T-norm based logics with n-contraction*

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Abstract: We consider two families of fuzzy propositional logics obtained by extending **MTL** and **IMTL** with the *n-contraction* axiom, for $n \geq 2$. These logics – called $\mathbf{C_n}$**-MTL** and $\mathbf{C_n}$**-IMTL** – range from Gödel and classical logic (when $n = 2$) to **MTL** and **IMTL** (when n tends to infinity), respectively. We investigate t-norm based semantics and proof theory for $\mathbf{C_n}$**-MTL** and $\mathbf{C_n}$**-IMTL**. We show standard completeness and suitable analytic hypersequent calculi for them.

1. Introduction

Hájek's Basic fuzzy Logic **BL** [9] is the logic axiomatizing the family of calculi where the (strong) conjunction & and the implication $\to$ are interpreted by a *continuous* t-norm and its residuum, respectively. In **BL**, $\overline{0}$ (falsum) is a primitive truth-constant while the negation $\neg$ and the truth-constant $\overline{1}$ (true) are defined as $\neg\varphi \equiv \varphi \to \overline{0}$ and $\overline{1} \equiv \varphi \to \varphi$. The lattice meet and join connectives $\wedge$ and $\vee$ are also definable in the logic as $\varphi \wedge \psi \equiv \varphi \& (\varphi \to \psi)$ and $\varphi \vee \psi \equiv ((\varphi \to \psi) \to \psi) \wedge ((\psi \to \varphi) \to \varphi)$.

Monoidal t-norm logic **MTL** has been introduced in [6] as the weakening of **BL** by not requiring the divisibility axiom

$$\varphi \wedge \psi \to \varphi \& (\varphi \to \psi) \qquad (Div)$$

which amounts to the continuity of the t-norm. Thus in **MTL** the lattice connective $\wedge$ is not derivable from the other ones and needs to be introduced as a primitive connective. An alternative axiomatization of **MTL** is obtained by extending Ono's system FL_{ew} [16] (or, equivalently, Höhle's Monoidal logic **ML** [11]) by the prelinearity axiom

$$(\varphi \to \psi) \vee (\psi \to \varphi)\ . \qquad (Lin)$$

FL_{ew} – also known as **aMAILL**[1] or H_{BCK} [15] – is an extension of Full Lambek calculus with weakening and exchange.

The corresponding algebraic structures of **MTL** are the so-called MTL algebras. A MTL algebra is nothing but a bounded residuated lattice $\mathcal{L} = \langle L, \star, \Rightarrow, \leq, 0, 1\rangle$

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[1] **aMAILL** stands for the affine Multiplicative Additive fragment of (propositional) Intuitionistic Linear Logic [8], i.e., intuitionistic linear logic without exponentials extended with weakening.

satisfying the linearity condition $(x \Rightarrow y) \vee (y \Rightarrow x) = 1$. The aim behind the introduction of **MTL** was to capture the logic of *left-continuous* t-norms and their residua. This conjecture has been proved in [12] where the authors show *standard* completeness for **MTL** i.e., completeness with respect to the class of MTL algebras on the unit real interval [0, 1], which are defined by left-continuous t-norms and their residua. Notice that in [0, 1], left-continuity is the necessary and sufficient condition for a t-norm to have a residuum. Therefore **MTL** turns out to be the weakest logic of the family of residuated logics with semantics on [0, 1] given by t-norms and their residua.

Several extensions of **MTL**, which are not extensions of **BL**, have been investigated in [10, 5]. In particular, [5] has proved standard completeness for **IMTL**, i.e., the extension of **MTL** obtained by adding to it the double negation axiom

$$\neg\neg\varphi \rightarrow \varphi \tag{Inv}$$

IMTL, already considered in [6], can be equivalently seen as **aMALL**[2] extended by axiom (Lin). IMTL algebras are MTL algebras satisfying the equation $\neg\neg x = x$.

Proof theory of **MTL** has been investigated in [3]. There, an analytic calculus for **MTL** has been introduced. This calculus, called **HMTL**, uses hypersequents, a natural generalization of Gentzen's sequents (see e.g. [2]).

In this paper we investigate t-norm based semantics and proof theory for the logics $\mathbf{C_n}$**-MTL** and $\mathbf{C_n}$**-IMTL**, namely, the extensions of **MTL** and **IMTL** with the *n-contraction* axiom (with $n \geq 2$)

$$\varphi^{n-1} \rightarrow \varphi^n \tag{C_n}$$

where φ^n stands for $\varphi \& \overset{n}{\ldots} \& \varphi$. $\mathbf{C_n}$**-MTL** and $\mathbf{C_n}$**-IMTL** were already introduced in [4] and called there IC_n and CC_n. Note that (C_2) is the "standard" contraction axiom, therefore $\mathbf{C_2}$**-MTL** and $\mathbf{C_2}$**-IMTL** turn out to be Gödel logic and classical logic, respectively.

In Section 2. we prove *standard* completeness for $\mathbf{C_n}$**-MTL** and $\mathbf{C_n}$**-IMTL** , while in Section 3., along the lines of [4], we present analytic hypersequent calculi for them. These calculi are obtained by extending both **HMTL** and the hypersequent calculus for **IMTL**, with suitable rules allowing to prove axiom (C_n).

Remark 1 One could pose the question why to only consider the resulting extensions by the n-contraction axiom over **MTL** and not over **BL**. The answer is clear from both perspectives of our interest, namely, t-norm based semantics and proof theory. Indeed, on the one hand it turns out that, in contrast with left-continuous t-norms, there is no continuous t-norm, but the minimum t-norm, satisfying the n-contraction condition. This follows by the well-known representation theorem [14, 13] saying that any continuous t-norm is an ordinal sum of Gödel (minimum), Łukasiewicz and product t-norms. It is clear that neither Łukasiewicz nor product t-norm satisfies n-contraction[3]. On the other hand, while there do exist analytic calculi for **MTL** (and these can be easily extended in order to formalize extensions of **MTL**), the question of finding a (purely syntactic) cut-free calculus for **BL** is still open.

[2] Linear logic without the exponentials connectives, extended with weakening.

[3] This does not mean that there are no linearly ordered BL-algebras satisfying n-contraction other than Gödel algebras, actually there are, and all of them are ordinal sums of Godel chains and MV chains with at most n elements.

2. New schematic extensions of MTL : $\mathbf{C_n}$-MTL and $\mathbf{C_n}$-IMTL

As mentioned above, for each $n \geq 2$, we denote by $\mathbf{C_n}$**-MTL** the schematic extension of **MTL** with the *n-contraction* axiom (C_n).

Correspondingly, let us define a C_n-MTL algebra as a MTL algebra $\mathcal{L} = \langle L, \star, \Rightarrow, \leq, 0, 1\rangle$ satisfying the n-contraction equation

$$x^n = x^{n-1} \tag{C_n}$$

where x^n is a shorthand for $x \star \overset{n}{\ldots} \star x$. Notice that if a MTL algebra satisfies the equation (C_n), then trivially it also satisfies the equation (C_m) for all $m > n$. Therefore C_n-MTL algebras will also be C_m-MTL algebras for any $m > n$. Actually, C_n-MTL algebras have been introduced in [4] under the name of $ILa^{c_n}_{lin}$-algebras ($ILa^{c_n}_{lin}$ stands for affine Intuitionistic linear algebras satisfying linearity and n-contraction) as the algebraic counterpart of the system IC_n.

Recall that a *continuous (left-continous) t-norm* $\star$ is a continuous (left-continuous, resp.) binary operation on the unit real interval [0, 1] which is associative, commutative, non decreasing in each variable and satisfying $x \star 1 = x$ for all $x \in [0, 1]$. In such a case, the residuum of $\star$ is defined as $x \Rightarrow_\star y = \max\{z \in [0, 1] \mid x \star z \leq y\}$. Obviously, a t-norm $\star$ is left-continuous if it holds $(\sup_i x_i) \star y = \sup_i(x_i \star y)$.

Here we show some genuine examples of left-continuous t-norms satisfying the equation (C_n) for some n's.

Example 1 Let $N : [0, 1] \to [0, 1]$ be a weak negation, i.e. a non-increasing function such that $N(1) = 0$ and $N(N(x)) \geq x$ for all $x \in [0, 1]$. The corresponding weak nilpotent minimum t-norm is defined by

$$x \wedge_N y = \begin{cases} 0, & \text{if } x \leq N(y) \\ \min(x, y), & \text{otherwise} \end{cases}$$

It is easy to check that $\wedge_N$ is a left-continuous t-norm satisfying the equation (C_n) for all $n \geq 3$. Indeed, one has $x \wedge_N x = 0$ if $x \leq N(x)$, $x \wedge_N x = x$ otherwise.

Example 2 Fix a natural m, and for each $x \in [0, 1]$ let us denote by $\bar{x}$ the rational $\frac{i}{m}$ such that x belongs to the semi-open interval $(\frac{i-1}{m}, \frac{i}{m}]$. We define the following left-continuous t-norm:

$$x \star y = \begin{cases} \min(x, y), & \text{if } \max(\bar{x}, \bar{y}) = 1 \\ \max(\bar{x} + \bar{y} - 1, 0), & \text{otherwise} \end{cases}$$

It is easy to see that $\star$ satisfies the equation (C_{m+1}) but not the equation (C_m). The structure of this operation, for $m = 4$ can be seen in Figure 1.A.

Let us denote by $\mathbf{C_n}$**-IMTL** the schematic extension of **IMTL** with the *n-contraction* axiom (C_n). Correspondingly, we define a C_n-IMTL algebra as an IMTL algebra $\mathcal{L} = \langle L, \star, \Rightarrow, \leq, 0, 1\rangle$ satisfying the equation (C_n).

Nilpotent minimum t-norms (those defined in Example 1 for which N is an involution, e.g. $N(x) = 1 - x$) are examples of left-continuous t-norms defining C_n-IMTL algebras on [0, 1] for all $n \geq 3$. A more interesting example is the following

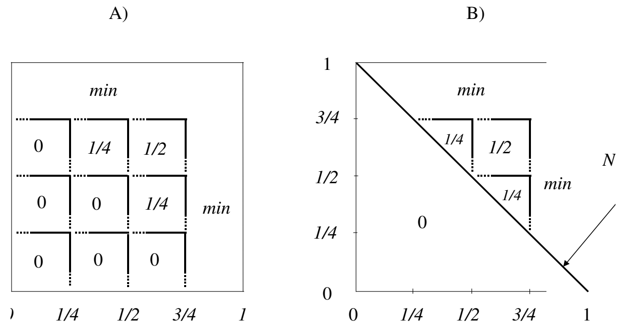

Fig. 1: A) Example of left-continuous t-norm satisfying C_5 and not C_4.; B) Example of left-continuous t-norm satisfying (Inv) and C_5 but not C_4.

Example 3 Take the left-continuous t-norm $\star$ defined in Example 2 and any involutive negation N such that $N(\frac{i}{m}) = \frac{m-i}{m}$. We define a new left-continuous t-norm as follows:

$$x \star_N y = \begin{cases} x \star y, & \text{if } x > N(y) \\ 0, & \text{otherwise} \end{cases}$$

It is easy to check that $\star_N$ satisfies the equation (C_{m+1}) but not the equation (C_m). Moreover its residuum is defined as $x \Rightarrow_N y = N(x \star_N N(y))$, and thus $x \Rightarrow_N 0 = N(x)$. Hence $\star_N$ and $\Rightarrow_N$ define a C_n-IMTL chain in [0, 1]. The structure of this operation for $m = 4$ and $N(x) = 1 - x$ can be seen in Figure 1.B.

2.1 Standard completeness for $\mathbf{C_n}$-MTL

In [4] it has been shown that $\mathbf{C_n}$**-MTL** is complete with respect to the class of linearly ordered C_n-MTL algebras. Here we show that $\mathbf{C_n}$**-MTL** is still complete when we restrict ourselves to a particular class of C_n-MTL algebras, namely, the ones defined by left-continuous t-norms satisfying the n-contraction property (standard completeness). The proof of this result is based on Jenei and Montagna's proof of the standard completeness for **MTL** [12]. In the latter, the authors show that if a formula φ does not get the value 1 for some evaluation on a countable linearly ordered MTL algebra $\mathcal{S}$, then we can always find a linearly ordered MTL algebra on the unit real interval [0, 1] and an evaluation e on this algebra such that $e(\varphi) < 1$. This is proved by showing that $\mathcal{S}$ can always be embedded into a MTL algebra on [0, 1] following the steps below:

1. Let $\mathcal{S} = \langle S, \star, \Rightarrow, \leq_S, 0_S, 1_S \rangle$ be a countable MTL chain, and define a dense structure $\langle X, \circ, \preceq, 0_X, 1_X \rangle$ as follows:

   (i) $X = \{(s, q) \mid s \in S, s \neq 0_S, q \in \mathbb{Q} \cap (0, 1]\} \cup \{(0_S, 1)\}$,

   (ii) $(s, q) \preceq (s', q')$ iff either $s <_S s'$, or $s = s'$ and $q \leq q'$,

   (iii) $0_X = (0_S, 1)$ and $1_X = (1_S, 1)$,

(iv)

$$(s,q)\circ(s',q')=\begin{cases}\min_X\{(s,q),(s',q')\}, & \text{if } s\star s'=\min_S(s,s')\\ (s\star s',1), & \text{otherwise}\end{cases}$$

where $\min_X$ and $\min_S$ are the lattice meets with respect to $\preceq$ and $\leq_S$ respectively.

2. Show that $\langle X,\circ,\preceq,0_X,1_X\rangle$ is a commutative linearly ordered integral monoid with null element and $\circ$ is left-continuous with respect to the order topology on $\langle X,\preceq\rangle$.

3. Show that the map $\Phi: S\to X$, defined as $\Phi(s)=(s,1)$, is an embedding of the structure $\langle S,\star,\leq_S,0_S,1_S\rangle$ into the structure $\langle X,\circ,\preceq,0_X,1_X\rangle$. Moreover, restricted to $\Phi(S)$, the residuum $\Rightarrow_\circ$ of $\circ$ exists, and $\Phi(s)\Rightarrow_\circ\Phi(s')=\Phi(s\Rightarrow s')$ for all $s,s'\in S$.

4. Show that the structure $\langle X,\circ,\preceq,0_X,1_X\rangle$ is order-isomorphic to a structure $\langle\mathbb{Q}\cap[0,1],\circ',\leq,0,1\rangle$. The latter can be embedded into an analogous structure $\langle[0,1],\hat{\circ},\leq,0,1\rangle$ on the real unit interval, where $\hat{\circ}$ is the extension of $\circ'$ defined by means of left continuity, i.e. $\alpha\hat{\circ}\beta=\sup\{x\circ' y\mid x,y\in\mathbb{Q}\cap[0,1],x\leq\alpha,y\leq\beta\}$. Moreover $\hat{\circ}$ is left-continuous.

5. Since $\hat{\circ}$ is left-continuous on [0, 1], its residuum –denoted by $\Rightarrow_{\hat{\circ}}$– exists. Then $\langle[0,1],\hat{\circ},\Rightarrow_{\hat{\circ}},\leq,0,1\rangle$ is a MTL chain in which the initial MTL chain $\mathcal{S}$ can be embedded.

Theorem 1 (standard completeness for $\mathbf{C_n}$-MTL) *A formula φ is provable in* $\mathbf{C_n}$**-MTL** *iff φ is a 1-tautology for all C_n-MTL algebras on [0, 1], i.e. those defined by left-continuous t-norms, satisfying (C_n), and their residua.*

Proof: In the above sketch of proof for **MTL** one can check that if we start with a MTL chain $\mathcal{S}=\langle S,\star,\Rightarrow,\leq_S,0_S,1_S\rangle$, where $\star$ satisfies the equation (C_n), then the extended operation $\circ$ on X defined on Step 1.(iv) still satisfies the equation (C_n). Indeed, if we denote $(s,q)\circ\overset{n}{\ldots}\circ(s,q)$ by $(s,q)^n$, one can check the following equality

$$(s,q)^n=\begin{cases}(s,q), & \text{if } s\star s=s\\ (s^n,1), & \text{otherwise}\end{cases}$$

where s^n denotes $s\star\overset{n}{\ldots}\star s$, and therefore we easily have $(s,q)^n=(s,q)^{n-1}$.

Finally, observe that $\hat{\circ}$, i.e. the extension of the left continuous t-norm $\circ'$ on $\mathbb{Q}\cap[0,1]$ to the whole real unit interval [0, 1] by means of suprema, preserves the property (C_n) of $\odot$. Namely, observe first that

$$\alpha\hat{\circ}\alpha=\sup_{q\leq\alpha,q\in\mathbb{Q}\cap[0,1]}\ \sup_{p\leq\alpha,p\in\mathbb{Q}\cap[0,1]} q\circ' p=\sup_{q\leq\alpha,q\in\mathbb{Q}\cap[0,1]} q\circ' q,$$

Therefore, since $\circ'$ satisfies (C_n) we have

$$\alpha\hat{\circ}\overset{n}{\ldots}\hat{\circ}\alpha=\sup_{q\leq\alpha,q\in\mathbb{Q}\cap[0,1]} q\circ'\overset{n}{\ldots}\circ' q=\sup_{q\leq\alpha,q\in\mathbb{Q}\cap[0,1]} q\circ'\overset{n-1}{\ldots}\circ' q=\alpha\hat{\circ}\overset{n-1}{\ldots}\hat{\circ}\alpha$$

$\square$

2.2 Standard completeness for $\mathbf{C_n}$-IMTL

To prove standard completeness for **$\mathbf{C_n}$-IMTL** we shall build on the same proof for **IMTL** given in [5]. The idea is the same as before. Since **$\mathbf{C_n}$-IMTL** is already complete with respect to linearly ordered C_n-IMTL algebras (see [4]), it suffices to show that it is possible to embed any countable linearly ordered C_n-IMTL algebra $\mathcal{S}$ into a C_n-IMTL algebra on the unit real interval [0, 1].

The proof of the standard completeness for **IMTL** given in [5] follows the same steps as the one for **MTL** except for a finer construction of a dense structure $\langle Y, \preceq\rangle$ where to embed a countable IMTL algebra $\mathcal{S} = \langle S, \star, \Rightarrow, \leq_S, 0_S, 1_S\rangle$ and in the extension of the monoidal operation $\star$ to Y. More precisely:

1. For each $s \in S$, let $suc(s)$ denote the successor of s if it exists, otherwise take $suc(s) = s$. Since the negation in S, defined as $\neg s = s \Rightarrow 0_S$ is involutive, $s = suc(\neg t)$ iff $t = suc(\neg s)$, and moreover, if $s <_S suc(s)$ then $suc(\neg suc(s)) = \neg s$.

2. Let $\langle Y, \preceq, 0_Y, 1_Y\rangle$ be the densely linearly ordered set defined by

$$Y = \{(s, 1) \mid s \in S\} \cup \{(s, r) \mid \exists s', s = suc(s') >_S s', r \in \mathbb{Q} \cap (0, 1)\}$$

   $0_Y = (1_S, 1)$, $1_Y = (0_S, 1)$ and $\preceq$ being the corresponding lexicographic ordering, that is, $(s, q) \preceq (t, r)$ iff either $s <_S t$, or $s = t$ and $q \leq r$.

3. Define on Y the following operation:

$$(s, q) \odot (t, r) = \begin{cases} (0_S, 1), & \text{if } s = suc(\neg t) \text{ and } q + r \leq 1 \\ (s, q) \circ (t, r), & \text{otherwise.} \end{cases}$$

   where $\circ$ is the operation defined in Step 1.(iv) in the previous subsection, restricted to Y.

4. Show that $\langle Y, \odot, \preceq, 0_Y, 1_Y\rangle$ is a commutative linearly ordered integral monoid with null element, with $\odot$ being left-continuous with respect to the order topology on $\langle Y, \preceq\rangle$.

5. Show that for all $y \in Y$, the residuum of y w.r.t. 0_Y –written $y \Rightarrow 0_Y$– exists. This defines an involutive negation $n(y) = y \Rightarrow 0_Y$ on Y.

6. As in the proof for **MTL**, finally show that the mapping $\Phi(s) = (s, 1)$ embeds $\langle S, \star, \neg, \leq_S, 0_S, 1_S\rangle$ into the densely ordered structure $\langle Y, \odot, \preceq, 0_Y, n, 1_Y\rangle$, which is order-isomorphic to a rational structure $\langle \mathbb{Q} \cap [0, 1], \odot', n' \leq, 0, 1\rangle$, and this can be embeddedable into $\langle [0, 1], \hat{\odot}, \hat{n}, \leq, 0, 1\rangle$ where $\hat{\odot}$ is left-continuous and $\hat{n}$ is an involutive negation. Then, $\langle [0, 1], \hat{\odot}, \Rightarrow_{\hat{\odot}}, \leq, 0, 1\rangle$, where $\Rightarrow_{\hat{\odot}}$ is the residuum of $\hat{\odot}$ and $y \Rightarrow_{\hat{\odot}} 0 = \hat{n}(y)$, is an IMTL chain in which the initial IMTL chain $\mathcal{S}$ can be embedded.

Now, we can basically reuse this proof to show standard completeness for **$\mathbf{C_n}$-IMTL**.

Theorem 2 (standard completeness for $\mathbf{C_n}$-IMTL) *A formula φ is provable in **$\mathbf{C_n}$-IMTL** iff φ is a 1-tautology for all C_n-IMTL algebras on [0, 1], i.e. those defined by left-continuous t-norms, satisfying (Inv) and (C_n), and their residua.*

Proof: Following the above steps, it suffices to show that if we start with a countable IMTL chain S satisfying the equation (C_n), then the operation $\odot$ defined on the dense set Y still satisfies this equation. Let us denote by $(s,q)^n$ the iterated operation $(s,q)\odot \overset{n}{\ldots} \odot(s,q)$. Then, by the definition of $\odot$, one can easily compute such an iteration:

$$(s,q)^n = \begin{cases} (0_S,1), & \text{if } s = suc(\neg s) \text{ and } q \leq 1/2 \\ (s,q), & \text{else if } s = s \star s \\ (s\star \overset{n}{\ldots} \star s, 1), & \text{otherwise.} \end{cases}$$

and check that $(s,q)^n = (s,q)^{n-1}$. The remaining part of the proof proceeds as the one for $\mathbf{C_n}$**-MTL**. □

3. Analytic Calculi for MTL and its extensions

In [3] an analytic calculus for **MTL** has been introduced. This calculus –called **HMTL** – uses hypersequents that are a simple and natural generalization of ordinary sequents.

Definition 1 *A* hypersequent *is an expression of the form*

$$\Gamma_1 \vdash \Delta_1 \mid \ \ldots \ \mid \Gamma_n \vdash \Delta_n$$

where for all $i = 1,\ldots n$, $\Gamma_i \vdash \Delta_i$ is an ordinary sequent. $\Gamma_i \vdash \Delta_i$ is called a component *of the hypersequent. A hypersequent is called* single-conclusion *if, for every $i = 1,\ldots,n$, Δ_i consists of at most one formula, otherwise the hypersequent is said to be* multiple-conclusion*.*

The intended interpretation of the symbol "|" is disjunctive.

Henceforth we consider sequents and hypersequents as *multisets* of formulas and *multisets*[4] of sequents, respectively. Like in ordinary sequent calculi, in a hypersequent calculus there are axioms and rules which are divided into two groups: *logical* and *structural rules*. The logical rules are essentially the same as those in sequent calculi, the only difference is the presence of dummy contexts G and H, called *side hypersequents* which are used as variables for (possibly empty) hypersequents. The structural rules are divided into *internal* and *external rules*. The internal rules deal with formulas within components. If they are present, they are the usual weakening and contraction. The external rules manipulate whole components within a hypersequent. These are external weakening (EW) and external contraction (EC):

$$(EW)\ \frac{H}{H \mid \Gamma \vdash A} \qquad\qquad (EC)\ \frac{H \mid \Gamma \vdash A \mid \Gamma \vdash A}{H \mid \Gamma \vdash A}$$

In hypersequent calculi it is possible to define further structural rules which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi with respect to ordinary sequent calculi. An example of such a kind of rule is Avron's

[4] If one prefers sequences over multisets as basic objects of inference then internal and external exchange rules have to be added to the calculus. This approach was used, e.g., in [2].

communication rule:

$$(com)\quad \frac{H \mid \Pi_1, \Gamma_1 \vdash A \qquad G \mid \Pi_2, \Gamma_2 \vdash B}{H \mid G \mid \Pi_1, \Pi_2 \vdash A \mid \Gamma_1, \Gamma_2 \vdash B}$$

By adding (com) to the hypersequent calculus for intuitionistic logic one gets a cut-free calculus for Gödel logic (see [1]).

HMTL has been defined by adding the (com) rule to the hypersequent calculus for intuitionistic logic without contraction (**aMAILL**). More precisely, axioms and rules of **HMTL** are the following:

$$A \vdash A \qquad \overline{0} \vdash A \qquad\qquad (cut)\quad \frac{H \mid \Gamma \vdash A \qquad G \mid A, \Gamma' \vdash C}{H \mid G \mid \Gamma, \Gamma' \vdash C}$$

Internal and External Structural Rules

$$(iw)\quad \frac{H \mid \Gamma \vdash C}{H \mid \Gamma, B \vdash C} \qquad\qquad (EC), \quad (EW), \quad (com)$$

Multiplicative fragment

$$(\&, l)\quad \frac{H \mid \Gamma, A, B \vdash C}{H \mid \Gamma, A\&B \vdash C} \qquad\qquad (\&, r)\quad \frac{H \mid \Gamma \vdash A \quad G \mid \Gamma' \vdash B}{H \mid G \mid \Gamma, \Gamma' \vdash A\&B}$$

$$(\rightarrow, l)\quad \frac{G \mid \Gamma \vdash A \quad H \mid \Gamma', B \vdash C}{G \mid H \mid \Gamma, \Gamma', A \rightarrow B \vdash C} \qquad\qquad (\rightarrow, r)\quad \frac{H \mid \Gamma, A \vdash B}{H \mid \Gamma \vdash A \rightarrow B}$$

Additive fragment

$$(\wedge, l_i)_{i=1,2}\quad \frac{H \mid \Gamma, A_i \vdash C}{H \mid \Gamma, A_1 \wedge A_2 \vdash C} \qquad\qquad (\wedge, r)\quad \frac{G \mid \Gamma \vdash A \quad H \mid \Gamma \vdash B}{G \mid H \mid \Gamma \vdash A \wedge B}$$

$$(\vee, l)\quad \frac{H \mid \Gamma, A \vdash C \quad G \mid \Gamma, B \vdash C}{H \mid G \mid \Gamma, A \vee B \vdash C} \qquad\qquad (\vee, r_i)_{i=1,2}\quad \frac{H \mid \Gamma \vdash A_i}{H \mid \Gamma \vdash A_1 \vee A_2}$$

Remark 2 As is well known, using the (internal) contraction rule, i.e.

$$(ic)\quad \frac{H \mid \Gamma, A, A \vdash C}{H \mid \Gamma, A \vdash C}$$

together with (iw), the above multiplicative and additive rules for conjunction are interderivable (see, e.g., [17]). Thus, by adding (ic) to **HMTL** one obtains Avron's hypersequent calculus for Gödel logic [1].

Theorem 3 ([3]) HMTL *is sound and complete for* **MTL***.*

Theorem 4 ([3]) HMTL *admits cut-elimination.*

3.1 Analytic Calculi for $\mathbf{C_n}$-MTL

Hypersequent calculi for the family of logics $\mathbf{C_n}$**-MTL** (for $n \geq 2$) can be uniformly defined by extending **HMTL** with the following rule

$$(nc)\quad \frac{G_1 \mid \Gamma, \Sigma_1, \overset{n}{\ldots}, \Sigma_1 \vdash C \qquad \ldots \qquad G_{n-1} \mid \Gamma, \Sigma_{n-1}, \overset{n}{\ldots}, \Sigma_{n-1} \vdash C}{G_1 \mid \ \ldots \ \mid G_{n-1} \mid \Gamma, \Sigma_1, \ldots, \Sigma_{n-1} \vdash C}$$

The above rule has been introduced in [4].

Let us call $\mathbf{HC_nMTL}$ the calculus obtained by adding (nc) to **HMTL**.

Definition 2 *Let $\mathcal{L} = \langle L, \star, \Rightarrow, \leq, 0_{\mathcal{L}}, 1_{\mathcal{L}}\rangle$ be a C_n-MTL algebra. We say that a* formula P *is* true *in $\mathcal{L}$ if for all evaluations v on $\mathcal{L}$, $v(P) = 1_{\mathcal{L}}$. A* formula *is* valid *in* $\mathbf{C_n}$**-MTL** *if it is true in every C_n-MTL algebra. A* sequent $\Gamma \vdash B$, *where $\Gamma = A_1, \ldots, A_k$, is* true *in $\mathcal{L}$ if the formula $A_1 \& \ldots \& A_k \to B$ is true in $\mathcal{L}$. ($\Gamma \vdash$ and $\vdash B$ are true in $\mathcal{L}$ if so are $A_1 \& \ldots \& A_k \to \overline{0}$ and $\overline{1} \to B$, respectively). A* hypersequent $G_1 \mid \ \ldots \ \mid G_m$ *is* true *in $\mathcal{L}$ iff for all evaluations v on $\mathcal{L}$, $v(G_1) \vee \cdots \vee v(G_m) = 1_{\mathcal{L}}$, where $\vee$ is the lattice join operation in $\mathcal{L}$. A* sequent *(resp.* hypersequent*) is* valid *in* $\mathbf{C_n}$**-MTL** *if its true in every C_n-MTL algebra.*

Theorem 5 (Soundness) *If a hypersequent H is derivable in* $\mathbf{HC_nMTL}$*, then it is valid in* $\mathbf{C_n}$**-MTL**.

Proof: By Theorem 1 it suffices to show that the rules of $\mathbf{HC_nMTL}$ preserve truth in all C_n-MTL algebras $\mathcal{L}$ on the unit real interval $[0, 1]$. We show the soundness of the (nc) rule. The soundness of the rules of **HMTL** follows by Theorem 3. To simplify the notation we can safely assume that there are no side hypersequents. Let v any evaluation on $\mathcal{L}$. By hypothesis, $v(\Gamma) \star v(\Sigma_i)^n \leq v(C)$, for every $i = 1, \ldots, n-1$. We show that $v(\Gamma) \star v(\Sigma_1) \star \cdots \star v(\Sigma_{n-1}) \Rightarrow v(C) = 1$. Let $v(\Sigma_m) = \max\{v(\Sigma_j) \mid j = 1, \ldots, n-1\}$, then $v(\Gamma) \star v(\Sigma_1) \star \cdots \star v(\Sigma_{n-1}) \leq v(\Gamma) \star v(\Sigma_m)^{n-1} = v(\Gamma) \star v(\Sigma_m)^n \leq v(C)$. Thus $v(\Gamma) \star v(\Sigma_1) \star \cdots \star v(\Sigma_{n-1}) \Rightarrow v(C) = 1$. □

Theorem 6 (Completeness) *If a formula P is valid in* $\mathbf{C_n}$**-MTL** *then $\vdash P$ is derivable in* $\mathbf{HC_nMTL}$.

Proof: As usual, we have to prove that the Lindenbaum algebra $L_n = \langle S_n, \star, \Rightarrow, \wedge, \vee, 0, 1\rangle$ determined by $\mathbf{HC_nMTL}$, is a C_n-MTL algebra, where S_n is the set of equivalence classes of formulas of $\mathbf{C_n}$**-MTL** w.r.t. the congruence relation $\approx_n$ defined as follows: For any two formulas A and B, $A \approx_n B$ iff $A \vdash B$ and $B \vdash A$ are both derivable in $\mathbf{HC_nMTL}$. By Theorem 3 we only have to show the identity $[A^{n-1}]_{\approx_n} \Rightarrow [A^n]_{\approx_n} = 1$. By definition, this holds iff the formula $A^{n-1} \to A^n$ is derivable in $\mathbf{HC_nMTL}$. The following derivation yields the desired result:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \vdash A \quad A \vdash A}{\vdots \qquad \vdots}\,(\&,r)}{A, \overset{n-1}{\ldots}, A \vdash A^{n-1}}\,(\&,r) \quad A \vdash A}{A, \overset{n}{\ldots}, A \vdash A^n}\,(\&,r) \qquad \overset{n-1}{\ldots} \qquad \dfrac{A \vdash A \quad \dfrac{\dfrac{A \vdash A \quad A \vdash A}{\vdots \qquad \vdots}\,(\&,r)}{A, \overset{n-1}{\ldots}, A \vdash A^{n-1}}\,(\&,r)}{A, \overset{n}{\ldots}, A \vdash A^n}\,(\&,r)}{A, \overset{n-1}{\ldots}, A \vdash A^n}\,(nc)}{A^{n-1} \vdash A^n}\,(\&,l)'s}{\vdash A^{n-1} \to A^n}\,(\to,r)}{}$$

□

Theorem 7 ([4]) $\mathbf{HC_nMTL}$ *admits cut-elimination.*

Remark 3 $(2c)$ is the internal contraction rule (ic) (see Remark 2). Thus $\mathbf{HC_2MTL}$ coincides with Avron's calculus for Gödel logic.

3.2 Analytic Calculi for IMTL and $\mathbf{C_n}$-IMTL

As is well known, Gentzen's sequent calculus LJ for intuitionistic logic is defined by simply restricting the right hand side of all sequents to contain at most one formula in the sequent calculus LK for classical logic (see [7]).

IMTL and $\mathbf{C_n}$**-IMTL** represent the classical (i.e. with involutive negation, as in classical logic) counterparts of **MTL** and $\mathbf{C_n}$**-MTL** respectively. Hypersequent calculi for them can be defined by considering the "classical" versions of **HMTL** and $\mathbf{HC_nMTL}$, namely, with multiple-conclusion hypersequents.

More precisely, **HIMTL** is obtained by adding to the hypersequent calculus for classical logic without contraction (**aMALL**) the following version of the communication rule

$$(lin)\quad \frac{G \mid \Gamma, \Gamma' \vdash \Delta, \Delta' \qquad H \mid \Gamma_1, \Gamma_1' \vdash \Delta_1, \Delta_1'}{G \mid H \mid \Gamma, \Gamma_1 \vdash \Delta, \Delta_1 \mid \Gamma', \Gamma_1' \vdash \Delta', \Delta_1'}$$

$\mathbf{HC_nIMTL}$ is defined by extending **HIMTL** with the following rule

$$\frac{G_0 \mid \Gamma, \Sigma_1, \overset{n}{\ldots}, \Sigma_1 \vdash \Pi_1, \overset{n}{\ldots}, \Pi_1, \Delta \ \ldots \ G_n \mid \Gamma, \Sigma_{n-1}, \overset{n}{\ldots}, \Sigma_{n-1} \vdash \Pi_{n-1}, \overset{n}{\ldots}, \Pi_{n-1}, \Delta}{G_0 \mid \ \ldots \ \mid G_n \mid \Gamma, \Sigma_1, \ldots, \Sigma_{n-1} \vdash \Pi_1, \ldots, \Pi_{n-1}, \Delta}$$

The semantic notions in Definition 2 are extended to multiple-conclusion (hyper)sequents as follows: A sequent $\Gamma \vdash \Delta$, where $\Gamma = A_1, \ldots, A_k$ and $\Delta = B_1, \ldots, B_m$, is true in an IMTL algebra (respectively, in a C_n-IMTL algebra) $\mathcal{L}$ if the formula $A_1 \& \ldots \& A_k \to B_1 \oplus \cdots \oplus B_m$ is true in $\mathcal{L}$, where $A \oplus B$ is an abbreviation for $\neg(\neg A \& \neg B)$.

Soundness and completeness for **HIMTL** and $\mathbf{HC_nIMTL}$ (w.r.t. **IMTL** and $\mathbf{C_n}$**-IMTL** , respectively), as well as the proofs of the cut-elimination theorem, proceed in a similar way to the corresponding proofs for their single-conclusion versions, see [4].

Concluding Remarks

In this paper we have investigated t-norm based semantics and proof theory for the two families of fuzzy propositional logics $\mathbf{C_n}$**-MTL** and $\mathbf{C_n}$**-IMTL** arising by extending **MTL** and **IMTL** respectively, with the n-contraction. In a future work we plan to perform a similar investigation for further useful extensions of **MTL**. This is of interest from both the semantical and the proof theoretical side. In particular, we would like to consider the family of extensions of **SMTL** [5], i.e. **MTL** with in addition the pseudo-complementation axiom $\neg(\varphi \wedge \neg\varphi)$, with the n-contraction. Left continuous t-norms obtained by ordinal sums of minimum and the t-norms described in Example 2 are examples of SMTL algebras satisfying n-contraction. Analytic calculi for these logics can be defined in a modular way, namely, by extending **HMTL** with both the rule (nc) and a suitable rule allowing to prove the pseudo-complementation axiom.

Other interesting families of fuzzy propositional logics are obtained by adding to **MTL**, **IMTL** and **SMTL** the n-law of excluded middle

$$\varphi \vee \neg\varphi^{n-1} \qquad (nW)$$

For $n = 2$ one gets classical logic, while extending **IMTL** with $(3W)$ yields 3-valued Łukasiewicz logic. Analytic hypersequent calculi for **MTL** and **IMTL** extended with (nW) have been defined in [4] while a (t-norm based) semantical analysis of them is still lacking. This is in contrast with the naturality of axiom (nW) that, algebraically, amounts to the condition $x^n = 0$ for all $x < 1$ in a linearly ordered MTL (IMTL or SMTL) algebra.

References

[1] A. Avron. Hypersequents, Logical Consequence and Intermediate Logics for Concurrency. *Annals of Mathematics and Artificial Intelligence*, **4**: 225–248, 1991.

[2] A. Avron. The Method of Hypersequents in the Proof Theory of Propositional Nonclassical Logics. In *Logic: from Foundations to Applications.* W. Hodges et al. Eds., European Logic Colloquium. Oxford Science Publications. Clarendon Press. Oxford. pp. 1-32. 1996.

[3] M. Baaz, A. Ciabattoni, F. Montagna. A Proof-Theoretical Investigation of Monoidal T-norm Based Logic. Submitted 2001.

[4] A. Ciabattoni. Bounded contraction in systems with linearity. In *Proceedings of International Conference TABLEAUX'99, Analytic Tableaux and Related Methods*, volume 1617 of *LNAI*, pages 113–128. Springer-Verlag, 1999.

[5] F. Esteva, J. Gispert, L. Godo, F. Montagna. On the Standard and Rational Completeness of some Axiomatic Extensions of the Monoidal T-norm Logic *Studia Logica* **71**: 393-420, 2002

[6] F. Esteva, L. Godo Monoidal t-norm based logic: towards a logic for left-continuous t-norms. *Fuzzy Sets and Systems* **124**(3): 271–288, 2001.

[7] G. Gentzen. Untersuchungen über das logische Schliessen I, II. *Mathematische Zeitschrift*, 39: 176–210, 405–431. 1934.

[8] J.Y. Girard. Linear Logic. *Theoretical Comp. Science* **50**: 1–102. 1987.

[9] P. Hájek. *Metamathematics of Fuzzy Logic.* Kluwer, 1998.

[10] P. Hájek. Observations on the monoidal t-norm logic. To appear in *Fuzzy Sets and Systems.*

[11] U. Höhle. Commutative, residuated l-monoids. In: U. Höhle and E.P. Klement eds., *Non-Classical Logics and Their Applications to Fuzzy Subsets*, Kluwer, 1995, 53-106.

[12] S. Jenei, F. Montagna. A proof of standard completeness for Esteva and Godo's logic MTL. *Studia Logica* **70**: 183-192, 2002.

[13] H.C. Ling. Representation of associative functions. *Publ. Math. Debrecen*, **12**: 182–212, 1965.

[14] P.S. Mostert, A.L. Shields. On the structure of semigroups on a compact manifold with boundary. *Ann. Math.* **68**: 117–143, 1957.

[15] H. Ono, Y. Komori. Logics without the contraction rule. *J. of Symbolic Logic*, **50**: 169–201. 1985.

[16] H. Ono. Logics without contraction rule and residuated lattices I. In *Festschrift on the occasion of R.K. Meyer 65th birthday* (edited by E. Mares), forthcoming.

[17] A.S. Troelstra, H. Schwichtenberg. *Basic Proof Theory.* Cambridge University Press. 1996.