A Natural Deduction System for Intuitionistic Fuzzy Logic

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Abstract. Intuitionistic fuzzy logic *IF* was introduced by Takeuti and Titani. This logic coincides with the first-order Gödel logic based on the real unit interval $[0,1]$ as set of truth-values. We present a natural deduction system **NIF** for *IF*. **NIF** is defined by suitably translating a first-order extension of Avron's hypersequent calculus for Gödel logic. Soundness, completeness and normal form theorems for **NIF** are provided.

1 Introduction

Intuitionistic fuzzy logic *IF* was defined by Takeuti and Titani [18] as the logic of the complete Heyting algebra over the real unit interval $[0,1]$. *IF* turned out to coincide with the first-order Gödel logic based on the truth-value set $[0,1]$.

The finite-valued propositional versions of this logic were introduced by Gödel in 1933 to show that intuitionistic logic does not admit a characteristic finite matrix [11]. Dummett later generalized these to the infinite set of truth-values $[0,1]$, and showed that the set of its tautologies LC is axiomatized as intuitionistic logic extended by the linearity axiom $(A \supset B) \vee (B \supset A)$.

Together with Łukasiewicz and product logic, LC is also one of the most important formalizations of fuzzy logic [12].

Cut-free sequent calculi for LC have been defined in [16, 1, 8]. However these calculi make use of rules with an arbitrary number of premises. A cut-free calculus for LC which does not have this drawback has been introduced in [2]. This calculus, called **GLC**, is based on *hypersequents* — a simple and natural generalization of Gentzen's sequents to *multisets* of sequents (see [3] for an overview). Contrary to the other calculi for LC, in **GLC** the rules introducing the connectives are exactly those of Gentzen's sequent calculus **LJ** for intuitionistic logic. In [7] the hypersequent calculus $\mathbf{GLC}^{\forall\exists}$ was defined by extending **GLC** with rules for quantifiers. It was shown that $\mathbf{GLC}^{\forall\exists}$ is sound and complete for *IF*.

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In this paper we introduce a natural deduction system **NIF** for *IF*. **NIF** is defined by suitably translating $\mathbf{GLC}^{\forall\exists}$. **NIF** is based on a natural deduction system for intuitionistic logic — called **NJ′** — that is different from the usual one. Albeit equivalent to Gentzen's original system **NJ** for intuitionistic logic [10], **NJ′** is more perspicuously related to the sequent calculus **LJ** than **NJ**.

We prove that any cut-free proof in $\mathbf{GLC}^{\forall\exists}$ translates into a normal natural deduction derivation in **NIF**, where "normal" essentially means that for no branch of the derivation tree an elimination of a formula follows an introduction of the same formula. Moreover we show that any derivation in **NIF** can be translated into a proof in $\mathbf{GLC}^{\forall\exists}$. These results, in particular, imply that **NIF** is sound and complete for *IF*.

In future work we will investigate the possibility of strong normalization for **NIF**. That is we aim at a non-deterministic procedure that stepwise converts any given derivation into a unique normal form. We also plan to investigate the relationships between *LC* and parallel computation (see [2] for some hints). The analysis of **NIF** can be seen as a first step to achieve these research goals.

2 Syntax and Semantic of Intuitionistic Fuzzy Logic

The language of Gödel logics is the same as that of classical and intuitionistic logic. More precisely, we use the binary *connectives* $\wedge, \vee$, and $\supset$ and the *truth constant* $\perp$ and abbreviate $A \supset \perp$ by $\neg A$. *Object variables* are denoted by x, y, $\ldots$; the usual existential and universal *quantifiers*, $\forall$ and $\exists$, refer to these variables. *Bound* and *free* occurrences of variables are defined as usual. Moreover, for every $n \geq 0$, there is an infinite supply of n-ary *predicate symbols* and *function symbols*. Constants are considered as 0-ary function symbols. *Terms* and *formulas* are inductively defined in the usual way. Propositional variables are identified with predicate symbols of arity 0.

In this work we consider intuitionistic fuzzy logic *IF*, that is the first-order Gödel logic defined over the real unit interval $[0, 1]$. In general, the truth-values of Gödel logic can be taken from a set W such that $\{0, 1\} \subseteq W \subseteq [0, 1]$. Note that whereas for propositional Gödel logic, the tautologies coincide for all infinite W, uncountably many different Gödel logics are induced by different infinite W. (This is proved in [6].)

An *interpretation* $\mathcal{I}$ consists of a non-empty *domain* D and a *valuation function* $\mathrm{val}_{\mathcal{I}}$ that maps constants and object variables to elements of D and n-ary function symbols to functions from D^n into D. $\mathrm{val}_{\mathcal{I}}$ extends in the usual way to function mapping all terms of the language to an element of the domain. Moreover, $\mathrm{val}_{\mathcal{I}}$ maps every n-ary predicate symbol P to a function from D^n into $[0, 1]$. The truth-value of an atomic formula $A \equiv P(t_1, \ldots, t_n)$ is thus defined as

$$\mathrm{val}_{\mathcal{I}}(A) = \mathrm{val}_{\mathcal{I}}(P)(\mathrm{val}_{\mathcal{I}}(t_1), \ldots, \mathrm{val}_{\mathcal{I}}(t_n)).$$

For the truth constant $\perp$ we have $\mathrm{val}_{\mathcal{I}}(\perp) = 0$.

The semantics of propositional connectives is given by

$$\mathrm{val}_{\mathcal{I}}(A \supset B) = \begin{cases} 1 & \text{if } \mathrm{val}_{\mathcal{I}}(A) \leq \mathrm{val}_{\mathcal{I}}(B) \\ \mathrm{val}_{\mathcal{I}}(B) & \text{otherwise,} \end{cases}$$

$$\mathrm{val}_{\mathcal{I}}(A \wedge B) = \min(\mathrm{val}_{\mathcal{I}}(A), \mathrm{val}_{\mathcal{I}}(B)),$$
$$\mathrm{val}_{\mathcal{I}}(A \vee B) = \max(\mathrm{val}_{\mathcal{I}}(A), \mathrm{val}_{\mathcal{I}}(B)).$$

To assist a concise formulation of the semantics of quantifiers we define the *distribution* of a formula A and a free variable x with respect to an interpretation $\mathcal{I}$ as $\mathrm{Distr}_{\mathcal{I}}(A(x)) = \{\mathrm{val}_{\mathcal{I}'}(A(x)) \mid \mathcal{I}' \sim_x \mathcal{I}\}$, where $\mathcal{I}' \sim_x \mathcal{I}$ means that $\mathcal{I}'$ is exactly as $\mathcal{I}$ with the possible exception of the domain element assigned to x. The semantics of quantifiers is given by the infimum and supremum of the corresponding distribution:

$$\mathrm{val}_{\mathcal{I}}((\forall x)A(x)) = \inf \mathrm{Distr}_{\mathcal{I}}(A(x)),$$

$$\mathrm{val}_{\mathcal{I}}((\exists x)A(x)) = \sup \mathrm{Distr}_{\mathcal{I}}(A(x)).$$

A formula A is a *tautology* iff for all $\mathrm{val}_{\mathcal{I}}$, $\mathrm{val}_{\mathcal{I}}(A) = 1$. Moreover A is a *logical consequence* of a set of formulas Γ (in symbols $\Gamma \models_{IF} A$) iff, for all $\mathrm{val}_{\mathcal{I}}$, $\min\{\mathrm{val}_{\mathcal{I}}(\gamma) \mid \gamma \in \Gamma\} \leq \mathrm{val}_{\mathcal{I}}(A)$. Note that, for Gödel logics, this condition is equivalent to the alternative definition: for all $\mathrm{val}_{\mathcal{I}}$, either there is a formula $\gamma \in \Gamma$ such that $\mathrm{val}_{\mathcal{I}}(\gamma) < 1$ or $\mathrm{val}_{\mathcal{I}}(A) = 1$.

Remark 1. Equivalent semantics for LC which stress the close relationship with intuitionistic logic are provided by linearly ordered Heyting algebras [13] and by the class of all rooted linearly ordered Kripke models [9].

A Hilbert style calculus for LC is obtained by extending the *intuitionistic propositional calculus*

I1	$A \supset (B \supset A)$	I6	$B \supset (A \vee B)$
I2	$(A \wedge B) \supset A$	I7	$(A \wedge \neg A) \supset B$
I3	$(A \wedge B) \supset B$	I8	$(A \supset \neg A) \supset \neg A$
I4	$A \supset (B \supset (A \wedge B))$	I9	$\bot \supset A$
I5	$A \supset (A \vee B)$	I10	$A \supset \top$

I11 $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$
I12 $((A \supset C) \wedge (B \supset C)) \supset ((A \vee B) \supset C)$

with the following axiom (see [9]) which expresses the linearity of the ordering of truth-values or linear Kripke models, respectively:

$$(\textit{Linearity}) \quad (A \supset B) \ \vee \ (B \supset A).$$

A Hilbert style calculus for IF is obtained by adding to the above calculus the following axioms and rules for quantifiers:

$$\forall x A(x) \supset A(t) \qquad\qquad A(t) \supset \exists x A(x)$$

$$\vdash B \supset A(x) \Longrightarrow \vdash B \supset \forall x A(x)$$

$$\vdash A(x) \supset B \Longrightarrow \vdash \exists x A(x) \supset B$$

together with axiom:

$$(\forall\text{-}\mathit{shift}) \quad \forall x(A(x) \lor B) \supset \forall x A(x) \lor B$$

The axioms and rules for quantifiers are subject to the usual variable conditions: t is free for x and x does not occur free in B.

Remark 2. The above Hilbert style axiomatization for *IF* was introduced by Horn [13]. Takeuti and Titani axiomatized *IF* in a different way, see [18]. Indeed they added to the sequent calculus for first-order intuitionistic logic a number of extra axioms as well as the following inference rule:

$$\frac{\Gamma \Rightarrow A \lor (C \supset p) \lor (p \supset B)}{\Gamma \Rightarrow A \lor (C \supset B)}$$

where p is a propositional variable that does not occur in the lower sequent. This rule expresses the fact that the set of truth-values is densely ordered.

3 A Hypersequent Calculus for *IF*

In this section we describe Avron's hypersequent calculus **GLC** for *LC* and its first-order version $\mathbf{GLC}^{\forall\exists}$.

Definition 3. *A* hypersequent *is a multiset*

$$\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \cdots \mid \Gamma_n \Rightarrow \Delta_n$$

where every $\Gamma_i \Rightarrow \Delta_i$ is an ordinary sequent, called component *of the hypersequent. We say that a hypersequent is* single-conclusioned *if for all $i = 1, \ldots, n$, Δ_i consists of at most one formula.*

The intended interpretation of the symbol " | " is disjunctive.

Henceforth we will only consider single-conclusioned hypersequents. As a matter of convenience, we follow [17] in assuming the left hand sides of sequents to be *sets* instead of *sequences* of formulas.

Like in ordinary sequent calculi, in a hypersequent calculus there are axioms and two types of rules: *logical rules* and *structural rules*. The logical rules are essentially the same as those in sequent calculi, the only difference being the presence of dummy contexts G and G' called *side hypersequents* which are used as meta-variables for (possibly empty) hypersequents. The structural rules are divided into *internal* and *external rules*. The former deal with formulas within components. If they are present, they are the same as in ordinary sequent calculi. The external rules manipulate whole components within a hypersequent. By the presence of these structural rules, *multiplicative* and *additive* versions of the rules are interderivable (see, e.g., [19] for this terminology). The connection to natural deduction as presented in Section 4, however, is more direct using additive logical rules with multiplicative side hypersequents.

We first display a hypersequent calculus for propositional intuitionistic logic. Note that the underlying sequent calculus is a variant of Gentzen's calculus **LJ**.

Axioms

$$(id)\quad A \Rightarrow A \qquad (\bot)\quad \bot \Rightarrow A$$

Cut Rule

$$\frac{G \mid \Gamma \Rightarrow A \quad G' \mid A, \Gamma \Rightarrow C}{G \mid G' \mid \Gamma \Rightarrow C}\ (cut)$$

Internal Structural Rule

$$\frac{G \mid \Gamma \Rightarrow C}{G \mid \Gamma, B \Rightarrow C}\ (w, l)$$

External Structural Rules

$$\frac{G \mid \Gamma \Rightarrow C}{G \mid \Gamma \Rightarrow C \mid \Gamma' \Rightarrow D}\ (EW) \qquad \frac{G \mid \Gamma \Rightarrow C \mid \Gamma \Rightarrow C}{G \mid \Gamma \Rightarrow C}\ (EC)$$

Logical Rules

$$\frac{G \mid \Gamma \Rightarrow A \quad G' \mid \Gamma, B \Rightarrow C}{G \mid G' \mid \Gamma, A \supset B \Rightarrow C}\ (\supset, l) \qquad \frac{G \mid \Gamma, A \Rightarrow B}{G \mid \Gamma \Rightarrow A \supset B}\ (\supset, r)$$

$$\frac{G \mid \Gamma, A \Rightarrow C \quad G' \mid \Gamma, B \Rightarrow C}{G \mid G' \mid \Gamma, A \vee B \Rightarrow C}\ (\vee, l) \qquad \frac{G \mid \Gamma \Rightarrow A_i}{G \mid \Gamma \Rightarrow A_1 \vee A_2}\ (\vee_i, r)_{i=1,2}$$

$$\frac{G \mid \Gamma, A_i \Rightarrow C}{G \mid \Gamma, A_1 \wedge A_2 \Rightarrow C}\ (\wedge_i, l)_{i=1,2} \qquad \frac{G \mid \Gamma \Rightarrow A \quad G' \mid \Gamma \Rightarrow B}{G \mid G' \mid \Gamma \Rightarrow A \wedge B}\ (\wedge, r)$$

Remark 4. The "hyperstructure" of the above calculus is redundant, in the sense that a hypersequent $\Gamma_1 \vdash A_1 | \ldots | \Gamma_k \vdash A_k$ is derivable iff for some $i \in \{1, \ldots, k\}$, $\Gamma_i \vdash A_i$ is derivable.

In hypersequent calculi it is possible to define further structural rules which simultaneously act on several components of one or more hypersequents. It is this type of rule which increases the expressive power of hypersequent calculi compared to ordinary sequent calculi.

Avron's calculus **GLC** for LC is obtained by adding to the above hypersequent calculus for intuitionistic logic the following two rules (see [2]):

$$\frac{G \mid \Pi, \Gamma \Rightarrow A}{G \mid \Pi \Rightarrow A \mid \Gamma \Rightarrow A}\ (S_I) \quad \text{and} \quad \frac{G \mid \Gamma_1 \Rightarrow A \quad G' \mid \Gamma_2 \Rightarrow B}{G \mid G' \mid \Gamma_2 \Rightarrow A \mid \Gamma_1 \Rightarrow B}\ (com)$$

Remark 5. As pointed out by Avron, a hypersequent can be thought of as a specification of a multi-process. With respect to this interpretation the (*com*) rule models the exchange of information within multi-processes.

Alternative cut-free hypersequent calculi for LC have been presented in [2] (see also [3]). These calculi are obtained by extending the hypersequent calculus for intuitionistic logic with either the rule

$$\frac{G \mid \Gamma_1, \Gamma_2 \Rightarrow A \quad G \mid \Gamma_1, \Gamma_2 \Rightarrow B}{G \mid \Gamma_1 \Rightarrow A \mid \Gamma_2 \Rightarrow B} \ (com')$$

or

$$\frac{G \mid \Gamma_1, \Gamma_1' \Rightarrow A \quad G \mid \Gamma_2, \Gamma_2' \Rightarrow B}{G \mid \Gamma_1, \Gamma_2 \Rightarrow A \mid \Gamma_1', \Gamma_2' \Rightarrow B} \ (com'')$$

It is not hard to show that all these calculi are equivalent.

As an example, we display a proof of the linearity axiom in **GLC**. Recall that this axiom is not valid intuitionistic logic.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{A \Rightarrow A \quad B \Rightarrow B}{A \Rightarrow B \mid B \Rightarrow A}\,(com)}{A \Rightarrow B \mid \ \Rightarrow B \supset A}\,(\supset, r)}{\Rightarrow A \supset B \mid \ \Rightarrow B \supset A}\,(\supset, r)}{\Rightarrow A \supset B \mid \ \Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee_2, r)}{\Rightarrow (A \supset B) \vee (B \supset A) \mid \ \Rightarrow (A \supset B) \vee (B \supset A)}\,(\vee_1, r)}{\Rightarrow (A \supset B) \vee (B \supset A)}\,(EC)$$

A cut-free calculus for IF is defined by adding to **GLC** the following rules for quantifiers (see [4])

$$\frac{G \mid \Gamma \Rightarrow F(x)}{G \mid \Gamma \Rightarrow \forall x F(x)} \ (\forall, r) \qquad \frac{G \mid \Gamma, F(t) \Rightarrow C}{G \mid \Gamma, \forall x F(x) \Rightarrow C} \ (\forall, l)$$

$$\frac{G \mid \Gamma \Rightarrow F(t)}{G \mid \Gamma \Rightarrow \exists x F(x)} \ (\exists, r) \qquad \frac{G \mid \Gamma, F(x) \Rightarrow C}{G \mid \Gamma, \exists x F(x) \Rightarrow C} \ (\exists, l)$$

where x does not occur as a free variable in the lower hypersequents of $(\forall, r)$ and $(\exists, l)$.

Henceforth we shall refer to this calculus as $\mathbf{GLC}^{\forall\exists}$.

Theorem 6. *[7]* $\mathbf{GLC}^{\forall\exists}$ *is sound and complete for IF.*

Theorem 7 (Cut-elimination). *[4, 7] If a hypersequent H is derivable in* $\mathbf{GLC}^{\forall\exists}$ *then it is derivable in* $\mathbf{GLC}^{\forall\exists}$ *without using the cut rule.*

4 Natural Deduction

In his seminal paper [10] Gentzen formulated the system **NJ** of natural deduction for intuitionistic logic. In correspondence with the sequent calculus **LJ** for intuitionistic logic, where the right hand side of a sequent is restricted to at most one formula, **NJ** deals with inference patterns ("Schlußweisen") of *one* conclusion from a set of assumptions. At the application of rules, assumptions of a certain form can be *cancelled* in parts of the proof. A proof of a formula is a derivation tree where all assumptions have been cancelled. A natural deduction system **NK** for classical logic is obtained by either adding to **NJ** some rule or axiom corresponding to *tertium non datur*, or by dropping the restriction to one formula in the conclusion and allowing multi-conclusions, i.e., sets of formulas as nodes of a derivation tree, instead. In [5] it has been shown how to generalize the multi-conclusion version of **NK** to natural deduction systems for each finite-valued logic.

Here, we first describe a version of natural deduction for intuitionistic logic called **NJ′**. **NJ′** is equivalent to the Gentzen's original system **NJ**, but more perspicuously related to the sequent calculus **LJ** than **NJ**. In a second step we build on **NJ′** to define a system of natural deduction **NIF** that is sound and complete for *IF*. **NIF** is defined by suitably translating $\mathbf{GLC}^{\forall\exists}$.

4.1 An alternative natural deduction system for intuitionistic logic

As usual, a *derivation* in **NJ′** consists in an upward rooted tree, where the inner nodes are formulas and the leaf nodes are multisets of formulas called *assumptions*. We say that A is derivable from assumptions Γ and write

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array}$$

if there is a derivation ν with root A such that the union of all assumptions of ν is a subset of the set Γ.

Derivations in **NJ′** are generated inductively as follows:

1. *Initial derivations* are of the form:

$$\frac{\alpha}{A}$$

   where A is any formula and α (i.e., the assumptions of this derivation) is any multiset of formulas containing A or $\bot$.

2. New derivations can be generated from given ones using the following rules for the *elimination* and *introduction* of *connectives*:

$$\dfrac{\begin{array}{cc}\mathit{\Gamma} & \mathit{\Gamma}\\ \vdots & \vdots\\ A & B\end{array}}{A \wedge B}\,(i\wedge) \qquad\qquad \dfrac{\begin{array}{cc}\mathit{\Gamma} & \mathit{\Gamma},[A_j]\\ \vdots & \vdots\\ A_1 \wedge A_2 & C\end{array}}{C}\,(e\wedge_j)\ j=1,2$$

$$\dfrac{\begin{array}{c}\mathit{\Gamma}\\ \vdots\\ A_j\end{array}}{A_1 \vee A_2}\,(i\vee_j)\ j=1,2 \qquad\qquad \dfrac{\begin{array}{ccc}\mathit{\Gamma} & \mathit{\Gamma},[A] & \mathit{\Gamma},[B]\\ \vdots & \vdots & \vdots\\ A \vee B & C & C\end{array}}{C}\,(e\vee)$$

$$\dfrac{\begin{array}{c}\mathit{\Gamma},[A]\\ \vdots\\ B\end{array}}{A \supset B}\,(i\supset) \qquad\qquad \dfrac{\begin{array}{ccc}\mathit{\Gamma} & \mathit{\Gamma} & \mathit{\Gamma},[B]\\ \vdots & \vdots & \vdots\\ A \supset B & A & C\end{array}}{C}\,(e\supset)$$

$$\dfrac{\bot}{A}(e\bot)$$

The application of a rule is to be understood as follows: The formula below the line is appended as a new root to the derivations rooted in the formulas indicated above the line. In writing

$$\begin{array}{c}\mathit{\Gamma},[F]\\ \vdots\\ G\end{array}$$

for the premise of a rule we mean that the formula F may be (but need not be) *cancelled* from some or all the assumptions (leaf nodes) of the derivation of G from $\mathit{\Gamma} \cup \{F\}$ when applying the rule. Cancelled formulas are marked as such in the leaf nodes and cannot be used anymore in later applications of rules.

3. The *elimination* and *introduction* rules of $\mathbf{NJ}'$ for *quantifiers* are:

$$\dfrac{\begin{array}{c}\mathit{\Gamma}\\ \vdots\\ A(x)\end{array}}{\forall x A(x)}\,(i\forall) \qquad\qquad \dfrac{\begin{array}{cc}\mathit{\Gamma} & \mathit{\Gamma},[A(t)]\\ \vdots & \vdots\\ \forall x A(x) & C\end{array}}{C}\,(e\forall)$$

where in $(i\forall)$ x neither occurs free in any assumption that $A(x)$ depends on, nor free in $\forall x A(x)$. Moreover:

$$\dfrac{\begin{array}{c}\mathit{\Gamma}\\ \vdots\\ A(t)\end{array}}{\exists x A(x)}\,(i\exists) \qquad\qquad \dfrac{\begin{array}{cc}\mathit{\Gamma} & \mathit{\Gamma},[A(x)]\\ \vdots & \vdots\\ \exists x A(x) & C\end{array}}{C}\,(e\exists)$$

where in $(e\,\exists)$ x neither occurs free in any assumption that C depends on except $A(x)$ nor free in $\exists x A(x), C$.

The application of these rules is defined as for the case of connectives above.

In an elimination rule — i.e., $(e\wedge_i)$, $(e\vee)$, $(e\supset)$, $(e\,\exists)$, or $(e\,\forall)$ — the premise containing the occurrence of the connective or quantifier being eliminated is called *major premise*; all the other premise(s) are called *minor premise(s)* of the rule. In the case of $(e\supset)$, we also refer to the exhibited occurrence of A in

$$\begin{array}{c} \Gamma \\ \vdots \\ A \end{array}$$

as the *left minor premise*.

Remark 8. In order to be able to recognize effectively wellformed derivations one should not only label the inner nodes of the derivation tree with the names of the applied rules as indicated above, but also point to the formulas that are cancelled at the application of the rules. This can be achieved, e.g., by labelling the formulas in assumptions and referring to these labels in denoting applications of rules. In our context it is not necessary to make this more explicit. We rather refer to [19] and [14] for more exact definitions.

Remark 9. The above system is sound and complete for first-order intuitionistic logic. Note that the $(e\wedge_i)$, $(e\supset)$ and $(e\,\forall)$ rules differ from those given by Gentzen for **NJ**. We follow [5] in the definition of them. They are more close to the corresponding rules of **LJ** than that of **NJ**. Moreover, as we shall see, the **NJ′**-rules allow for a direct translation of cut-free **LJ**-proof into normal derivations. This transfers also to the level of hypersequents and hyperderivations, respectively, as shown in Theorem 19, below.

Is easy to check that **NJ** and **NJ′** are equivalent. For instance, the usual version for the elimination rules for the conjunction connective are

$$\dfrac{\begin{array}{c} \Gamma \\ \vdots \\ A_1 \wedge A_2 \end{array}}{A_1}\,(e\wedge_1)' \qquad \text{and} \qquad \dfrac{\begin{array}{c} \Gamma \\ \vdots \\ A_1 \wedge A_2 \end{array}}{A_2}\,(e\wedge_2)'$$

These rules are obtained from $(e\wedge_1)$ and $(e\wedge_2)$ by simply instantiating C with A_1 and A_2, respectively, and disregarding the redundant minor premise.

On the other hand, an application of $(e\wedge_i)$ can be replaced by an application of $(e\wedge_i)'$ by substituting each assumption (i.e., each leaf node) in

$$\begin{array}{c} \Gamma \cup \{A_i\} \\ \vdots \\ C \end{array}$$

that contains A_i, by an instance of a derivation

$$\dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A_1 \wedge A_2\end{array}}{A_i}\,(e\wedge_i)'$$

The equivalence of the rules $(e \supset)$ and $(e\,\forall)$ to the corresponding formulations in **NJ** is handled similarly.

Remark 10. As usual, in $\mathbf{NJ'}$ we have no rules corresponding to the (w, l) rule of $\mathbf{GLC}^{\forall\exists}$. This is in accordance to the traditional understanding of natural deduction: A derivation of a formula A from assumptions Γ is, by definition, also a derivation of A from assumptions *containing* Γ.

As usual in natural deduction we are interested in *normal proofs*; i.e., proofs that do not contain redundancies corresponding to cuts and enjoying a properly defined subformula property (see [14, 19]).

Definition 11. *A* maximum segment *or* cut sequence *in a derivation ν of* $\mathbf{NJ'}$ *is a sequence of consecutive occurrences $A^1, \ldots, A^n$ of a formula A in ν such that A^1 is the conclusion of an introduction rule and A^n is the major premise of an elimination rule.*

Note that while in **NJ**, for $i < n$, A^i can only be a minor premise of an elimination rule for either $\vee$ or $\exists$, in $\mathbf{NJ'}$ it can also be a minor premise of the rules $(e\wedge_i)$, $(e \supset)$ or $(e\,\forall)$.

Definition 12. *An* $\mathbf{NJ'}$*-derivation ν is called* normal *when it contains no maximum segment.*

It is not hard to see that both the proof transformations described in Remark 9 — that from **NJ** to $\mathbf{NJ'}$ and that from $\mathbf{NJ'}$ to **NJ** — preserve normality.

To define paths in $\mathbf{NJ'}$, we suitably adapt the notion of *tracks* in [19].

Definition 13. *A* path *in a derivation ν of* $\mathbf{NJ'}$ *is a sequence of occurrences of formulas $A_0, \ldots, A_n$ such that*

1. *A_0 is a formula in an assumption (leaf node) of ν that is not cancelled by an application of an elimination rule in ν;*
2. *A_i for $i < n$ is not the left minor premise of an instance of $(e \supset)$, and* either
   (a) *A_i is not the major premise of an application of an elimination rule and A_{i+1} is the conclusion for which A_i is a premise,* or
   (b) *A_i is the major premise of an application of an elimination rule, and A_{i+1} is a formula in an assumption cancelled by this rule application;*
3. *A_n is* either
   (a) *the left minor premise of an instance of $(e \supset)$,* or
   (b) *the conclusion of ν,* or

(c) *the major premise of an application of an elimination rule, and no formulas are cancelled by this rule application.*

Definition 14. *A* path of order 0 *in a normal derivation* ν *of* $\mathbf{NJ'}$ *is a path ending in a conclusion of* ν. *A* path of order $n+1$ *is a path ending in the left minor premise of* $(e \supset)$, *with major premise belonging to a path of order* n.

Theorem 15 (Subformula Property). *Let* ν *be a normal derivation in* $\mathbf{NJ'}$ *of* A *from* Γ. *Then each formula in* ν *is a subformula of a formula in* $\Gamma \cup \{A\}$.

Proof. By an easy induction on the order of paths in $\mathbf{NJ'}$.

4.2 Extending natural deduction to the hyper-level

In this section we extend $\mathbf{NJ'}$ to a natural deduction system for *IF* in analogy to the extension of **LJ** to $\mathbf{GLC}^{\forall\exists}$. To this aim we have to generalize natural deduction derivations to operate on *multisets* of derivation trees called *hyperderivations*. We call the resulting calculus **NIF**. We use the symbol $*$ to separate the components (i.e., the derivation trees) which a hyperderivation consists of.

Naturally, the definition of **NIF**-hyperderivations refers to that for $\mathbf{NJ'}$-derivations above. In fact, as long as no rules corresponding to the external structural rules of $\mathbf{GLC}^{\forall\exists}$ is applied, the constructed hyperderivation will consist of a single component which is an ordinary $\mathbf{NJ'}$-derivation.

1. Initial $\mathbf{NJ'}$-derivations are also initial **NIF**-hyperderivations.
2. The rules for introducing and eliminating connectives and quantifiers are the same as for $\mathbf{NJ'}$. However, their interpretation is slightly changed in the context of **NIF**. Applying, e.g., the rule $(e\,\exists)$ here means that the two hyperderivations $N_1 * \nu_1$ and $N_2 * \nu_2$ are replaced by the hyperderivation $N_1 * N_2 * \nu$, where N_1 and N_2 denote arbitrary *side hyperderivations* and ν denotes the derivation resulting from the application of $(e\,\exists)$ to ν_1 and ν_2, where the eigenvariable conditions refer to the whole hyperderivation. (Analogously for all the other rules of $\mathbf{NJ'}$).
3. The additional rules of **NIF** correspond to the external structural rules (EW), (EC), (S_I), and (com) of $\mathbf{GLC}^{\forall\exists}$. In their formulation the double bar shall indicate that the upper hyperderivation is *replaced* by the lower one. Let N and N' be metavariables for hyperderivations.
   (a) For *external weakening* we have the following rule:

$$\frac{N}{N * \dfrac{\Gamma}{A}}\ (EW)$$

   Observe that the new component is a tree with exactly two nodes: a leaf node consisting in a finite multiset of formulas Γ and a root node consisting in the formula A.

(b) For *external contraction* we have:

$$
\dfrac{N \;*\; \begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \;*\; \begin{array}{c}\Gamma'\\ \vdots\\ A\end{array}}{N \;*\; \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \quad \begin{array}{c}\Gamma'\\ \vdots\\ A\end{array}}{A}\;(EC)}
$$

Applying this rule means the following: Given a hyperderivation in which there are two components that end with occurrences of the same formula A we combine these two components into one by appending another occurrence of A as a common root node to the components.

(c) In applying the *splitting* rule

$$
\dfrac{N \;*\; \begin{array}{c}\Gamma, \Gamma'\\ \vdots\\ A\end{array}}{N \;*\; \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{A}\;(S_I) \;*\; \dfrac{\begin{array}{c}\Gamma'\\ \vdots\\ A\end{array}}{A}\;(S_I)}
$$

the set of formulas occurring in leaf nodes of the indicated component of the upper hyperderivation is split into two (not necessarily disjoint) subsets Γ and Γ'. The application itself can be described in two steps. First, the indicated component of the upper derivation is duplicated. Second, we remove formulas not occurring in Γ' from the assumptions of the first copy, and formulas not occurring in Γ from the assumptions of the second copy of the original component, respectively.

(d) In applying the *communication* rule

$$
\dfrac{N \;*\; \begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad N' \;*\; \begin{array}{c}\Gamma'\\ \vdots\\ B\end{array}}{N \;*\; N' \;*\; \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array}}{B}\;(com) \;*\; \dfrac{\begin{array}{c}\Gamma'\\ \vdots\\ B\end{array}}{A}\;(com)}
$$

two hyperderivations — one containing a derivation of A from Γ, the other one containing a derivation of B from Γ' — are replaced by a hyperderivation consisting of the two indicated components with their conclusions interchanged, together with the union of the respective side hyperderivations.

As an example we give a proof of the linearity axiom in **NIF**. It starts by taking the two initial derivations:

$$\frac{\{A\}}{A} \qquad \text{and} \qquad \frac{\{B\}}{B}$$

and applying (*com*) to obtain the hyperderivation

$$\dfrac{\dfrac{\{A\}}{A}}{B}\,(com) \;\; * \;\; \dfrac{\dfrac{\{B\}}{B}}{A}\,(com)$$

In both components we now apply $(i \supset)$, thereby discarding the assumptions:

$$\dfrac{\dfrac{\dfrac{\{[A]\}}{A}}{B}\,(com)}{A \supset B}\,(i \supset) \;\; * \;\; \dfrac{\dfrac{\dfrac{\{[B]\}}{B}}{A}\,(com)}{B \supset A}\,(i \supset)$$

We then introduce $\vee$ in both components and obtain

$$\dfrac{\dfrac{\dfrac{\dfrac{\{[A]\}}{A}}{B}\,(com)}{A \supset B}\,(i \supset)}{(A \supset B) \vee (B \supset A)}\,(i \vee_1) \;\; * \;\; \dfrac{\dfrac{\dfrac{\dfrac{\{[B]\}}{B}}{A}\,(com)}{B \supset A}\,(i \supset)}{(A \supset B) \vee (B \supset A)}\,(i \vee_2)$$

Since the two components end with the same formula we can apply external contraction (*EC*) to obtain the following proof (i.e., a hyperderivation with a single component in which all assumptions are discarded).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\{[A]\}}{A}}{B}\,(com)}{A \supset B}\,(i \supset)}{(A \supset B) \vee (B \supset A)}\,(i \vee_1) \quad \dfrac{\dfrac{\dfrac{\dfrac{\{[B]\}}{B}}{A}\,(com)}{B \supset A}\,(i \supset)}{(A \supset B) \vee (B \supset A)}\,(i \vee_2)}{(A \supset B) \vee (B \supset A)}\,(EC)$$

Remark 16. The rules (S_I) and (*com*) allow to move assumptions and formulas, respectively, between different components. Thus the components of a hyperderivation need not be **NJ′**-derivations generated according to the rules of Section 4.1.

A similar remark holds for the definition of a *maximum segment*. In the communication rule, as formulated above, the lower occurrence of B is *not* to be understood as "consecutive" to the upper occurrence of A but rather is defined to be consecutive to the upper occurrence of B. (The same also holds for A and B interchanged, of course.) Given this modification, the definition of *normality* for **NIF**-hyperderivations is the same as for **NJ′**-derivations (see Definition 12).

Definition 17. *A* path *in an* **NIF***-hyperderivation is defined exactly as a path for* **NJ′** *(Definition 13) except for keeping track with the exchange of assumptions and formulas between different components. More exactly, let A be a premise of* (com)*, i.e., A is moved from one component* ν *of a hyperderivation to another component* ν' *by an application of* (com)*. Then any path containing the indicated occurrence of A crosses from* ν *to* ν' *at the position of A. I.e., the predecessor of A (occurring in* ν'*, by the application of* (com)*) in the path is the occurrence of the premise (possible assumption) in* ν *that was used to deduce A. Similarly, for* (S_I)*.*

Another small modification concerns external contraction. In any path $A_0, \ldots, A_n$ *the successor* A_{i+1} *of* A_i *for* $i < n$ *may also be the conclusion in the new component arising from an application of* (EC)*, where* A_i *is the conclusion of one of the combined components.*

Theorem 15 (subformula property) also holds for **NIF**.

Definition 18. *The* height *of a derivation in* $\mathbf{GLC}^{\forall\exists}$ *is the maximal number of consecutive internal nodes (i.e., inferences) in it.*

Theorem 19. *Every cut-free* $\mathbf{GLC}^{\forall\exists}$*-proof of a hypersequent*

$$\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_m \Rightarrow A_m$$

can be translated into a normal hyperderivation in **NIF** *of the form*

$$\begin{array}{ccc} \Gamma_1 & & \Gamma_m \\ \vdots & * \ \ldots \ * & \vdots \\ A_1 & & A_m \end{array}$$

Proof. We describe the construction of an appropriate hyperderivation π^* by induction on the height h of the cut-free proof π of $\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_m \Rightarrow A_m$.

If $h = 0$, then π is an axiom either of the form $A \Rightarrow A$ or of the form $\bot \Rightarrow A$. The corresponding hyperderivations π^* in **NIF** are the initial derivations of A from the assumptions $\{A\}$ and $\{\bot\}$, respectively.

For $h > 0$ we distinguish the following cases according to the last inference rule applied in π:

1. π ends with an internal weakening rule:

$$\dfrac{\begin{array}{c} \vdots\ \pi_1 \\ G \mid \Gamma \Rightarrow C \end{array}}{G \mid \Gamma, B \Rightarrow C}\ (w, l)$$

By the induction hypothesis there exists a normal hyperderivation ${\pi_1}^*$ corresponding to the $\mathbf{GLC}^{\forall\exists}$-derivation π_1 of $G \mid \Gamma \Rightarrow C$. The hyperderivation π^* is then generated by adding (everywhere) B to the assumptions in

$$\begin{array}{c} \Gamma \\ \vdots \\ C \end{array}$$

which, by definition, is a component of ${\pi_1}^*$. Obviously π^* is normal, too.

2. π ends in a right hand side rule for either a connective or a quantifier. We present the case for conjunction. The other cases are analogous.
Suppose π ends with an application of the $(\wedge r)$ rule:

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ G \mid \Gamma \Rightarrow A\end{array} \quad \begin{array}{c}\vdots\ \pi_2\\ G' \mid \Gamma \Rightarrow B\end{array}}{G \mid G' \mid \Gamma \Rightarrow A \wedge B}\ (\wedge, r)$$

By the induction hypothesis we have the normal hyperderivations ${\pi_1}^*$ and ${\pi_2}^*$ of the form

$$N_G * \begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \qquad \text{and} \qquad N_{G'} * \begin{array}{c}\Gamma\\ \vdots\\ B\end{array}$$

corresponding to π_1 and π_2, respectively. By applying the **NIF**-rule $(i\wedge)$ to these two hyperderivations we obtain π^* as

$$N_G * N_{G'} * \dfrac{\begin{array}{c}\Gamma\\ \vdots\\ A\end{array} \quad \begin{array}{c}\Gamma\\ \vdots\\ B\end{array}}{A \wedge B}\ (i\wedge)$$

Since π^* is defined by adding an introduction rule at the end of two normal derivations π^* is normal.

3. π ends in a left hand side rule for either a connective or a quantifier. Then π^* is obtained by applying the corresponding elimination rule to the hyperderivation(s) corresponding to the $\mathbf{GLC}^{\forall\exists}$-proofs of the premise(s) of this rule application. Again, we illustrate the case for conjunction.
Suppose that π is of the form

$$\dfrac{\begin{array}{c}\vdots\ \pi_1\\ G \mid \Gamma, A_1 \Rightarrow C\end{array}}{G \mid \Gamma, A_1 \wedge A_2 \Rightarrow C}\ (\wedge_1, l)$$

Then, by induction hypothesis there is a normal hyperderivation ${\pi_1}^*$ of the form

$$N_G * \begin{array}{c}\Gamma, A_1\\ \vdots\\ C\end{array}$$

By applying the rule $(e\wedge_1)$ to ${\pi_1}^*$ and the initial derivation

$$\dfrac{\{A_1 \wedge A_2\}}{A_1 \wedge A_2}$$

we obtain π^* as

$$N_G * \dfrac{\dfrac{\{A_1 \wedge A_2\}}{A_1 \wedge A_2} \qquad \begin{array}{c}\Gamma, [A_1]\\ \vdots\\ C\end{array}}{C}\ (e\wedge)$$

Since the elimination rule is applied immediately below the initial derivation, π^* is normal.

4. π ends with one of the rules: (EC), (EW), (S_I), or (com). Like in the cases above, π^* is obtained by applying the corresponding **NIF**-rule to the hyperderivation(s) that correspond to the $\mathbf{GLC}^{\forall\exists}$-proof(s) of the premise(s) of this application of the rule. It is easy to see that π^* is normal.

Theorem 20. *Every hyperderivation ν in* **NIF**

$$\begin{array}{ccccc} \Gamma_1 & & & & \Gamma_m \\ \vdots & * & \ldots & * & \vdots \\ A_1 & & & & A_m \end{array}$$

can be translated into a $\mathbf{GLC}^{\forall\exists}$-proof $\sigma(\nu)$ of the hypersequent

$$\Gamma_1 \Rightarrow A_1 \mid \ldots \mid \Gamma_m \Rightarrow A_m$$

Proof. We define $\sigma(\nu)$ by translating each inference step of ν into a corresponding inference of $\mathbf{GLC}^{\forall\exists}$. More exactly, the proof is by induction on the number of inference steps applied to construct the derivation ν.

1. If ν is an initial derivation
$$\frac{\alpha}{A}$$
then either A or $\bot$ occurs in α. In the first case $\sigma(\nu)$ starts with axiom $A \Rightarrow A$; in the second case with axiom $\bot \Rightarrow A$. The other formulas of α are then added to the left hand side of these sequents by internal weakenings.
2. If the last inference step in deriving ν is the application of an introduction rule (for either a connective or a quantifier). Then $\sigma(\nu)$ simply ends with the corresponding right hand side rule in $\mathbf{GLC}^{\forall\exists}$.
3. If the last rule applied to derive ν is an elimination rule we combine the corresponding left hand side rule of $\mathbf{GLC}^{\forall\exists}$ with an application of cut to define $\sigma(\nu)$. We illustrate this for the case for implication. The other cases are analogous. ν is of the form

$$\dfrac{\begin{array}{c}\Gamma \\ \vdots\, \nu_1 \\ A \supset B\end{array} \quad \begin{array}{c}\Gamma \\ \vdots\, \nu_2 \\ A\end{array} \quad \begin{array}{c}\Gamma, [B] \\ \vdots\, \nu_3 \\ C\end{array}}{C}\,(e\supset)$$

The corresponding derivation $\sigma(\nu)$ in $\mathbf{GLC}^{\forall\exists}$ is

$$\dfrac{\begin{array}{c}\vdots\, \sigma(\nu_1) \\ \Gamma \Rightarrow A \supset B\end{array} \quad \dfrac{\begin{array}{c}\vdots\, \sigma(\nu_2) \\ \Gamma \Rightarrow A\end{array} \quad \begin{array}{c}\vdots\, \sigma(\nu_3) \\ \Gamma, B \Rightarrow C\end{array}}{\Gamma, A \supset B \Rightarrow C}\,(\supset, l)}{\Gamma \Rightarrow C}\,(cut)$$

4. If the last rule applied to derive ν is (EC), (EW), (S_I) or (com), then $\sigma(\nu)$ simply ends with the corresponding external structural rule in $\mathbf{GLC}^{\forall\exists}$.

Corollary 21. **NIF** *is sound and complete for IF.*

Proof. Since $\mathbf{GLC}^{\forall\exists}$ is sound and complete for *IF*, the claim follows from Theorems 19 and 20.

Corollary 22 (Normal Form Property). *For every derivation in* **NIF***, there exists a normal derivation in* **NIF** *of the same formula from the same assumptions.*

Proof. If there exists a derivation in **NIF** of A from assumptions Γ, then, by Theorem 20 there exists a proof in $\mathbf{GLC}^{\forall\exists}$ of $\Gamma \Rightarrow A$. By Theorem 7 there exists also a cut-free proof in in $\mathbf{GLC}^{\forall\exists}$ of $\Gamma \Rightarrow A$ whose translation according to Theorem 19 yields a normal derivation of A from assumptions Γ.

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