Quantified Propositional Gödel Logics*

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Abstract. It is shown that $\mathbf{G}_\uparrow^{\mathrm{qp}}$, the quantified propositional Gödel logic based on the truth-value set $V_\uparrow = \{1 - 1/n : n \geq 1\} \cup \{1\}$, is decidable. This result is obtained by reduction to Büchi's theory S1S. An alternative proof based on elimination of quantifiers is also given, which yields both an axiomatization and a characterization of $\mathbf{G}_\uparrow^{\mathrm{qp}}$ as the intersection of all finite-valued quantified propositional Gödel logics.

1 Introduction

In 1932, Gödel [10] introduced a family of finite-valued propositional logics to show that intuitionistic logic does not have a characteristic finite matrix. Dummett [7] later generalized these to an infinite set of truth-values, and showed that the set of its tautologies **LC** is axiomatized by intuitionistic logic extended by the linearity axiom $(A \supset B) \vee (B \supset A)$. Gödel-Dummett logic naturally turns up in a number of different areas of logic and computer science. For instance, Dunn and Meyer [8] pointed out its relation to relevance logic; Visser [15] employed it in investigations of the provability logic of Heyting arithmetic; Pearce used it to analyze inference in extended logic programming [13]; and eventually it was recognized as one of the most important formalizations of fuzzy logic [11].

The propositional Gödel logics are well understood: Any infinite set of truth-values characterizes the same set of tautologies. **LC** is also characterized as the intersection of the sets of tautologies of all finite-valued Gödel logics $\mathbf{G}_k$ [7], and as the logic determined either by linearly ordered Kripke frames or linearly ordered Heyting algebras [12].

When Gödel logic is extended beyond pure propositional logic, however, the situation is more complex. For the cases of propositional entailment and extension to first-order validity, infinite truth-value sets with different order types determine different logics with different properties. There are infinitely many sets of truth values which give rise to distinct logics. As an example, consider

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the truth-value sets

$$\begin{aligned} V_\infty &= [0,1] \\ V_\downarrow &= \{0\} \cup \{1/n : n \geq 1\} \\ V_\uparrow &= \{1\} \cup \{1 - 1/n : n \geq 1\} \\ V_k &= \{1\} \cup \{1 - 1/n : n = 1, \dots, k-1\} \end{aligned}$$

Propositional entailment with respect to V_∞ is compact, but not with respect to $V_\downarrow$ or $V_\uparrow$. If a formula A is entailed by a set Γ with respect to V_k for every k, then it is also entailed with respect to $V_\uparrow$, but not necessarily with respect to V_∞ or $V_\downarrow$ [5]. Similarly, the first-order logic based on V_∞ is axiomatizable (this is Takeuti and Titani's intuitionistic fuzzy logic [14]), while those based on $V_\uparrow$ and $V_\downarrow$ are not [2]. The first-order Gödel logic based on $V_\uparrow$ is the intersection of all finite-valued first-order Gödel logics.

Another interesting generalization of propositional logic is obtained by adding quantifiers over propositional variables. In classical logic, propositional quantification does not increase expressive power per se. It does, however, allow expressing complicated properties more naturally and succinctly, e.g., satisfiability and validity of formulas are easily expressible within the logic once such quantifiers are available. This fact can be used to provide efficient proof search methods for several non-monotonic reasoning formalisms [9].

For Gödel logic the increase in expressive power is witnessed by the fact that statements about the topological structure of the set of truth-values (taken as infinite subsets of the real interval $[0,1]$) can be expressed using propositional quantifiers [4]. In [4] it is also shown that there is an uncountable number of different quantified propositional infinite-valued Gödel logics. The same paper investigates the quantified propositional Gödel logic $\mathbf{G}^{\mathrm{qp}}_\infty$ based on the set of truth-values $[0,1]$, which was shown to be decidable. It is of some interest to characterize the intersection of all finite-valued quantified propositional Gödel logics. As was pointed out in [4], $\mathbf{G}^{\mathrm{qp}}_\infty$ does not provide such a characterization.

In this paper we study the quantified propositional Gödel logic $\mathbf{G}^{\mathrm{qp}}_\uparrow$ based on the truth-value set $V_\uparrow$. We show that $\mathbf{G}^{\mathrm{qp}}_\uparrow$ is decidable. In general, it is not obvious that a quantified propositional logic is decidable or even axiomatizable. For instance, neither the closely related quantified propositional intuitionistic logic, nor the set of valid first-order formulas on the truth-value set $V_\uparrow$ are r.e. Although our result can be obtained by reduction to Büchi's monadic second order theory of one successor S1S [6], we also give a more informative proof based on elimination of propositional quantifiers. This proof allows us to characterize $\mathbf{G}^{\mathrm{qp}}_\uparrow$ as the intersection of all finite-valued quantified propositional Gödel logics, and moreover yields an axiomatization of $\mathbf{G}^{\mathrm{qp}}_\uparrow$.

A remark is in order about the relationship between the approach taken here using truth-value semantics and Kripke semantics. As was pointed out above, **LC** is often defined as the propositional logic of linearly ordered Kripke frames. In Kripke semantics, quantified propositional **LC** would then result by adding quantifiers over propositions (subsets of the set of worlds closed under accessibility). Here different classes of linear Kripke structures which all define **LC** in the pure propositional case in general do not define the same quantified propositional logic. In particular, the logic obtained by just taking Kripke models of order type ω is not the same as that defined by the class of all finite linear orders. It follows from the results of this paper that the logic of all finite linear Kripke structures coincides with $\mathbf{G}_{\uparrow}^{\mathrm{qp}}$.

2 Gödel Logics

Syntax. We work in the language of propositional logic containing a countably infinite set $Var = \{p, q, \dots\}$ of (propositional) variables, the constants $\bot, \top$, as well as the connectives $\wedge, \vee$, and $\supset$. Propositional variables and constants are considered atomic formulas. Uppercase letters will serve as meta-variables for formulas. If $A(p)$ is a formula containing the variable p free, then $A(X)$ denotes the formula with all occurrences of the variable p replaced by the formula X. $Var(A)$ is the set of variables occurring in the formula A. We use the abbreviations $\neg A$ for $A \supset \bot$ and $A \leftrightarrow B$ for $(A \supset B) \wedge (B \supset A)$.

Semantics. The most important form of Gödel logic is defined over the real unit interval $V_\infty = [0, 1]$; in a more general framework, the truth-values are taken from a set V such that $\{0, 1\} \subseteq V \subseteq [0, 1]$. In the case of k-valued Gödel logic $\mathbf{G}_k$, we take $V_k = \{1 - 1/i : i = 1, \dots, k-1\} \cup \{1\}$. The logic we will be most interested in is based on the set $V_\uparrow = \{1 - 1/i : i \geq 1\} \cup \{1\}$.

A *valuation* $v\colon Var \to V$ is an assignment of values in V to the propositional variables. It can be extended to formulas using the following truth functions introduced by Gödel [10]:

$$\begin{aligned} v(\bot) &= 0 & v(A \vee B) &= \max(v(A), v(B)) \\ v(\top) &= 1 & & \\ v(A \wedge B) &= \min(v(A), v(B)) & v(A \supset B) &= \begin{cases} 1 & \text{if } v(A) \leq v(B) \\ v(B) & \text{otherwise} \end{cases} \end{aligned}$$

A formula A is a *tautology* over a truth-value set $V \subseteq [0, 1]$ if for all valuations $v\colon Var \to V$, $v(A) = 1$. The *propositional logics* $\mathbf{LC}$, $\mathbf{G}_\uparrow$ and $\mathbf{G}_k$ are the sets of tautologies over the corresponding truth value sets, e.g., $\mathbf{LC} = \mathbf{G}_\infty = \{A : A \text{ a tautology over } V_\infty\}$. We also write $\mathbf{G} \models A$ for $A \in \mathbf{G}$ ($\mathbf{G} \in \{\mathbf{LC}, \mathbf{G}_\uparrow, \mathbf{G}_k\}$).

It is easily seen that $\mathbf{LC} \supseteq \mathbf{G}_\uparrow \supseteq \mathbf{G}_k$. Dummett [7] showed that $\mathbf{LC} = \mathbf{G}_\uparrow$ and that $\mathbf{LC} = \bigcap_{k \geq 2} \mathbf{G}_k$.

The abbreviation $A \prec B$ for $(A \supset B) \wedge ((B \supset A) \supset A)$ will be used extensively below. It expresses strict linear order in the sense that

$$v(A \prec B) = \begin{cases} 1 & \text{if } v(A) < v(B) \text{ or } v(B) = 1 \\ \min(v(A), v(B)) & \text{otherwise} \end{cases}$$

Propositional Quantification. In *classical* propositional logic we define $(\exists p)A(p)$ by $A(\bot) \vee A(\top)$ and $(\forall p)A(p)$ by $A(\bot) \wedge A(\top)$. In other words, propositional quantification is semantically defined by the supremum and infimum, respectively, of truth functions (with respect to the usual ordering "$0 < 1$" over the classical truth-values $\{0, 1\}$). This can be extended to Gödel logic by using *fuzzy quantifiers*. Syntactically, this means that we allow formulas $(\forall p)A$ and $(\exists p)A$ in the language. Free and bound occurrences of variables are defined in the usual way. Given a valuation v and $w \in V$, define $v[w/p]$ by $v[w/p](p) = w$ and $v[w/p](q) = v(q)$ for $q \not\equiv p$. The semantics of fuzzy quantifiers is then defined as follows:

$$v((\exists p)A) = \sup\{v[w/p](A) : w \in V\} \qquad v((\forall p)A) = \inf\{v[w/p](A) : w \in V\}$$

When we consider quantifiers, V has to be closed under infima and suprema, since otherwise truth values for quantified formulas are not defined.

We also add the additional unary connective $\circ$ to the language. The truth function for $\circ$ is given by $v(\circ A) = v((\forall p)((p \supset A) \vee p))$. In $\mathbf{G}_\uparrow^{\mathrm{qp}}$, this makes

$$v(\circ A) = \begin{cases} 1 & \text{if } v(A) = 1 \\ 1 - \frac{1}{n+1} & \text{if } v(A) = 1 - \frac{1}{n} \end{cases}$$

We abbreviate $\circ \ldots \circ A$ (n occurrences of $\circ$) by $\circ^n A$.

Using the above definitions, it is straightforward to extend the notion of tautologyhood to the new language. We write $\mathbf{G}_\uparrow^{\mathrm{qp}}$ ($\mathbf{G}_\infty^{\mathrm{qp}}$, $\mathbf{G}_k^{\mathrm{qp}}$) for the set of tautologies in the extended language over $V_\uparrow$ (V_∞, V_k).

We will show below that every quantified propositional formula is equivalent in $\mathbf{G}_\uparrow^{\mathrm{qp}}$ to a quantifier-free formula, which in general can contain $\circ$. $\circ A$ itself (or the equivalent formula $(\forall p)((p \supset A) \vee p)$), however, is not in general equivalent to a quantifier-free formula not containing $\circ$. Inspection of the truth tables shows that a quantifier-free formula containing only the variable q takes one of 0, $v(q)$, or 1 as its value under a given valuation v, and thus no such formula can define $\circ q$.

3 Hilbert-style Calculi

All the calculi we consider are based on the following set of axioms:

I1	$A \supset (B \supset A)$	I7	$(A \wedge \neg A) \supset B$
I2	$(A \wedge B) \supset A$	I8	$(A \supset \neg A) \supset \neg A$
I3	$(A \wedge B) \supset B$	I9	$\bot \supset A$
I4	$A \supset (B \supset (A \wedge B))$	I10	$A \supset \top$
I5	$A \supset (A \vee B)$	I11	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
I6	$B \supset (A \vee B)$	I12	$((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$

These axioms, together with the rule of modus ponens, define the system IPC that is sound and complete for intuitionistic propositional logic. The system LC is obtained by adding to IPC the linearity axiom

$$\text{LC} \quad (A \supset B) \vee (B \supset A).$$

It is well known [7] that IPC and LC are sound for all propositional Gödel logics, and that LC is complete for all infinite-valued propositional Gödel logics. We will make frequent use of this fact below, and omit derivations of formulas which are (instances of) quantifier- and $\circ$-free tautologies in $\mathbf{G}_\uparrow$. These omissions are indicated by pointing out that the formula follows already in LC or IPC. In particular, familiar inference patterns such as the chain rule or case distinction are derivable in LC and its extensions.

When we turn to quantified propositional logics, a natural system $\mathsf{IPC}^{\mathrm{qp}}$ to start with is obtained by adding to IPC the following two axioms:

$$\supset\exists \quad A(C) \supset (\exists p)A(p) \qquad\qquad \supset\forall \quad (\forall p)A(p) \supset A(C)$$

and the rules:

$$\frac{A(p) \supset B^{(p)}}{(\exists p)A(p) \supset B^{(p)}}\mathrm{R}\exists \qquad\qquad \frac{B^{(p)} \supset A(p)}{B^{(p)} \supset (\forall p)A(p)}\mathrm{R}\forall$$

where for any formula C, the notation $C^{(p)}$ indicates that p does not occur free in C, i.e., p is a (propositional) *eigenvariable.*

Let $\mathsf{QG}_\uparrow^{\mathrm{qp}}$ be the system obtained by adding to $\mathsf{IPC}^{\mathrm{qp}}$ the axioms (LC),

$$\forall\vee \qquad (\forall p)[A \vee B(p))] \supset [A \vee (\forall p)B(p)]$$

where $p \notin A$, and the following:

G1	$\circ(A \supset B) \leftrightarrow (\circ A \supset \circ B)$	G4	$(A \supset \circ B) \supset ((A \supset C) \vee (C \supset B))$
G2	$A \prec \circ A$	G5	$(A \leftrightarrow \bot) \vee (\exists p)(A \leftrightarrow \circ p)$
G3	$(\circ A \supset \circ B) \supset ((A \supset B) \vee \circ B)$	G6	$(A \prec B) \supset (\circ A \supset B)$

Proposition 1. *The system* $\mathsf{QG}_\uparrow^{\mathrm{qp}}$ *is sound for* $\mathbf{G}_k^{\mathrm{qp}}$ *and* $\mathbf{G}_\uparrow^{\mathrm{qp}}$.

Proof. It is easily seen that the rules of inference preserve validity. For instance, if $B \supset A(p)$ is valid, then, for any valuation v, $v[w/p](B) \le v[w/p](A(p))$ where $w \in V$. If p does not occur in B, then $v(B) = v[w/p](B)$ and we have $v(B) \le \inf\{v[w/p](A(p)) : w \in V\}$. That LC is sound for arbitrary Gödel logics was shown in [7]. The tedious but straightforward verification that the remaining axioms $(\vee\forall)$ and (G1)–(G6) are valid is left to the reader.

Remark 2. In [4] it was shown that a system sound and complete for $\mathbf{G}_\infty^{\mathrm{qp}}$, the quantified propositional Gödel logic based on the truth-value set $[0, 1]$, is obtained by extending $\mathsf{IPC}^{\mathrm{qp}}$ with (LC), $(\vee\forall)$ and the axiom

$$(\forall p)[(A^{(p)} \supset p) \vee (p \supset B^{(p)})] \supset \ (A^{(p)} \supset B^{(p)}).$$

This schema is not valid in $\mathbf{G}_\uparrow^{\mathrm{qp}}$ (it comes out $= 0$ under any v with $v(A) = 1/2$ and $v(B) = 0$). On the other hand, it is easy to see that $v(\circ A) = v(A)$ in V_∞, and hence axiom (G2) is not valid in $\mathbf{G}_\infty^{\mathrm{qp}}$. Thus neither of $\mathbf{G}_\infty^{\mathrm{qp}}$ and $\mathbf{G}_\uparrow^{\mathrm{qp}}$ is included in the other. This is in contrast to the situation in propositional entailment and first-order logic, where V_∞ defines the smallest Gödel logic and is included in all others.

4 Decidability

In this section we prove that $\mathbf{G}_{\uparrow}^{\mathrm{qp}}$ is decidable. This is done by defining a reduction of tautologyhood in $\mathbf{G}_{\uparrow}^{\mathrm{qp}}$ to S1S, the monadic theory of one successor, which was shown to be decidable by Büchi [6].

S1S is the set of second-order formulas in the language with second-order quantification restricted to monadic set variables X, Y, ... with one unary function $'$ (successor) which are true in the model $\langle \omega, ' \rangle$. For the purposes of this section we consider $\circ A$ to be an abbreviation of $(\forall p)((p \supset A) \vee p)$.

Suppose A is a quantified propositional formula, and B is a formula in the language of S1S with only x free. Let $TV(B(x))$ abbreviate $(\forall z)(B(z') \supset B(z))$. We define A^x by:

$$\begin{aligned}
p^x &= X_p(x) \\
\bot^x &= X_\bot(x) \\
\top^x &= (\forall z)(z = z) \\
(B \wedge C)^x &= B^x \wedge C^x \\
(B \vee C)^x &= B^x \vee C^x \\
(B \supset C)^x &= (\forall y)(B^y \supset C^y) \vee (\exists y)(B^y \wedge \neg C^y) \wedge C^x \\
(\forall p)B^x &= (\forall X_p)(TV(X_p(x)) \supset B^x) \\
(\exists p)B^x &= (\exists X_p)(TV(X_p(x)) \wedge B^x)
\end{aligned}$$

Consider the following reduction:

$$\Phi(A) = (\forall X_\bot)((\forall x)\neg X_\bot(x) \supset (\forall x)A^x)$$

The idea behind this is to correlate truth-values in $V_\uparrow$ with subsets of ω which are closed under predecessor, i.e., predicates in

$$TV = \{P \subseteq \omega : \text{if } n \in P \text{ then } m \in P \text{ for all } m \leq n\}.$$

Under this correlation, 1 corresponds to ω, and $1 - 1/n$ corresponds to $\{1, \ldots, n\}$.

Let s be an interpretation of the language of S1S, mapping variables to elements or subsets of ω. We denote by $s[n/x]$ the interpretation which is just like s except that it assigns n to x. Then $TV(A(x))$ obviously expresses the condition that the predicate $A(x)[s] = \{n : S1S \models A(x)[s[n/x]]\}$ defined by $A(x)$ in s is closed under predecessor. If a monadic predicate P is closed under predecessor, we define its truth value by

$$tv(P) = \sup\{1 - \frac{1}{n} : 1^n \in P\}.$$

Conversely, every truth-value $v \in V_\uparrow$ corresponds to a monadic predicate

$$mp(v) = \begin{cases} \{k : k \leq n\} & \text{if } v = 1 - 1/n \\ \omega & \text{if } v = 1. \end{cases}$$

Note that for $P, Q \in TV$, $P \subseteq Q$ iff $tv(P) \leq tv(Q)$, and conversely, for $v, w \in V_\uparrow$, $v \leq w$ iff $mp(v) \subseteq mp(w)$.

Lemma 3. *Let v be a valuation and s be the interpretation defined by $s(X_p) = mp(v(p))$ and $s(X_\perp) = \emptyset$. Then we have $tv(A^x[s]) = v(A)$.*

Proof. By induction on the complexity of A. The claim is obvious for atomic formulas, conjunction and disjunction. If $A \equiv B \supset C$ we have to distinguish two cases. Suppose first that $v(B) \leq v(C)$. By induction hypothesis, $B^x[s] = mp(v(B)) \subseteq mp(v(C)) = C^x[s]$, and hence the first disjunct in the definition of $(B \supset C)^x$ is true. Thus $(B \supset C)^x$ defines ω and $tv((B \supset C)^x[s]) = 1$. Now suppose that $v(B) > v(C)$. Then $tv(B^x[s]) \supsetneq tv(C^x[s])$, $S1S \not\models (\forall y)(B^y \supset C^y)\ [s]$ and $S1S \models (\exists y)(B^y \wedge \neg C^y)\ [s]$, and thus $(B \supset C)^x[s] = C^x[s]$.

If $A \equiv (\exists p)B$, let $v[w/p]$ be the valuation which is just like v except that $v[w/p](p) = w$, and let $s[mp(w)/X_p]$ be the corresponding interpretation which is like s except that it assigns $mp(w)$ to X_p.

By induction hypothesis, $tv(B^x[s[mp(w)/X_p]]) = v[w/p](B)$. We again have two cases. Suppose first that $\sup\{v[w/p](B) : w \in V_\uparrow\} = 1 - 1/n$. For all $m > n$, $S1S \not\models B^x[m/x, mp(w)/X_p]$, since $v[w/p](B^x) < 1 - 1/m$ by induction hypothesis. On the other hand, $S1S \models TV(P_p) \supset B^x\ [s[k/x, mp(1-1/n)/P_p]]$ for all $k \leq n$, and so $tv((\exists p)B^x[s]) = 1 - 1/n$. Now consider the case where $\sup\{v[w/p](B) : w \in V_\uparrow\} = 1$. Here there is no bound n on the the members of sets defined by $B^x[s[mp(w)/X_p]]$ where $w \in V_\uparrow$. Hence, $mp((\exists p)B)^x[s]) = \omega$ and $tv((\exists p)B^x[s]) = 1$.

The case $A \equiv (\forall p)B$ is similar. □

Lemma 4. *Let s be an interpretation with $s(X_\perp) = \emptyset$ and $s(X_p) \in TV$. Let v be defined by $v(p) = tv(s(X_p))$. Then $A^x[s] \in TV$, and $v(A) = tv(A^x[s])$.*

Proof. By induction on the complexity of A. The claim is again trivial for atomic formulas, conjunctions or disjunctions. If $A \equiv B \supset C$, two cases occur. If $S1S \models (\forall y)(B^y \supset C^y)$, then $B^y[s] \subseteq C^y[s]$. By induction hypothesis, $v(B) \leq v(C)$, and hence $v(B \supset C) = 1 = tv((B \supset C)^x[s])$. Otherwise, for some n we have $n \in B^y[s]$ but $n \notin C^y[s]$. So $(\exists y)(B^y \wedge \neg C^y)$ must be true and the predicate defined is the same as $C^y[s]$.

Now for the case $A \equiv (\exists p)B$: If $S1S \models (\exists X_p)(TV(X_p) \supset B^x)[s[n/x]]$, then there is a prefix closed witness P so that $S1S \models B^x[s[n/x, P/X_p]]$. By induction hypothesis, $B^x[s[P/X_p]] \in TV$, and hence $S1S \models TV(X_p) \supset B^x\ [s[m/x, P/X_p]]$ for all $m \leq n$, and thus $((\exists p)B)^x[s] \in TV$ as well.

Consider $N = ((\exists p)B)^x[s]$. First, suppose that $\sup N = k$. That means that for some $P \in TV$, $1^k \in B^x[s[P/X_p]]$, and for no $Q \in TV$ and no $j > k$, $j \in B^x[s[Q/X_p]]$. By induction hypothesis, $v[tv(P)/p](B) = 1 - 1/k$ and for all $w \in V_\uparrow$, $v[w/p](B) \leq 1 - 1/k$. Hence $v((\exists p)B) = 1 - 1/k$.

If $\sup N$ does not exist, for each k there is a witness $Q_k \in TV$ with $k \in B^x[s[Q_k/X_p]]$. By induction hypothesis, for each k we have $v[tv(Q_k)/p](B) \geq 1 - 1/k$, and so $v((\exists p)B) = 1$.

The case $A \equiv (\forall p)B$ is similar. □

Theorem 5. $\mathbf{G}_\uparrow^{\mathrm{qp}}$ *is decidable.*

Proof. If there is a valuation v such that $v(A) < 1$, then by Lemma 3 there is an s with $s(P_\bot) = \emptyset$ and n so that $n \notin A^x[s]$, and hence $S1S \not\models \Phi(A)$.

Conversely, suppose $S1S \not\models \Phi(A)$. We may assume, without loss of generality, that all propositional variables in A are bound. Then there is an interpretation s with $X_\bot(x)[s] = \emptyset$ so that some $n \notin A^x[s]$. By Lemma 4, $A^x[s] \in TV$. Hence, if $n \notin A^x[s]$, then $k \notin A^x[s]$ for all $k \geq n$, and, also by Lemma 4, $v(A) = tv(A^x[s]) < 1$.

Thus a formula A is a tautology in $\mathbf{G}_\uparrow^{\mathrm{qp}}$ iff $S1S \models \Phi(A)$. The claim follows by the decidability of $S1S$. □

5 Properties and Normal Forms

In this section we introduce suitable normal forms for formulas of $\mathsf{QG}_\uparrow^{\mathrm{qp}}$ and prove some useful properties of $\mathsf{QG}_\uparrow^{\mathrm{qp}}$. These results will be crucial in the proof of the elimination of quantifiers.

Proposition 6. *1.* $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (A \supset B) \supset (\circ A \supset \circ B)$
2. $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash \circ(A \wedge B) \leftrightarrow (\circ A \wedge \circ B)$
3. $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash \circ(A \vee B) \leftrightarrow (\circ A \vee \circ B)$

Proof. (1) From (G2) we have $(A \supset B) \supset \circ(A \supset B)$, which, together with the left-to-right direction of (G1) yields the result.

(2) The left-to-right implication immediately follows from axioms (I2) and (I3) together with Prop. 6(1). For the converse, replace B by $B \supset (A \wedge B)$ in Prop. 6(1) and use (I4) to derive $\circ A \supset \circ(B \supset (A \wedge B))$. Then, using (G1), one has $\circ A \supset (\circ B \supset \circ(A \wedge B))$. The claim follows by IPC.

(3) In LC, we have $(A \vee B) \leftrightarrow (A \supset B) \supset B) \wedge (B \supset A) \supset A)$. Replacing A by $\circ A$ and B by $\circ B$, we have $(\circ A \vee \circ B) \leftrightarrow (\circ A \supset \circ B) \supset \circ B) \wedge (\circ B \supset \circ A) \supset \circ A)$. The result follows using (G1) and IPC. □

Proposition 7. *1. If p does not occur boind in $C(p)$, then*

$$\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (\forall \bar{q})(A \leftrightarrow B) \supset (C(A) \supset C(B))$$

where $\bar{q}$ are the propositional variables occurring free in A and B.
2. If $C(p)$ is quantifier-free, we also have

$$\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (A \leftrightarrow B) \supset (C(A) \supset C(B))$$

Proof. By induction on the complexity of C. Cases for $\wedge$, $\vee$, and $\supset$ are easy. If $C(p) \equiv \circ D(p)$, we use the induction hypothesis and Prop. 6(1). If $C(p) \equiv$

$(\exists r)D(p,r)$, we argue:

(1)	$(\forall\bar{q})(A \leftrightarrow B) \supset (D(A,r) \supset D(B,r))$	by IH
(2)	$((\forall\bar{q})(A \leftrightarrow B) \wedge D(A,r)) \supset D(B,r))$	(1), IPC
(3)	$D(B,r) \supset (\exists r)D(B,r)$	$\supset\exists$
(4)	$(\forall\bar{q})(A \leftrightarrow B) \wedge D(A,r)) \supset (\exists r)D(B,r)$	(2), (3)
(5)	$D(A,r) \supset ((\forall\bar{q})(A \leftrightarrow B) \supset (\exists r)D(B,r))$	(4), IPC
(6)	$(\exists r)(D(A,r) \supset ((\forall\bar{q})(A \leftrightarrow B) \supset (\exists r)D(B,r)))$	(5), R$\exists$
(7)	$(\forall\bar{q})(A \leftrightarrow B) \supset ((\exists r)D(A,r) \supset (\exists r)D(B,r))$	(6), IPC

The case of $C \equiv (\forall r)D(p,r)$ is handled similarly. □

Definition 8. A formula A of $\mathsf{QG}_{\uparrow}^{\mathrm{qp}}$ is in $\circ$-*normal form* if it is quantifier-free and for all subformulas $\circ B$ of A, $B \in \{\bot, \top\} \cup \mathit{Var}$ or $B \equiv \circ B'$.

Proposition 9. *Let A be a quantifier-free formula of $\mathsf{QG}_{\uparrow}^{\mathrm{qp}}$. Then there exists a formula A' of $\mathsf{QG}_{\uparrow}^{\mathrm{qp}}$ in $\circ$-normal form such that $\mathsf{QG}_{\uparrow}^{\mathrm{qp}} \vdash A \leftrightarrow A'$.*

Proof. Follows from axiom (G1), Prop. 6(2) and (3) using Prop. 7(2). □

Proposition 10. *For every $n \geq 0$, $\mathsf{QG}_{\uparrow}^{\mathrm{qp}} \vdash \circ^n\top \leftrightarrow \top$.*

Proof. $\circ^n\top \supset \top$ is already derivable intuitionistically. For $\top \supset \circ^n\top$, use (G2), Prop. 6(1), and induction on n. □

For propositional Gödel logic, a normal form similar to the disjunctive normal form of classical logic has been introduced in [1] (see also [3, 4]). This so-called *chain normal form* is based on the fact that, in a sense, the truth value of a formula only depends on the ordering of the variables occurring in the formula induced by the valuation under consideration. The chain normal form can then be constructed by enumerating all such orderings (using $\prec$ and $\leftrightarrow$ to encode the ordering) in a way similar to how one constructs a disjunctive normal form by enumerating all possible truth value assignments. We extend the notion of chain normal form and the results of [3] in order to deal with the $\circ$ connective. This is possible, since by Prop. 9 we can always push the $\circ$ in front of atomic subformulas, so we only need to consider orderings of subformulas of the form $\circ^j B$ with B atomic. Let Γ be a finite subset of $\{\circ^j p, \circ^j\bot : p \in \mathit{Var}, j \in \omega\} \cup \{\top\}$ and $\top, \bot \in \Gamma$.

Definition 11. A $\circ$-*chain* over Γ is an expression of the form

$$(S_1 \star_1 S_2) \wedge \cdots \wedge (S_{n-1} \star_{n-1} S_n)$$

such that $\Gamma = \{S_1, \ldots, S_n\}$, $S_1 \equiv \bot$, $S_n \equiv \top$, and $\star_i \in \{\leftrightarrow, \prec\}$, for all $i = 1, \ldots, n$.

Every $\circ$-chain C uniquely determines a partition $\Pi_1^C, \ldots, \Pi_k^C$ of Γ so that $\Pi_i^C = \{S_{j_i}, \ldots, S_{j_{i+1}-1}\}$ where $j_1 = 1$, $j_{k+1} = n+1$, $j_i < j_{i+1}$, $\star_{j_i} = \cdots = \star_{j_{i+1}-2} = \leftrightarrow$, and $\star_{j_{i+1}-1} = \prec$. Conversely, every such partition determines a $\circ$-chain up to provable equivalences. It is easily seen that if C is such a chain, then $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \supset (S_i \leftrightarrow S_j)$ if $S_i, S_j \in \Pi_l^C$ for some l, and $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \supset (S_i \prec S_{i'})$ if $S_i \in \Pi_j^C$, $S_{i'} \in \Pi_{j'}^C$ and $j < j'$. Thus C also uniquely corresponds to an ordering of Γ which we denote $<_C$, defined by $S_i <_C S_{i'}$ iff $S_i \in \Pi_j^C$, $S_{i'} \in \Pi_{j'}^C$ and $j < j'$. This order is total, the Π_i^C are maximal anti-chains, $\bot$ is minimal, and $\top$ is maximal.

Suppose now that A is in $\circ$-normal form, and that Γ contains all the subformulas of A of the form $\circ^j p$ or $\circ^j \bot$, as well as $\top$; that C is an $\circ$-chain on Γ; and that the valuation v agrees with $<_C$, i.e., $S_i <_C S_j$ iff $v(S_i) < v(S_j)$. Using the same idea as in the proof of Lemma 3 in [3], one can find $A^C \in \Gamma$, the "value" of A under C, so that $v(A^C) = v(A)$, and the choice of A^C depends only on $<_C$, not on v itself. Specifically, A^C can be constructed as follows: (1) If $A \in \Gamma$, then $A^C \equiv A$. (2) If $A \equiv D \wedge E$, then $A^C \equiv D^C$ if $D^C <_C E^C$ and $\equiv E^C$ otherwise. (3) If $A \equiv D \vee E$, then $A^C \equiv D^C$ if $E^C <_C D^C$, and $\equiv E^C$ otherwise. (4) If $A \equiv D \supset E$, then $A^C \equiv E^C$ if $E^C <_C D^C$, and $\equiv \top$ otherwise. This "evaluation" of A is provable in the sense that $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \supset (A \leftrightarrow A^C)$. This follows easily using the following theorems of **LC**:

$$
\begin{array}{cc}
(D \prec E) \supset (D \wedge E \leftrightarrow D) & (E \prec D) \supset (D \wedge E \leftrightarrow E) \\
(D \leftrightarrow E) \supset (D \wedge E \leftrightarrow D) & (D \prec E) \supset (D \vee E \leftrightarrow E) \\
(E \prec D) \supset (D \vee E \leftrightarrow D) & (E \leftrightarrow D) \supset (D \vee E \leftrightarrow E) \\
(D \prec E) \supset (D \supset E \leftrightarrow \top) & (E \prec D) \supset (D \supset E \leftrightarrow E) \\
(E \leftrightarrow D) \supset (D \supset E \leftrightarrow \top) &
\end{array}
$$

Definition 12. Let A be a quantifier free formula in $\circ$-normal form, Γ_A be the set of all subformulas of A of the form $\circ^j p, \circ^k \bot, \top$, $\Gamma \supseteq \Gamma_A$, and $C(\Gamma)$ the set of all possible $\circ$-chains over Γ. Then

$$\bigvee_{C \in C(\Gamma)} C \wedge A^C$$

is the $\circ$-*chain normal form* for A over Γ.

Theorem 13. *Let A and Γ be as above, and A' be the $\circ$-chain normal form for A over Γ. Then $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash A \leftrightarrow A'$.*

Proof. (See also Thm. 4 of [3].) First note that $\bigvee_{C \in C(\Gamma)} C$ is a tautology and provable in LC. Since for each $C \in C(\Gamma)$ we have $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (C \wedge A^C) \supset A$, the right-to-left implication $A' \supset A$ follows by case distinction.

For the left-to-right implication, consider $A \supset (A \wedge \bigvee_{C \in C(\Gamma)} C)$. This is provable, since $\bigvee_{C \in C(\Gamma)} C$ is provable. By distributivity of $\wedge$ over $\vee$, we have $A \supset \bigvee_{C \in C(\Gamma)} (A \wedge C)$. We also have $(A \wedge C) \supset (C \wedge A^C)$ for each $C \in C(\Gamma)$ from $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \supset (A \leftrightarrow A^C)$. Together we get $A \supset \bigvee_{C \in C(\Gamma)} (C \wedge A^C)$. $\square$

We now strengthen the $\circ$-normal form result so that only $\circ$-chains that are intuitively "possible" need to be considered. For this, we have to verify that we can exclude chains C which result in orders which, e.g., have $\circ S <_C S$.

Definition 14. A formula A is in *minimal normal form* over Γ if it is of the form $\bigvee_{C \in \mathcal{C} \subseteq C(\Gamma)} C$, where each C is a $\circ$-chain over Γ, and so that the corresponding ordered partition $\Pi_1^C, \ldots, \Pi_k^C$ satisfies

1. for no $i < j$ and $S \in \Gamma$ do we have $\circ^{r+s} S \in \Pi_i^C$ and $\circ^r S \in \Pi_j^C$ with $s > 0$;
2. for all $S \in \Gamma$, if $\circ^s S \in \Pi_i^C$ $(i < k)$, then $\circ^r S \notin \Pi_i^C$ if $r \neq s$; and
3. for no j, j' and $S \in \Gamma$ do we have both $\circ^i S \in \Pi_j^C$ and $\circ^{i+1} S \in \Pi_{j'}^C$ with $j' > j + 1$.

Theorem 15. *Let A be in $\circ$-normal form. There exists a formula A^{nf} in minimal normal form such that $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash A \leftrightarrow A^{\mathrm{nf}}$.*

Proof. By Thm. 13, $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash A \leftrightarrow A'$ where A' is a $\circ$-chain normal form over Γ. Consider a disjunct of A' of the form $C \wedge A^C$, where $\Pi_1^C, \ldots, \Pi_k^C$ is the ordered partition of Γ corresponding to C. If $A^C \in \Pi_k^C$, then $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (C \wedge A^C) \leftrightarrow C$, since $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash A^C \leftrightarrow (A^C \leftrightarrow \top)$. Otherwise, $A^C \in \Pi_i^C$ with $i < k$. Then the sequence $\Pi_i^C, \ldots, \Pi_k^C$ corresponds to a conjunction

$$C' \equiv (A^C \star_1 S_1') \wedge \ldots \wedge (S_{j-1}' \star_j \top)$$

where for at least one $l \leq j$, $\star_j = \prec$, and $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \leftrightarrow C'' \wedge C'$, where C'' is the part of C corresponding to $\Pi_1^C, \ldots, \Pi_{i-1}^C$. Since $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash A^C \leftrightarrow (A^C \leftrightarrow \top)$, we have

$$\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (C' \wedge A^C) \leftrightarrow (C' \wedge (\top \leftrightarrow A^C)) \tag{1}$$

As is easily seen, the right-hand side of (1) is provably equivalent to

$$C''' \equiv (A^C \leftrightarrow S_1') \wedge \ldots \wedge (S_{j-1}' \leftrightarrow \top)$$

In sum, $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (C \wedge A^C) \leftrightarrow (C'' \wedge C''')$, and $C'' \wedge C'''$ is a $\circ$-chain.

By induction on the number of disjuncts in A' one shows that there is A'' which is a disjunction of $\circ$-chains such that $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash A \leftrightarrow A''$. Now we have to prove that there exists a disjunction of $\circ$-chains A^{nf} satisfying 1–3 of Def. 14 so that $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash A'' \leftrightarrow A^{\mathrm{nf}}$.

Suppose that for some disjunct C in A'' we have $\circ^{r+s} S \in \Pi_i^C$ and $\circ^r S \in \Pi_j^C$ where $s > 0$ and $i < j$. Then, since $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (\circ^{r+s} A \prec \circ^r A) \leftrightarrow \circ^r A$ we have $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \leftrightarrow C'$ where C' is the $\circ$-chain corresponding to $\Pi_1^C, \ldots, \Pi_{i-1}^C, \Pi_i^C \cup \ldots \cup \Pi_k^C$.

Consider a disjunct C of A'' where for some $i < k$, both $\circ^r S \in \Pi_i^C$ and $\circ^s S \in \Pi_i^C$ where $r < s$. Then $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \supset (\circ^s S \leftrightarrow \top)$. To see this, recall that $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash \circ^r v \prec \circ^s S$ if $r < s$. By definition of $\prec$, that means that

$$\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash ((\circ^s S \supset \circ^r S) \supset \circ^r S) \wedge (\circ^r S \supset \circ^s S). \tag{2}$$

Since $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \supset (\circ^s S \leftrightarrow \circ^r S)$, we have $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \supset (\circ^s S \supset \circ^r S)$ which together with the left conjunct of (2) gives $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \supset \circ^r S$. Thus, as before, C is provably equivalent to the $\circ$-chain corresponding to $\Pi_1^C, \ldots, \Pi_i^C \cup \ldots \cup \Pi_k^C$.

Lastly, suppose that for a disjunct C of A'' we have both $\circ^i S \in \Pi_j^C$ and $\circ^{i+1} S \in \Pi_{j'}^C$ for some j, j' such that $j' > j + 1$. Then by axiom (G6) together with transitivity we get $C \supset (\circ^{i+1} S \prec \circ^{i+1} S)$, and since $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (B \prec B) \leftrightarrow B$ we have $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash C \leftrightarrow C'$ where C' is the $\circ$-chain corresponding to $\Pi_1^C, \ldots, \Pi_{j-1}^C, \Pi_j^C \cup \ldots \cup \Pi_{j'}^C \ldots \cup \Pi_k^C$.

By induction on the number of disjuncts in A'' we obtain the desired A^{nf}. □

6 Quantifier Elimination

In this section we prove quantifier elimination for $\mathsf{QG}_\uparrow^{\mathrm{qp}}$. As a corollary of this result we show that the system $\mathsf{QG}_\uparrow^{\mathrm{qp}}$ is sound and complete for $\mathbf{G}_\uparrow^{\mathrm{qp}}$ and that the latter is the intersection of all finite-valued quantified propositional Gödel logics $\mathbf{G}_k^{\mathrm{qp}}$.

Proposition 16. *1.* $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (\forall p)A(p) \leftrightarrow (A(\bot) \wedge (\forall p)A(\circ p))$
2. $\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (\exists p)A(p) \leftrightarrow (A(\bot) \vee (\exists p)A(\circ p))$.

Proof. (1) The left-to-right implication follows easily from the two instances of $(\supset\forall)$

$$(\forall p)A(p) \supset A(\bot) \qquad \text{and} \qquad (\forall p)A(p) \supset A(\circ p).$$

For right-to-left, consider

$$(q \leftrightarrow \bot) \supset (A(\bot) \wedge (\forall p)A(\circ p)) \supset A(q) \tag{3}$$
$$(q \leftrightarrow \circ p) \supset (A(\bot) \wedge (\forall p)A(\circ p)) \supset A(q) \tag{4}$$

which are derived easily from Prop. 7(2) using $\mathsf{IPC}^{\mathrm{qp}}$. Use $(\mathrm{R}\exists)$ to introduce the existential quantifier in the antecedent of (4), and then (I12) to obtain

$$[(q \leftrightarrow \bot) \vee (\exists p)(q \leftrightarrow \circ p)] \supset (A(\bot) \wedge (\forall p)A(\circ p)) \supset A(q) \tag{5}$$

The antecedent of (5) is an instance of (G5), and so

$$\mathsf{QG}_\uparrow^{\mathrm{qp}} \vdash (A(\bot) \wedge (\forall p)A(\circ p)) \supset A(q)$$

from which the right-to-left direction of (1) follows by $(\mathrm{R}\forall)$.

(2) The argument is analogous to the derivation of (1). □

Definition 17. For $\Gamma \subseteq Var \cup \{\bot, \top\}$, let $OP_\Gamma(A)$ be the set of formulas inductively defined as follows:

$$OP_\Gamma(A * B) = OP_\Gamma(A) \cup OP_\Gamma(B), \quad \text{where } * \in \{\vee, \wedge, \supset\}$$
$$OP_\Gamma((\mathsf{Q}p)A) = OP_\Gamma(A), \quad \text{where } \mathsf{Q} \in \{\forall, \exists\}$$
$$OP_\Gamma(\circ^k v) = \begin{cases} \{\circ^k v\} & \text{if } v \in \Gamma \\ \emptyset & \text{otherwise} \end{cases}$$

Then $\exp_\Gamma(A) = \{k : \circ^k q \in OP_\Gamma(A)\}$

Definition 18. The *quantifier depth* $\mathrm{qd}(A)$ of a formula is defined by:

$$\mathrm{qd}(p) = \mathrm{qd}(\bot) = 0 \qquad \mathrm{qd}((\forall p)B) = \mathrm{qd}((\exists p)B) = \mathrm{qd}(B) + 1$$
$$\mathrm{qd}(B * C) = \max(\mathrm{qd}(B), \mathrm{qd}(C)) \text{ for } * \in \{\wedge, \vee, \supset\}$$

Lemma 19. *Let A be a closed formula such that (a) every quantifier free subformula of A is in $\circ$-normal form and (b) no two quantifier occurrences bind the same variable. Let $\Delta = \{p_1, \dots, p_j\}$ be the set of variables belonging to the innermost quantifiers in A, and $\Gamma = Var(A) \setminus \Delta$. Then there is a formula $A^\sharp$ so that*

1. $\mathsf{QG}^{\mathrm{qp}}_\uparrow \vdash A \leftrightarrow A^\sharp$,
2. $\max \exp_\Delta(A^\sharp) \leq \min \exp_\Gamma(A^\sharp)$,
3. $\max \exp_{Var(A^\sharp)}(A^\sharp) \leq 2 \cdot \max \exp_{Var(A)}(A)$,
4. $\mathrm{qd}(A^\sharp) \leq \mathrm{qd}(A)$.

Proof. Suppose $\Gamma = \{q_1, \dots, q_l\}$. Let $A_0 = A$, $m = \max \exp_\Delta(A)$. At stage i, pick the non-innermost quantified subformula $(\forall q_i)B_i(q_i)$ or $(\exists q_i)B_i(q_i)$ of A_i corresponding to q_i and replace

$$(\forall q_i)B_i(q_i) \;\text{ by }\; B_i(\bot) \wedge \ldots \wedge B_i(\circ^{m-1}\bot) \wedge (\forall p)B_i(\circ^m q_i)$$
$$(\exists q_i)B_i(p) \;\text{ by }\; B_i(\bot) \vee \ldots \vee B_i(\circ^{m-1}\bot) \vee (\exists q_i)B_i(\circ^m q_i)$$

to obtain A_{i+1}. The procedure terminates with $A_l = A^\sharp$.

At each stage $\mathsf{QG}^{\mathrm{qp}}_\uparrow \vdash A_i \leftrightarrow A_{i+1}$ follows by induction on m from Prop. 16. The lower bounds are obvious from the construction of $A^\sharp$. ☐

Lemma 20. *Suppose $A(p)$ is in $\circ$-normal form and*

$$\max \exp_{\{p\}} A \leq \min \exp_{Var(A) \setminus \{p\}} A.$$

There is a formula $A^\exists$, with $Var(A^\exists) \subseteq Var(A) \setminus \{p\}$ so that

$$\mathsf{QG}^{\mathrm{qp}}_\uparrow \vdash (\exists p)A \leftrightarrow A^\exists$$

and $\max \exp_{Var(A^\exists) \cup \{\bot\}} A^\exists \leq \max \exp_{Var(A) \cup \{\bot\}} A + 1$.

Proof. Let $m = \max\exp_{Var(A)\cup\{\bot\}} A$ be the maximal exponent of a subformula $\circ^j S$ and let $\Gamma = \{\circ^i S : S \in Var \cup \{\bot\}, i \leq m\}$.

Theorem 15 provides us with A^{nf} in minimal normal form over Γ so that $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\exists p)A \leftrightarrow (\exists p)A^{\mathrm{nf}}$. Since $\exists$ distributes over $\lor$, we only have to consider formulas of the form $(\exists p)C$ where C is a $\circ$-chain and satisfies the conditions of Thm. 15. C corresponds to an ordered partition $\Pi_1, \dots, \Pi_k$ over Γ. We prove that $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\exists p)C \leftrightarrow C'$ for some quantifier-free C' by induction on k.

If $k = 2$, then either $p \in \Pi_1$ or $p \in \Pi_k$. In the first case, $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\exists p)C(p) \leftrightarrow C(\bot)$, in the second one, $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\exists p)C(p) \leftrightarrow C(\top)$.

Now suppose $k > 2$. Three cases arise, according to how the equivalence classes containing p are distributed.

(1) The partition corresponding to C is of the form

$$\Pi_1, \dots, \Pi_i, \{p\}, \{\circ p\}, \dots, \{\circ^j p\} \cup \Pi_k$$

Then $C(p)$ is of the form

$$B \land \underbrace{(v \prec p) \land (p \prec \circ p) \land \dots \land (\circ^j p \leftrightarrow \top)}_{D(p)} \land E$$

Since $D(\top)$ is provable, $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\exists p)C \leftrightarrow B \land v \prec \top \land E$.

(2) The partition corresponding to C is of the form

$$\Pi_1, \dots, \Pi_i, \{p\}, \{\circ p\}, \dots, \{\circ^j p\}, \Pi_{i'}, \dots, \Pi_k$$

and $\circ^j p \notin \Pi_{i'}$. Then $C(p)$ is of the form

$$B \land \underbrace{(S \prec p) \land (p \prec \circ p) \land \dots \land (\circ^j p \prec S')}_{D(p)} \land E$$

We first show that $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\exists p)D(p) \leftrightarrow (\circ^{j+1} S \prec S')$. For the right-to-left direction, observe that

$$\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\circ^{j+1} S \prec S') \supset [(S \prec \circ S) \land \dots \land (\circ^j S \prec \circ^{j+1} S) \land (\circ^{j+1} S \prec S'),$$

from which the claim follows by (R$\exists$). The left-to-right direction is proved by induction on j, using axiom (G6). In sum, we have

$$\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\exists p)C(p) \leftrightarrow (B \land (\circ^{j+1} S \prec S') \land E)$$

(3) The partition corresponding to C is of the form

$$\Pi_1, \dots, \Pi_i, \{p\}, \{\circ p\}, \dots, \{\circ^j p\} \cup \Pi, \Pi_{i'}, \dots, \Pi_k$$

with $S \in \Pi$, $S \neq \circ^j p$. Because of the condition on $\max\exp_{\{p\}} A$ we can assume that $S \equiv \circ^n q$ with $n \geq j$.

We proceed by induction on j. If $j = 0$, then we have a conjunct $p \leftrightarrow S$, and $(\exists p)C \equiv C(S)$. Otherwise, we have a conjunct $\circ^j p \leftrightarrow \circ^n q$ with $n \geq j$. Using

(G3), this conjunct is provably equivalent to $(\circ^{j-1}p \leftrightarrow \circ^{n-1}q) \vee (\circ^j p \wedge \circ^n q)$. Hence, C is equivalent to the disjunction of two $\circ$-chains corresponding to

$$\begin{array}{ll} \Pi_1, \ldots, \Pi_i, \{p\}, \{\circ p\}, \ldots, & \{\circ^{j-1}p, \circ^{n-1}q\}, \Pi, \Pi_{i'}, \ldots, \Pi_k \\ \Pi_1, \ldots, \Pi_i, \{p\}, \{\circ p\}, \ldots, & \{\circ^j p\} \cup \Pi \cup \Pi_{i'} \cup \ldots \cup \Pi_k \end{array}$$

For the first $\circ$-chain, the maximum exponent of p is smaller and hence the induction hypothesis of the present subcase applies. The second $\circ$-chain is shorter overall, and hence the induction hypothesis based on number of equivalence classes applies. □

Lemma 21. *Let $A(p)$ be in $\circ$-normal form, and so that*

$$\max\exp_{\{p\}} A \leq \min\exp_{Var(A)\setminus\{p\}} A.$$

There is a formula $A^\forall$, with $Var(A^\forall) \subseteq Var(A) \setminus \{p\}$ so that

$$\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\forall p)A \leftrightarrow A^\forall$$

and $\max\exp_{Var(A^\forall)\cup\{\bot\}} A^\forall \leq \max\exp_{Var(A)\cup\{\bot\}} A + 1$.

Proof. Let A^{nf} be the minimal normal form of A. It is provably equivalent to the formula obtained from A^{nf} by replacing each element of a chain $S \prec S'$ by $\circ S \supset S'$. By distributivity then, $A \leftrightarrow A'$ where A' is a conjunction of disjunctions of implications of the form $\circ^i S \supset \circ^j S'$. Any such disjunct of the form $\circ^i p \supset \circ^j p$ is provably equivalent to $\top$ if $i \leq j$ (in which case the entire disjunction can be deleted), or to $\top \supset \circ^j p$ if $i > j$. The part of a disjunction in A' containing p thus can be assumed to be of the form

$$\bigvee_i (D_i \supset \circ^{n_i} p) \vee \bigvee_j (\circ^{m_j} p \supset E_j)$$

where $p \notin D_i, E_i$. This, in turn, is equivalent to a conjunction of disjunctions of the form

$$\bigvee_i (D \supset \circ^{n_i} p) \vee \bigvee_j (\circ^{m_j} p \supset E)$$

This can again be simplified by taking $n = \max\{n_i\}$ and $m = \min\{m_j\}$, since $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (A \supset B) \vee (A \supset C) \leftrightarrow (A \supset C)$ if $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash B \supset C$.

Since $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\forall p)(A \wedge B) \leftrightarrow (\forall p)A \wedge (\forall p)B$ and $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\forall p)(A(p) \vee B) \leftrightarrow (\forall p)A(p) \vee B$ if $p \notin B$, it suffices to show that a formula of the form

$$F \equiv (\forall p)(D \supset \circ^n p) \vee (\circ^m p \supset E))$$

is equivalent to a quantifier free formula. We distinguish three cases:

(1) $E \equiv \circ^k \top$, $k \geq 0$. Then $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\circ^m p \supset E)$ and hence $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash F \leftrightarrow \top$.

(2) $E \equiv \circ^k\bot$, $k < m$. Then $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\circ^m p \supset E) \leftrightarrow E$, and hence $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash F \leftrightarrow (A \supset \circ^n\bot) \vee E$.

(3) Since $\max\exp_{\{p\}} A \leq \min\exp_{Var(A)\setminus\{p\}} A$ by assumption, this leaves only the case $E \equiv \circ^m S$. Then $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash F \leftrightarrow (A \supset \circ^{n+1}S) \vee \circ^m S$. The left-to-right implication is obvious by ($\supset\forall$), instantiating p by $\circ S$. For the right-to-left implication two cases arise:

(a) $n \leq m$. By (G4), we have $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (A \supset \circ^{n+1}S) \supset [(A \supset \circ^n p) \vee (\circ^n p \supset \circ^n S)]$. Furthermore, $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (\circ^n p \supset \circ^n S) \supset (\circ^m p \supset \circ^m S)$. In sum, we have

$$[(A \supset \circ^{n+1}S) \vee \circ^m S] \supset [(A \supset \circ^n p) \vee (\circ^m p \supset \circ^m S) \vee \circ^m S]$$

Since $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash \circ^m S \supset (\circ^m p \vee \circ^m S)$, we have $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash [(A \supset \circ^{n+1}S) \vee \circ^m S] \supset F$.

(b) $n > m$. By (G2), $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash \circ^m S \supset \circ^{n+1}S$, and so $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash [(A \supset \circ^{n+1}S) \vee \circ^m S] \supset (A \supset \circ^{n+1}S]$. Using induction and (G4), it is easy to show that

$$\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash (A \supset \circ^{n+1}S) \supset [(A \supset \circ^n p) \vee \underbrace{\bigvee_{i=m}^{n-1} (\circ^{i+1} p \supset \circ^i p) \vee (\circ^m p \supset \circ^m S}_{D}].$$

Each of the disjuncts $\circ^{i+1}p \supset \circ^i p$ implies $\circ^i p$, which in turn implies $A \supset \circ^n p$, so $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash D \supset (A \supset \circ^n p)$. In sum, we have again $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash [(A \supset \circ^{n+1}S) \vee \circ^m S] \supset F$.

The bound on $\max\exp_{Var(A^\forall)\cup\{\bot\}} A$ follows by inspection. □

Theorem 22. *For every closed formula A of $\mathsf{QG}^{\mathrm{qp}}_{\uparrow}$ there exists a variable-free formula A^{qf} such that $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash A \leftrightarrow A^{\mathrm{qf}}$, and $\max\exp_{\{\bot\}} A^{\mathrm{qf}} \leq 2^{\mathrm{qd}(A)+l}$ where $l = \max\exp_{Var(A)\cup\{\bot\}}$.*

Proof. We may assume, renaming variables if necessary, that each variable in A is bound by only one quantifier occurrence. By induction on $\mathrm{qd}(A)$. If $\mathrm{qd}(A) = 0$, there is nothing to prove. If $\mathrm{qd}(A) > 0$, let $A^\sharp$ be as in Lemma 19. Replace each innermost quantified formula $(\exists p)B$, $(\forall p)B$ by $B^\exists$ or $B^\forall$, respectively. The resulting formula A' satisfies $\mathrm{qd}(A') \leq \mathrm{qd}(A) - 1$ and $\max\exp_{Var(A)\cup\{\bot\}} A' \leq 2\max\exp_{Var(A)\cup\{\bot\}} A + 1$. □

Proposition 23. *Let A be variable-free, and in $\circ$-normal form. Then either $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash A \leftrightarrow \top$ or $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash A \leftrightarrow \circ^k(\bot)$ where $k \leq \max\exp_{\{\bot\}} A = n$.*

Proof. Consider the minimal normal form A^{nf} of A over $\{\circ^k(\bot) : k \leq n\}$. Each chain in A^{nf} is of one of two forms

$$\begin{aligned} C &= (\bot \prec \circ(\bot)) \wedge (\circ(\bot) \prec \circ\circ(\bot)) \wedge \ldots \wedge (\circ^{n-1}\bot \prec \circ^n(\bot)) \\ C_m &= (\bot \prec \circ(\bot)) \wedge (\circ(\bot) \prec \circ\circ(\bot)) \wedge \ldots \wedge (\circ^{m-1}\bot \prec \circ^m(\bot)) \wedge \bigwedge_{k=m}^{n} \circ^k(\bot) \end{aligned}$$

C is provable, so $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash C \leftrightarrow \top$, and $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash C_m \leftrightarrow \circ^m(\bot)$. So if A^{nf} contains C, then $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash A \leftrightarrow \top$, otherwise $\mathsf{QG}^{\mathrm{qp}}_{\uparrow} \vdash A \leftrightarrow \circ^k(\bot)$, where k is the maximum of C_i occurring in A^{nf}. □

Corollary 24. *Let A be closed and not containing $\circ$. Then either $\mathsf{QG}_{\uparrow}^{\mathrm{qp}} \vdash A$ or $\mathsf{QG}_{\uparrow}^{\mathrm{qp}} \vdash A \leftrightarrow \circ^k(\bot)$, where $k \leq 2^{\mathrm{qd}(A)}$.*

Corollary 25. *The calculus $\mathsf{QG}_{\uparrow}^{\mathrm{qp}}$ is complete for $\mathbf{G}_{\uparrow}^{\mathrm{qp}}$.*

Proof. If $\mathsf{QG}_{\uparrow}^{\mathrm{qp}} \nvdash A$, then $\mathsf{QG}_{\uparrow}^{\mathrm{qp}} \vdash A \leftrightarrow \circ^k\bot$ for some k. Since $\mathbf{G}_{\uparrow}^{\mathrm{qp}} \nvDash \circ^k\bot$ for all k, $\mathbf{G}_{\uparrow}^{\mathrm{qp}} \nvDash A$.

Theorem 26. *$\mathbf{G}_{\uparrow}^{\mathrm{qp}}$ is the intersection of all finite-valued quantified propositional Gödel logics.*

Proof. $\mathsf{QG}_{\uparrow}^{\mathrm{qp}}$ is sound for each finite-valued Gödel logic, so $\mathbf{G}_{\uparrow}^{\mathrm{qp}} \subseteq \mathbf{G}_{k}^{\mathrm{qp}}$ for each k. Conversely, if $\mathbf{G}_{\uparrow}^{\mathrm{qp}} \not\models A$, then $\mathsf{QG}_{\uparrow}^{\mathrm{qp}} \vdash A \leftrightarrow \circ^k(\bot)$ for some k. Since $\mathsf{QG}_{\uparrow}^{\mathrm{qp}}$ is sound for $\mathbf{G}_{k+2}$, we have $\mathbf{G}_{k+2} \not\models A$ as obviously $\mathbf{G}_{k+2} \nvDash \circ^k\bot$.

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