigma_1, A, (\mu_1 - 1)\bot$ which is derivable by Lemma 1. For $h > 0$ we have several cases according to the last rule r applied.

- r is (WL). There are two subcases.

  (a) We step to: $Q'' = \Gamma_1, (\lambda_1 - 1) q \vdash \Delta_1 | \ldots | \Gamma_k, \lambda_k q \vdash \Delta_k$

  Since $\mu_1 \geq \lambda_1 \geq 1$ we have by Lemma 2 that $Q''_1 = G_1 | \Pi_1 \vdash \Sigma_1, (\mu_1 - 1) q$ is derivable in **GŁ**. Hence by the induction hypothesis applied to $Q'', Q''_1, Q'_2, \ldots, Q'_k$ we get that Q' is derivable in **GŁ**.

  (b) We step to: $Q'' = \Gamma_1 - \{A\}, \lambda_1 q \vdash \Delta_1 | \ldots | \Gamma_k, \lambda_k q \vdash \Delta_k$

  and the claim follows by the induction hypothesis and (WL).

- r is (EW), (EC), $(\supset, l)$ or $(\supset, r)$. These involve straightforward applications of the induction hypothesis and the corresponding rule.

- r is (M). We step to:

  $Q_1 = \Gamma_1^1, \lambda_1^1 q \vdash \Delta_1^1 | \ldots | \Gamma_k, \lambda_k q \vdash \Delta_k$ and

  $Q_2 = \Gamma_1^2, \lambda_1^2 q \vdash \Delta_1^2 | \ldots | \Gamma_k, \lambda_k q \vdash \Delta_k$

  By the induction hypothesis applied to Q_1 and $Q'_1, \ldots, Q'_k$ we get that:

  $Q'' = G | G_1 | \Gamma_1^1, \Pi_1 \vdash \Delta_1^1, \Sigma_1, (\mu_1 - \lambda_1^1) q$ where

  $G = G_2 | \ldots | G_k | \Gamma_2, \Pi_2 \vdash \Delta_2, \Sigma_2, (\mu_2 - \lambda_2) q | \ldots | \Gamma_k, \Pi_k \vdash \Delta_k, \Sigma_k, (\mu_k - \lambda_k) q$

[7] Note that an atomic formula is either a propositional variable or $\bot$.

is derivable in **GŁ**. Now we apply the induction hypothesis *again*, this time to Q_2 and Q'', Q'_2,..., Q'_k, giving that:

$$G|G_1|\Gamma_1^1, \Gamma_1^2, \Pi_1 \vdash \Delta_1^1, \Delta_1^2, \Sigma_1, ((\mu_1 - \lambda_1^1) - \lambda_1^2)q|G$$

is derivable in **GŁ**. Hence Q' is derivable in **GŁ** using (EC).

– r is (S). We step to:

$$Q_1 = \Gamma_1, \Gamma_2, \lambda_1 q, \lambda_2 q \vdash \Delta_1, \Delta_2| \ldots |\Gamma_k, \lambda_k q \vdash \Delta_k$$

Since $G_i|\Pi_i \vdash \Sigma_i, \mu_i q$ is derivable in **GŁ** for $i = 1, 2$ we have that:

$$Q'_{1,2} = G_1|G_2|\Pi_1, \Pi_2 \vdash \Sigma_1, \Sigma_2, (\mu_1 + \mu_2)q$$

is derivable in **GŁ** by (M) and (EW). So now by the induction hypothesis applied to Q_1 and $Q'_{1,2}, Q'_3, \ldots, Q'_k$ we get that:

$$G_1| \ldots |G_k|\Gamma_1, \Gamma_2, \Pi_2, \Pi_2 \vdash \Delta_1, \Delta_2, \Sigma_1, \Sigma_2, ((\mu_1 + \mu_2) - (\lambda_1 + \lambda_2))q|G$$

is derivable in **GŁ** where

$$G = \Gamma_3, \Pi_3 \vdash \Delta_3, \Sigma_3, (\mu_3 - \lambda_3)q| \ldots |\Gamma_k, \Pi_k \vdash \Delta_k, \Sigma_k, (\mu_k - \lambda_k)q$$

Hence Q' is derivable in **GŁ** by (S). $\square$

We now show that applications of $(gencut)$ on atomic cut formulae can be eliminated.

Proposition 6. *If $Q = \Gamma_1, \lambda_1 q \vdash \Delta_1, \lambda_1 q| \ldots |\Gamma_k, \lambda_k q \vdash \Delta_k, \lambda_k q$ is derivable in **GŁ** for q atomic and $\lambda_1, \ldots, \lambda_k \geq 0$, then $Q' = \Gamma_1 \vdash \Delta_1| \ldots |\Gamma_k \vdash \Delta_k$ is derivable in **GŁ**.*

Proof. We proceed by induction on h, the height of a proof of Q. We assume that $\lambda_i > 0$ for some i, $1 \leq i \leq k$, as otherwise the claim is trivial. If $h = 0$ then $Q = q \vdash q$ and $Q' = \vdash$ which is derivable by (Λ). For $h > 0$ we have several possibilities for the last rule r applied. We outline the only non-trivial case, where r is (M) and we step to:

$$Q_1 = \Gamma_1^1, \mu q \vdash \Delta_1^1, \gamma q| \ldots |\Gamma_k, \lambda_k q \vdash \Delta_k, \lambda_k q \quad \text{and}$$

$$Q_2 = \Gamma_1^2, (\lambda_1 - \mu)q \vdash \Delta_1^2, (\lambda_1 - \gamma)q| \ldots |\Gamma_k, \lambda_k q \vdash \Delta_k, \lambda_k q$$

Without loss of generality we assume $\mu \geq \gamma$. By the induction hypothesis twice we get that:

$$Q'_1 = \Gamma_1^1, (\mu - \gamma)q \vdash \Delta_1^1| \ldots |\Gamma_k \vdash \Delta_k \text{ and } Q'_2 = \Gamma_1^2 \vdash \Delta_1^2, (\mu - \gamma)q| \ldots |\Gamma_k \vdash \Delta_k$$

are derivable in **GŁ**, so Q' is derivable in **GŁ** by Proposition 5 and (EC). $\square$

Finally we show that cut-elimination holds for **GŁ** + $(gencut)$.

Theorem 5. *If $G|\Gamma, A \vdash A, \Delta$ is derivable in **GŁ**, then $G|\Gamma \vdash \Delta$ is derivable in **GŁ**.*

Proof. We proceed by induction on c, the complexity of A. If $c = 1$ then the claim holds by Proposition 6. For $c > 1$ we use Propositions 3 and 4 and apply the appropriate logical rules to $A = B \supset C$ on both sides, giving that:

$Q_1 = G|\Gamma \vdash \Delta|\Gamma, C \vdash \Delta, B$ and $Q_2 = G|\Gamma, B, C \vdash \Delta, B, C|\Gamma, B \vdash \Delta, C$

are derivable in **GŁ**. By the induction hypothesis twice for Q_2 we have that:

$Q_3 = G|\Gamma \vdash \Delta|\Gamma, B \vdash \Delta, C$

is derivable in **GŁ**. By (M) applied to Q_1 and Q_3 we get that:

$Q_4 = G|\Gamma \vdash \Delta|\Gamma, \Gamma, B, C \vdash \Delta, \Delta, B, C$

is derivable in **GŁ**. Hence, applying the induction hypothesis twice to Q_4:

$G|\Gamma \vdash \Delta|\Gamma, \Gamma \vdash \Delta, \Delta$

is derivable in **GŁ**. The claim now follows using (EC) and (S). $\Box$

Corollary 2. *Cut-elimination holds for* ***GŁ*** *+ (cut).*

Proof. By Theorem 5 and the derivability of (*cut*) in **GŁ** + (gencut). $\Box$

5 Hypersequent Calculi for Bounded Łukasiewicz Logics

We now return our attention to bounded Łukasiewicz logics, defining the validity of a hypersequent in $\mathbf{ŁB_n}$ in terms of its validity in the k-valued Łukasiewicz logics for $k = 2, \ldots, n$.

Definition 11 (Interpretation of Hypersequents for $\mathbf{Ł_k}$). *A hypersequent $G = \Gamma_1 \vdash \Delta_1| \ldots |\Gamma_n \vdash \Delta_n$ is valid in $\mathbf{Ł_k}$, in symbols $\models^*_{Ł_k} G$, iff for all valuations v for $[-1, 0]_{Ł_k}$ there exists i such that $\Sigma_{A \in \Gamma_i} v(A) \leq \Sigma_{B \in \Delta_i} v(B)$, where $\Sigma_{A \in \emptyset} v(A) = 0$.*

Definition 12 (Interpretation of Hypersequents for $\mathbf{ŁB_n}$). *A hypersequent G is valid in $\mathbf{ŁB_n}$, in symbols $\models^*_{ŁB_n} G$, iff $\models^*_{Ł_k} G$ for $k = 2, \ldots, n$.*

Remark 6. It follows from Corollary 1 that A is a theorem of $\mathbf{ŁB_n}$ iff $\models^*_{\text{ŁB}_n} \vdash A$.

We introduce analytic calculi for $\mathbf{ŁB_n}$ based on this interpretation.

Definition 13 ($\mathbf{GŁB_n}$). $\boldsymbol{GŁB_n}$ *has the same axioms and rules as* ***GŁ*** *and:*

$$\frac{G|\overbrace{\Pi, \ldots, \Pi}^{n-1}, \Gamma, \bot \vdash \overbrace{\Sigma, \ldots, \Sigma}^{n-1}, \Delta}{G|\Pi \vdash \Sigma|\Gamma \vdash \Delta}\ (nC)$$

Remark 7. To obtain a true *subformula property* for the calculus we can replace the occurence of $\bot$ in the above rule by an atomic formula q with the added condition that q must occur in $G \cup \Sigma \cup \Delta$.

Example 2. As an example we prove the characteristic axiom of $\mathbf{ŁB_n}$.

$$
\dfrac{A \vdash A \qquad \dfrac{\dfrac{\neg A \vdash \neg A \qquad \dfrac{\dfrac{\bot \vdash A \qquad \dfrac{\bot \vdash \bot \quad A \vdash A}{\bot, A \vdash A, \bot}\,(M)}{\bot \vdash A, \neg A}\,(\supset, r)}{\dfrac{\vdash}{(n-3)\neg A, \bot \vdash A, (n-2)\neg A}}\,(M)}{(n-2)\neg A, \bot \vdash A, (n-1)\neg A}\,(M)}{(n-2)\neg A \vdash A | \vdash \neg A}\,(nC)}{(n-2)\neg A \vdash A | A \vdash A, \neg A}\,(M)
$$

$$
\dfrac{\dfrac{\neg A \vdash \neg A \qquad \dfrac{\vdots}{(n-2)\neg A \vdash A|(n-1).A, (n-3)\neg A \vdash A}\,(\supset, l)}{(n-2)\neg A \vdash A|(n-1).A, (n-2)\neg A \vdash A, \neg A}\,(M)}{n.A, (n-2)\neg A \vdash A}\,(\supset, l)
$$

with $(\supset, l)$ applied below $(n-2)\neg A \vdash A | A \vdash A, \neg A$, and

$$
\dfrac{\dfrac{\dfrac{\dfrac{n.A, (n-2)\neg A \vdash A}{\vdots}\,(\supset, r)}{n.A, \neg A \vdash (n-2).A}\,(\supset, r)}{n.A \vdash (n-1).A}\,(\supset, r)}{\vdash n.A \supset (n-1).A}\,(\supset, r)
$$

Theorem 6 (Soundness of $\mathbf{GŁB_n}$). *If a hypersequent G is derivable in $\boldsymbol{GŁB_n}$, then $\models^*_{ŁB_n} G$.*

Proof. We proceed by induction on the height of a proof of G, checking that the axioms are valid in $\mathbf{GŁB_n}$ and that the rules preserve validity, i.e. that if the premises are valid in $\mathbf{GŁB_n}$ then so is the conclusion. Since all but one of the cases are essentially the same as for the proof of Theorem 4 given in [10], we just check the rule (nC). Suppose that:

(1) $\models^*_{\text{Ł}_k} \overbrace{\Pi, \ldots, \Pi}^{n-1}, \Gamma, \bot \vdash \overbrace{\Sigma, \ldots, \Sigma}^{n-1}, \Delta$ for $k = 2, \ldots, n$

We check that $\models^*_{\text{Ł}_k} \Pi \vdash \Sigma | \Gamma \vdash \Delta$ for $k = 2, \ldots, n$.

Consider a valuation v for $[-1, 0]_{\text{Ł}_k}$, $2 \leq k \leq n$. If $\Sigma_{A \in \Pi} v(A) \leq \Sigma_{B \in \Sigma} v(B)$ then we are done. If $\Sigma_{A \in \Pi} v(A) > \Sigma_{B \in \Sigma} v(B)$ then we get:

$0 > \Sigma_{B \in \Sigma} v(B) - \Sigma_{A \in \Pi} v(A)$ and, since there are just k truth values:

$-1/(n-1) \geq -1/(k-1) \geq \Sigma_{B \in \Sigma} v(B) - \Sigma_{A \in \Pi} v(A)$

But now, since by (1), we have:

$(n-1)(\Sigma_{A \in \Pi} v(A)) + \Sigma_{A \in \Gamma} v(A) + (-1) \leq (n-1)(\Sigma_{B \in \Sigma} v(B)) + \Sigma_{B \in \Delta} v(B)$

we get that $\Sigma_{A \in \Gamma} v(A) \leq \Sigma_{B \in \Delta} v(B)$ and $\models^*_{\text{Ł}_k} \Pi \vdash \Sigma | \Gamma \vdash \Delta$ as required. $\square$

We now turn our attention to proving the *completeness* of $\mathbf{GŁB_n}$. As a first step we show that $\mathbf{GŁB_n}$ + (*gencut*) is complete.

Proposition 7. *If* $\vdash A$ *and* $\vdash A \supset B$ *are derivable in* $\boldsymbol{G\L B_n}$ + *(gencut) then* $\vdash B$ *is derivable in* $\boldsymbol{G\L B_n}$ + *(gencut).*

Proof. We have that $\vdash A$ and $A \vdash B$ are derivable in $\mathbf{G\L B_n}$ + $(gencut)$. Hence by (M), $A \vdash A, B$ is derivable in $\mathbf{G\L B_n}$ + $(gencut)$, and by $(gencut)$, $\vdash B$ is derivable in $\mathbf{G\L B_n}$ + $(gencut)$. $\Box$.

Theorem 7 (Completeness). *If a formula* A *is a theorem of* $\boldsymbol{\L B_n}$*, then* $\vdash A$ *is derivable in* $\boldsymbol{G\L B_n}$ + *(gencut).*

Proof. We use the completeness of the axiomatisation for $\mathbf{\L B_n}$ of Definition 6. It is easy to check that $\mathbf{G\L B_n}$ + $(gencut)$ proves all the axioms of $\mathbf{\L B_n}$, and by Proposition 7 also (mp) is admissible in $\mathbf{G\L B_n}$ + $(gencut)$. $\Box$

We now show that $(gencut)$ can be *eliminated* from derivations in $\mathbf{G\L B_n}$ + $(gencut)$, by checking that the rule (nC) does not spoil the cut-elimination procedure for $\mathbf{G\L}$ outlined in Section 4.1. It is easy to prove that the logical rules remain invertible in $\mathbf{G\L B_n}$ by extending the proofs of Propositions 3 and 4. The next step is to show that applications of (cut) on atomic cut formulae can be eliminated.

Proposition 8. *If* $G|\Gamma, \lambda q \vdash \Delta$ *and* $G'|\Pi \vdash \Sigma, \lambda q$ *are derivable in* $\boldsymbol{G\L B_n}$ *for* q *atomic, then* $G|G'|\Gamma, \Pi \vdash \Delta, \Sigma$ *is derivable in* $\boldsymbol{G\L B_n}$*.*

Proof. We follow exactly the proof of Proposition 5 and prove the more general result that:

If $Q = \Gamma_1, \lambda_1 q \vdash \Delta_1 | \dots | \Gamma_k, \lambda_k q \vdash \Delta_k$ and $Q'_i = G_i | \Pi_i \vdash \Sigma_i, \mu_i q$ for $i = 1, \dots, k$ are derivable in $\mathbf{G\L B_n}$ for q atomic and $\mu_i \geq \lambda_i \geq 0$ for $i = 1, \dots, k$, then $Q' = G_1 | \dots | G_k | \Gamma_1, \Pi_1 \vdash \Delta_1, \Sigma_1, (\mu_1 - \lambda_1)q | \dots | \Gamma_k, \Pi_k \vdash \Delta_k, \Sigma_k, (\mu_k - \lambda_k)q$ is derivable in $\mathbf{G\L B_n}$.

As before we proceed by induction on the height of a proof of Q. Here we just check the case where the last step is an application of the rule (nC). We step to:

$$Q_1 = (n-1)\Gamma_1, \Gamma_2, ((n-1)\lambda_1 + \lambda_2)q, \bot \vdash (n-1)\Delta_1, \Delta_2 | \dots | \Gamma_k, \lambda_k q \vdash \Delta_k$$

Now, since $G_i | \Pi_i \vdash \Sigma_i, \mu_i q$ is derivable in $\mathbf{G\L B_n}$ for $i = 1, 2$ we have that:

$$Q'_{1,2} = G_1 | G_2 | (n-1)\Pi_1, \Pi_2 \vdash (n-1)\Sigma_1, \Sigma_2, ((n-1)\mu_1 + \mu_2)q$$

is derivable in $\mathbf{G\L B_n}$ by (M) and (EW). So by the induction hypothesis applied to Q_1 and $Q'_{1,2}, Q'_3, \dots, Q'_k$ we get that:

$$G_1 | \dots | G_k | (n-1)\Gamma_1, (n-1)\Pi_1, \Gamma_2, \Pi_2, \bot \vdash (n-1)\Delta_1, (n-1)\Sigma_1, \Delta_2, \Sigma_2, (n-1)(\mu_1 - \lambda_1)q, (\mu_2 - \lambda_2)q | \dots | \Gamma_k, \Pi_k \vdash \Delta_k, \Sigma_k, (\mu_k - \lambda_k)q$$

is derivable in $\mathbf{G\L B_n}$. Hence Q' is derivable in $\mathbf{G\L B_n}$ by (nC) as required. $\Box$

We now check that applications of $(gencut)$ on atomic cut formulae can be eliminated in $\mathbf{G\L B_n}$.

Proposition 9. *If $Q = \Gamma_1, \lambda_1 q \vdash \Delta_1, \lambda_1 q | \ldots | \Gamma_k, \lambda_k q \vdash \Delta_k, \lambda_k q$ is derivable in* ***GŁB$_\mathbf{n}$*** *for q atomic and $\lambda_1, \ldots, \lambda_k \geq 0$, then $Q' = \Gamma_1 \vdash \Delta_1 | \ldots | \Gamma_k \vdash \Delta_k$ is derivable in* ***GŁB$_\mathbf{n}$***.

Proof. We follow the proof of Proposition 6 and just check the extra case where the last step in the proof of Q is an application of (nC). We step to:

$(n-1)\Gamma_1, \Gamma_2, ((n-1)\lambda_1 + \lambda_2)q, \perp \vdash (n-1)\Delta_1, \Delta_2, ((n-1)\lambda_1 + \lambda_2)q | \ldots | \Gamma_k, \lambda_k q \vdash \Delta_k, \lambda_k q$. By the induction hypothesis we get that:

$(n-1)\Gamma_1, \Gamma_2, \perp \vdash (n-1)\Delta_1, \Delta_2 | \ldots | \Gamma_k \vdash \Delta_k$.

is derivable in **GŁB$_\mathbf{n}$**. Hence Q is derivable in **GŁB$_\mathbf{n}$** by (nC). $\Box$

We arrive at the following cut-elimination theorem for **GŁB$_\mathbf{n}$**.

Theorem 8. *If $G|\Gamma, A \vdash \Delta, A$ is derivable in* ***GŁB$_\mathbf{n}$***, *then $G|\Gamma \vdash \Delta$ is derivable in* ***GŁB$_\mathbf{n}$***.

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