On Urquhart's C Logic

Agata Ciabattoni
Dipartimento di Informatica
Via Comelico, 39 – 20135 Milano, Italy
ciabatto@dsi.unimi.it

Abstract

In this paper we investigate the basic many-valued logics introduced by Urquhart in [15] and [16], here referred to as $\mathbf{C}$ *and* $\mathbf{C}_{new}$*, respectively. We define a cut-free hypersequent calculus for* $\mathbf{C}_{new}$ *and show the following results: (1)* $\mathbf{C}$ *and* $\mathbf{C}_{new}$ *are distinct versions of Gödel logic without contraction. (2)* $\mathbf{C}_{new}$ *is decidable. (3) In* $\mathbf{C}_{new}$ *the family of axioms* $((A^k \to C) \wedge (B^k \to C)) \to ((A \vee B)^k \to C)$*, with* $k \geq 2$*, is in fact redundant.*

1 Introduction

The logic $\mathbf{C}$ was introduced by Urquhart in the chapter devoted to many-valued logic of the Handbook of Philosophical Logic [15].

$\mathbf{C}$ turns out to be a *basic many-valued logic* being contained in the most important formalizations of fuzzy logic [7], namely infinite-valued Gödel, Łukasiewicz and product logic (see [3]).

In [9, 10] $\mathbf{C}$ was shown to be a particular Gödel logic without contraction. A cut-free calculus for $\mathbf{C}$ was defined in [3]. This calculus uses hypersequents that are a natural generalization of Gentzen sequents.

Due to semantical motivations, in the second edition of his handbook paper [16], Urquhart presented a new version of $\mathbf{C}$. We shall refer to this logic as $\mathbf{C}_{new}$. Semantically, $\mathbf{C}_{new}$ is characterized by model structures on ordered abelian monoids and is therefore contained in Łukasiewicz logic which is characterized by ordered Abelian groups [4, 16].

In this paper we introduce a cut-free hypersequent calculus for $\mathbf{C}_{new}$. As a corollary, we prove that $\mathbf{C}_{new}$ is decidable. Moreover, for each $k \geq 2$, we show the redundancy in $\mathbf{C}_{new}$ of the axiom $((A^k \to C) \wedge (B^k \to C)) \to ((A \vee B)^k \to C))$, where $A^k \to C$ stands for $\underbrace{A \to (A \to (\ldots (A}_{k \text{ times}} \to C) \ldots))$.

In order to define a calculus for $\mathbf{C}_{new}$ we shall consider three different contraction-free versions of Avron's hypersequent calculus for Gödel logic [1] and we shall relate them with their corresponding logics, here after named $\mathbf{C}^+$, $\mathbf{C}^*$ and $\mathbf{C}^\#$. $\mathbf{C}^+$ and $\mathbf{C}^*$, already investigated in [3], are respectively Urquhart's logic $\mathbf{C}$ and its residuated version. As we shall show, $\mathbf{C}^\#$ turns out to coincide with $\mathbf{C}_{new}$.

2 Urquhart's logics C and $\mathbf{C}_{new}$

Urquhart's basic many-valued logic $\mathbf{C}$ consists of the following axioms (see [15]):

Ax1. $A \to (B \to A)$

Ax2. $(A \to B) \to [(C \to A) \to (C \to B)]$

Ax3. $[A \to (C \to B)] \to [C \to (A \to B)]$

Ax4. $(A \wedge B) \to A$

Ax5. $(A \wedge B) \to B$

Ax6. $A \to [C \to (A \wedge C)]$

Ax7. $A \to (A \vee B)$

Ax8. $B \to (A \vee B)$

Ax9. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

Lin. $(A \to B) \vee (B \to A)$

together with the rule of modus ponens.

In [16] Urquhart introduced a new version of $\mathbf{C}$. We shall refer to this logic as $\mathbf{C}_{new}$. Semantically, $\mathbf{C}_{new}$ is characterized by model structures on ordered commutative monoids.

More precisely, let $\mathcal{A} = (A, +, 0)$ be a (totally) ordered commutative monoid and $\leq$ a total ordering of A satisfying the condition $x \leq y$ implies $x+z \leq y+z$ for all $x, y, z \in A$. Let A_0 be the set $\{x \in A \mid x \geq 0\}$.

Definition 2.1 ([16]) *A model structure over* $\mathcal{A}$ *consists of a subset* $[P] \subseteq A_0$ *for each atomic formula* P *in the language. The sets* $[P]$ *are required to be increasing (a subset* $B \subseteq A$ *is increasing if* $x \in B$ *and* $x \leq y$ *imply that* $y \in B$*). Given a model structure over* $\mathcal{A}$*, we define* truth at a point x *in* A_0 *by the following inductive stipulation:*

1. $x \models P$ *iff* $x \in [P]$;

2. $x \models (\phi \wedge \psi)$ *iff* $x \models \phi$ *and* $x \models \psi$;

3. $x \models (\phi \vee \psi)$ *iff either* $x \models \phi$ *or* $x \models \psi$;

4. $x \models (\phi \rightarrow \psi)$ *iff, for all* $y \in A_0$, $y \models \phi$ *implies* $x + y \models \psi$.

A formula Φ *is* valid *iff* $0 \models \Phi$ *in every model structure over an ordered commutative monoid.*

In [16] a Hilbert-style axiomatization of $\mathbf{C}_{new}$ is defined by adding to the above axioms $\{Ax2, \ldots, Ax8, Ax9, Lin\}$ both:

U1. $((A \rightarrow B) \wedge (A \rightarrow C)) \rightarrow (A \rightarrow (B \wedge C))$

U2. $((A^k \rightarrow C) \wedge (B^k \rightarrow C)) \rightarrow ((A \vee B)^k \rightarrow C)$

for every $k \geq 2$, where $A^k \rightarrow C$ stands for $\underbrace{A \rightarrow (A \rightarrow (\ldots (A}_{k \text{ times}} \rightarrow C) \ldots))$.

Urquhart proved that a formula Φ is valid in the model theory of Definition 2.1 if and only if $\mathbf{C}_{new} \vdash \Phi$.

Remark 2.2 *If in Definition 2.1 one makes the assumption that the underlying algebra is an ordered abelian group, one obtains model structures characterizing infinite-valued Łukasiewicz logic [4, 16].*

To put the logics $\mathbf{C}$ and $\mathbf{C}_{new}$ into a broader context, let us introduce the following axioms:

Res1. $[(A \wedge B) \rightarrow C] \rightarrow [A \rightarrow (B \rightarrow C)]$

Res2. $[(A \rightarrow (B \rightarrow C)] \rightarrow [(A \wedge B) \rightarrow C]$

Abs. $\perp \rightarrow A$

Taking *Modus Ponens* as the only rule of derivation, we consider the logics:

$\mathbf{I}^-$	:	$\{Ax1, \ldots, Ax9, Abs\}$	$\mathbf{C}^+$	:	$\mathbf{I}^- \cup \{Lin\}$
$\mathbf{I}^{-\#}$	:	$\mathbf{I}^- \cup \{U1\}$	$\mathbf{C}^\#$	:	$\mathbf{I}^{-\#} \cup \{Lin\}$
$\mathbf{I}^{-*}$	:	$\mathbf{I}^- \cup \{Res1, Res2\}$	$\mathbf{C}^*$	:	$\mathbf{I}^{-*} \cup \{Lin\}$

$\mathbf{I}^-, \mathbf{I}^{-*}, \mathbf{C}^+$ and $\mathbf{C}^*$ have been investigated in [3, 5].

$\mathbf{I}^-$, $\mathbf{I}^{-\#}$ and $\mathbf{I}^{-*}$, are different versions of intuitionistic logic IL without contraction. Notice that *Res1* is already derivable in $\mathbf{I}^-$. In fact, this axiom turns out to be redundant in the presence of $Ax1, Ax2$ and $Ax3$. $\mathbf{I}^{-\#}$ and $\mathbf{I}^{-*}$ are incomparable and they strictly extend $\mathbf{I}^-$ (Corollary 3.13). $\mathbf{I}^{-\#}$ leads to IL by adding axiom *Res2* to the former.

$\mathbf{C}^+$ coincides with $\mathbf{C} \cup \{Abs\}$. $\mathbf{C}^*$ and $\mathbf{C}^\#$ are incomparable versions of Gödel logic without contraction (Corollary 3.17) and they strictly extend $\mathbf{C}^+$. In Section 4 we show that $\mathbf{C}^\# = \mathbf{C}_{new} \cup \{Abs\}$.

Figure 1 shows proper inclusions among all the previously mentioned logics as well as of other well known logics (Corollary 4.6).

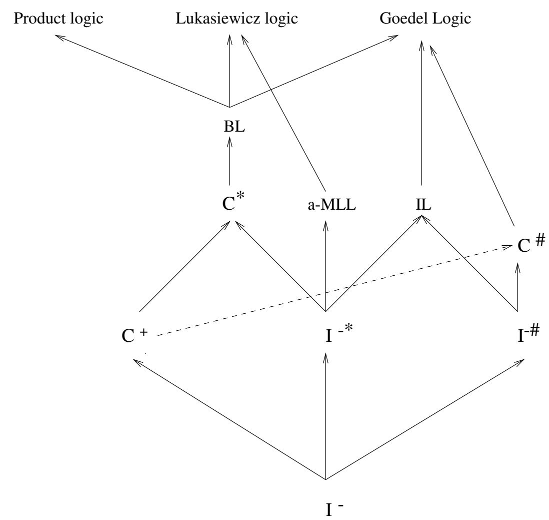

Figure 1. Inclusions among various logics

Indeed, by extending $\mathbf{I}^{-*}$ with axiom $Dneg$ (involutivity of negation), i.e., $((A \rightarrow \perp) \rightarrow \perp) \rightarrow A$ one gets the affine multiplicative fragment of linear logic a-MLL[1] [6]. As shown in [13], Łukasiewicz logic [8] is obtainable by adding to a-MLL the axiom $((A \rightarrow B) \rightarrow B) \rightarrow ((B \rightarrow A) \rightarrow A)$. Hájek's Basic Logic BL was introduced in [7] as the logical counterpart of continuous *t*-norms. As proved in [3], by adding to $\mathbf{C}^*$ the axiom $[A \wedge (A \rightarrow B)] \rightarrow [B \wedge (B \rightarrow A)]$ one gets BL. If we add to BL axiom $Dneg$ we get Łukasiewicz logic, while by extending BL with both axioms $\neg\neg A \rightarrow ((B \wedge A \rightarrow C \wedge A) \rightarrow (B \rightarrow C))$ and $\neg(A \wedge (A \rightarrow \neg A))$ we obtain product logic (see [7]).

Each of $\mathbf{C}^+, \mathbf{C}^*$, $\mathbf{C}^\#$ and BL yields Gödel logic by adding the axiom $[A \rightarrow (A \rightarrow B)] \rightarrow (A \rightarrow B)$ (contraction).

Remark 2.3 *Henceforth we shall abuse the names* $\mathbf{C}_{new}$ *and* $\mathbf{C}$ *to refer to their proper extensions* $\mathbf{C}_{new} \cup \{Abs\}$ *and* $\mathbf{C} \cup \{Abs\}$ *(i.e.,* $\mathbf{C}^+$*), respectively.*

3 Sequent and Hypersequent Calculi

In this section we define sequent (and hypersequent) calculi for $\mathbf{I}^{-\#}$ and $\mathbf{C}^\#$.

Hypersequent calculi are a simple and natural generalization of Gentzen sequent calculi. See [2] for an overview.

Definition 3.1 *A* hypersequent *is an expression of the form* $\Gamma_1 \vdash \Delta_1 | \cdots | \Gamma_n \vdash \Delta_n$*, where for all* $i = 1, \ldots n$, $\Gamma_i \vdash \Delta_i$ *is an ordinary sequent.*

The intended meaning of the symbol $|$ is disjunctive.

Hypersequent calculi are particularly useful to formalize logics containing axiom Lin (see [3, 5]).

[1] In fact, we get a-MLL extended with the additive disjunction.

In [3] cut-free calculi for the logics $\mathbf{I}^-, \mathbf{I}^{-*}, \mathbf{C}$ and $\mathbf{C}^*$ have been introduced. They are called $\mathbf{LJ}^-, \mathbf{LJ}^{-*}, \mathbf{HC}$ and $\mathbf{HC}^*$, respectively.

$\mathbf{LJ}^-$ and $\mathbf{LJ}^{-*}$ turn out to be different contraction-free fragments of the $\mathbf{LJ}$ sequent calculus for intuitionistic logic, while $\mathbf{HC}$ and $\mathbf{HC}^*$ are their hypersequent versions with Avron's communication rule [1].

More precisely, let us consider the $\mathbf{LJ}$ sequent calculus for intuitionistic logic, i.e.,

$Axioms$

$$A \Rightarrow A \qquad \bot \Rightarrow A \qquad (cut)\ \frac{\Gamma_1 \Rightarrow A \quad A, \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow B}$$

$Structural\ Rules$

$$(w)\ \frac{\Gamma \Rightarrow C}{\Gamma, A \Rightarrow C} \qquad (c)\ \frac{\Gamma, A, A \Rightarrow C}{\Gamma, A \Rightarrow C}$$

$Logical\ Rules$

$$(\to, r)\ \frac{\Gamma, A \Rightarrow B}{\Gamma \Rightarrow A \to B} \qquad (\to, l)\ \frac{\Gamma_1 \Rightarrow A \quad B, \Gamma_2 \Rightarrow C}{\Gamma_1, \Gamma_2, A \to B \Rightarrow C}$$

$$(\wedge, r)\ \frac{\Gamma_1 \Rightarrow A \quad \Gamma_2 \Rightarrow B}{\Gamma_1, \Gamma_2 \Rightarrow A \wedge B} \qquad (\wedge, l)\ \frac{\Gamma, A, B \Rightarrow C}{\Gamma, A \wedge B \Rightarrow C}$$

$$(\vee_i, r)\ \frac{\Gamma \Rightarrow A_i}{\Gamma \Rightarrow A_1 \vee A_2} \qquad (\vee, l)\ \frac{\Gamma, A \Rightarrow C \quad \Gamma, B \Rightarrow C}{\Gamma, A \vee B \Rightarrow C}$$

When investigating contraction-free fragments of $\mathbf{LJ}$, we can consider the following alternative formulations for the $(\wedge, r)$ and $(\wedge, l)$ rules:

$$(\wedge', r)\ \frac{\Gamma \Rightarrow A \quad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B} \qquad (\wedge'_i, l)\ \frac{\Gamma, A_i \Rightarrow C}{\Gamma, A_1 \wedge A_2 \Rightarrow C}$$

for $i = 1, 2$. The above rules are called *additive*, while the corresponding ones in $\mathbf{LJ}$ are called *multiplicative* (see [14]). As is well known, using (c) and (w), the additive rules are interderivable with the multiplicative ones, while in absence of (c), they are distinct.

Remark 3.2 *See [12] for a general investigation on intuitionistic logic without contraction.*

Let us consider the calculi obtained substituting the rules for conjunction in the contraction-free $\mathbf{LJ}$ calculus by one of the four possible combinations of the above additive and multiplicative rules. The calculus defined by choosing $(\wedge, l)$ and $(\wedge', r)$ as rules for conjunction shows undesirable properties: it allows to derive contraction, and cuts are not eliminable (see [14]). The calculi obtained using the pairs of rules $(\wedge'_i, l), (\wedge, r)$ and $(\wedge, l), (\wedge, r)$ are respectively $\mathbf{LJ}^-$ and $\mathbf{LJ}^{-*}$ of [3]. Henceforth we shall investigate the fourth possible system, which we shall call $\mathbf{LJ}^{-\#}$. In $\mathbf{LJ}^{-\#}$ the rules for conjunction are $(\wedge'_i, l)$ and $(\wedge', r)$ and the remaining ones are those of $\mathbf{LJ}$ *except contraction* (c).

Let us introduce the hypersequent versions of the above calculi. In $\mathbf{HC}$ and $\mathbf{HC}^*$ axioms, internal structural rules and logical rules are directly obtained from those of $\mathbf{LJ}^-$ and $\mathbf{LJ}^{-*}$, respectively: the only difference is the presence of dummy contexts G and G' representing (possibly empty) hypersequents. For instance, the rules for implication are

$$(\to, r)\ \frac{G|\Gamma, A \Rightarrow B}{G|\Gamma \Rightarrow A \to B} \qquad (\to, l)\ \frac{G|\Gamma_1 \Rightarrow A \quad G'|B, \Gamma_2 \Rightarrow C}{G|G'|\Gamma_1, \Gamma_2, A \to B \Rightarrow C}$$

Additional structural rules are external weakening (EW) and contraction (EC)

$$(EW)\ \frac{G|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta|\Gamma' \Rightarrow \Delta'} \qquad (EC)\ \frac{G|\Gamma \Rightarrow \Delta|\Gamma \Rightarrow \Delta}{G|\Gamma \Rightarrow \Delta}$$

together with Avron's communication rule [1]

$$(com)\ \frac{G|\Pi_1, \Gamma_1 \Rightarrow A \quad G'|\Pi_2, \Gamma_2 \Rightarrow B}{G|G'|\Pi_1, \Pi_2 \Rightarrow A|\Gamma_1, \Gamma_2 \Rightarrow B}$$

See [3] for a detailed description on these calculi.

The hypersequent version of $\mathbf{LJ}^{-\#}$ shall be called $\mathbf{HC}^{\#}$. In $\mathbf{HC}^{\#}$ axioms, internal structural rules and logical rules are obtained from those of $\mathbf{LJ}^{-\#}$, as seen for $\mathbf{HC}$ and $\mathbf{HC}^*$. Additional structural rules are (EW), (EC) and (com).

Remark 3.3 *By adding the* (c) *rule to* $\mathbf{HC}, \mathbf{HC}^*$ *or* $\mathbf{HC}^{\#}$ *one gets Avron's* $\mathbf{GLC}$ *calculus [1] for Gödel logic.*

3.1 Cut-elimination

In [3] it was proved that the cut-elimination theorem holds for $\mathbf{LJ}^-, \mathbf{LJ}^{-*}$, $\mathbf{HC}$ and $\mathbf{HC}^*$. This is true also for $\mathbf{HC}^{\#}$. Indeed

Theorem 3.4 $\mathbf{LJ}^{-\#}$ *admits cut-elimination.*

Proof: See, e.g., [12].

Definition 3.5 *A sequent calculus is* single-conclusion *iff in all the involved sequents* $\Gamma \Rightarrow \Delta$*, the succedent* Δ *contains at most one formula.*

Theorem 3.6 *If a single-conclusion sequent calculus admits cut-elimination then its hypersequent version with* (com) *as additional rule admits cut-elimination.*

Proof: See [3, 5].

Corollary 3.7 $\mathbf{HC}^{\#}$ *admits cut-elimination.*

Definition 3.8 *Let $A_1, \ldots, A_n \Rightarrow B$ be a sequent. Then the* generic interpretation $\mathcal{I}$ *of $A_1, \ldots, A_n \Rightarrow B$ is defined as follows:*

$$\mathcal{I}(\Rightarrow B) := B$$

$$\mathcal{I}(A_1, \ldots, A_n \Rightarrow B) := (A_1 \to \ldots \to (A_n \to B) \ldots)$$

$$\mathcal{I}(A_1, \ldots, A_n \Rightarrow) := (A_1 \to \ldots \to (A_n \to \perp) \ldots).$$

Let $S_1 | \cdots | S_n$ be a hypersequent. Its generic interpretation *is defined by*

$$\mathcal{I}(S_1 | \cdots | S_n) := \mathcal{I}(S_1) \vee \ldots \vee \mathcal{I}(S_n).$$

Definition 3.9 *Sequent (or hypersequent) rules*

$$\frac{S}{S^*} \quad \textit{and} \quad \frac{T \quad T'}{T^*}$$

are called sound *for a (Hilbert-style) calculus $\mathbf{H}$ iff $\mathbf{H} \vdash \mathcal{I}(S) \to \mathcal{I}(S^*)$ and $\mathbf{H} \vdash \mathcal{I}(T) \to (\mathcal{I}(T') \to \mathcal{I}(T^*))$, respectively. If all the rules of a sequent (or hypersequent) calculus $\mathbf{L}_{\mathcal{L}}$ are sound for $\mathbf{H}$ and $\mathbf{H} \vdash \mathcal{I}(S)$ for all the axioms S of $\mathbf{L}_{\mathcal{L}}$, then $\mathbf{L}_{\mathcal{L}}$ is said to be* sound for $\mathbf{H}$.

A sequent (or hypersequent) calculus $\mathbf{L}_{\mathcal{L}}$ is complete for $\mathbf{H}$ *iff $\mathbf{L}_{\mathcal{L}} \vdash (\Rightarrow A)$ whenever $\mathbf{H} \vdash A$.*

Corollary 3.10

1. $\mathbf{LJ}^{-\#}$ *and* $\mathbf{LJ}^{-*}$ *are proper extensions of* $\mathbf{LJ}^{-}$.
2. $\mathbf{LJ}^{-\#}$ *and* $\mathbf{LJ}^{-*}$ *are incomparable.*
3. $\mathbf{HC}^{\#}$ *is a proper extension of* $\mathbf{HC}$.
4. $\mathbf{HC}^{*}$ *is a proper extension of* $\mathbf{HC}$.
5. $\mathbf{HC}^{\#}$ *and* $\mathbf{HC}^{*}$ *are incomparable.*

Proof: *1.* It is easy to see that

$$\mathbf{LJ}^{-\#} \vdash \Rightarrow A \to A \wedge A \quad \text{and} \quad \mathbf{LJ}^{-*} \vdash \Rightarrow \textit{Res2}.$$

By the cut-free completeness of $\mathbf{LJ}^{-}$ it follows immediately that the above two axioms are not derivable in $\mathbf{LJ}^{-}$.

2. This trivially follows from the non-derivability of *Res2* in $\mathbf{LJ}^{-\#}$ and of $A \to A \wedge A$ in $\mathbf{LJ}^{-*}$.

3. It suffices to prove that the formula $A \to A \wedge A$ cannot be derived in $\mathbf{HC}$. Indeed, $\mathbf{HC} \vdash \Rightarrow A \to A \wedge A$ if and only if the hypersequent

$$\Rightarrow A | A \Rightarrow | \Rightarrow A \wedge A$$

is derivable in $\mathbf{HC}$. By Definition 3.9, this hypersequent translates to the formula P given by

$$A \vee (A \to \perp) \vee (A \wedge A)$$

which evaluates to $v(P) < 1$, for instance, in Gödel logic if $v(A) = 0.5$. Since for each hypersequent S which is derivable in $\mathbf{HC}$, $\mathcal{I}(S)$ must be valid in Gödel logic, this concludes the proof.

4. Follows by the fact that axiom *Res2* is not derivable in $\mathbf{HC}$ (see [3]).

5. It is immediate to see that

$$\mathbf{HC}^{\#} \vdash \Rightarrow A \to A \wedge A \quad \text{and} \quad \mathbf{HC}^{*} \vdash \Rightarrow \textit{Res2}.$$

However, neither $\mathbf{HC}^{*}$ derives $A \to A \wedge A$, nor $\mathbf{HC}^{\#}$ derives *Res2*. The proofs proceed as in cases *3* and *4* above, respectively.

Corollary 3.11 *Derivability in* $\mathbf{LJ}^{-\#}$ *and* $\mathbf{HC}^{\#}$ *is decidable.*

Proof: (Sketch) Proof search based decision algorithms for sequent calculi without contraction are well known (see, e.g., [11]).

In the case of hypersequents first note that by the occurrence of the external structural rules, any cut-free proof can be transformed in an equivalent one in which every hypersequent has at most two equal components. Thus a bound for the number of hypersequents occurring in the cut-free proofs, as well as for the number of the components in the involved hypersequents, can be obtained in terms of the size of the end hypersequent.

3.2 Correspondences

In [3] it was proved that the calculi $\mathbf{LJ}^{-}, \mathbf{LJ}^{-*}$, $\mathbf{HC}^{*}, \mathbf{HC}$ are sound and complete for $\mathbf{I}^{-}, \mathbf{I}^{-*}$, $\mathbf{C}^{*}$ and $\mathbf{C}$, respectively. In this section we relate the logics $\mathbf{I}^{-\#}$ and $\mathbf{C}^{\#}$, as given by their Hilbert-style axiomatizations, to the calculi $\mathbf{LJ}^{-\#}$ and $\mathbf{HC}^{\#}$.

Theorem 3.12 $\mathbf{LJ}^{-\#}$ *is sound and complete for* $\mathbf{I}^{-\#}$.

Proof: (*Soundness*) The axioms of $\mathbf{LJ}^{-\#}$ translate into $A \to A$ and $\perp \to A$, respectively. The corresponding derivation in $\mathbf{I}^{-\#}$ is straightforward.

In order to show the soundness of the rules of $\mathbf{LJ}^{-\#}$ it suffices to prove that their generic interpretations are derivable in $\mathbf{I}^{-\#}$.

As an example we consider the $(\wedge', r)$ rule. Its generic interpretation $(*)$ is: $(X_1 \to (\ldots (X_n \to A) \ldots)) \to [(X_1 \to (\ldots (X_n \to B) \ldots)) \to (X_1 \to (\ldots (X_n \to (A \wedge B) \ldots)))]$. For $n = 1$ this formula coincides with axiom $Ax6$. Thus let $n \geq 2$. In axiom $U1$ let us replace A with X_1, B with $(X_2 \to (\ldots (X_n \to A) \ldots))$ and C with $(X_2 \to (\ldots (X_n \to B) \ldots))$. Let us denote with $U1^*$ the resulting formula. By induction on n one can derive the formula $(**)$ $U1^* \to [[(X_1 \to (X_2 \to (\ldots (X_n \to$

$A)\ldots) \wedge (X_1 \to (X_2 \to (\ldots(X_n \to B)\ldots)] \to (X_1 \to \ldots(X_n \to (A \wedge B)\ldots))]$.
For the basic step $n = 1$, the derivation is trivial. For the inductive step, assuming the claim is true for $n = k$, applying axioms $Ax1, Ax2, Ax3$ together with modus ponens we can derive $(**)$ for $n = k+1$. Then $(*)$ follows by applying to $(**)$ both axioms $U1$ and $Res1$ together with modus ponens.

For the remaining rules see either [3] or Corollary 2.8.1 in [12].
(*Completeness*) Observe that Modus Ponens — the only rule of $\mathbf{I}^{-\#}$ — corresponds to the derivability of $A, A \to B \Rightarrow B$ and of the cut rule. It thus suffices to show that all the axioms of $\mathbf{I}^{-\#}$ are derivable in $\mathbf{LJ}^{-\#}$. This is straightforward.

Corollary 3.13 *The logics $\mathbf{I}^{-\#}$ and $\mathbf{I}^{-*}$ are incomparable extensions of $\mathbf{I}^{-}$.*

Proof: Immediate from soundness and completeness of the $\mathbf{LJ}^{-*}$ calculus with respect to $\mathbf{I}^{-*}$ (see [3]), together with Corollary 3.10 and Theorem 3.12.

Lemma 3.14 $\mathbf{HC}^{\#} \vdash \Rightarrow Lin$.

Proof: The derivation proceeds as follows (see [1, 3, 5]):

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow A \qquad B \Rightarrow B}{A \Rightarrow B | B \Rightarrow A}\,{\scriptstyle (\mathrm{com})}}{A \Rightarrow B | \Rightarrow B \to A}\,{\scriptstyle (\to,\mathrm{r})}}{\Rightarrow A \to B | \Rightarrow B \to A}\,{\scriptstyle (\to,\mathrm{r})}}{\Rightarrow A \to B | \Rightarrow (A \to B) \vee (B \to A)}\,{\scriptstyle (\vee,\mathrm{r})}}{\Rightarrow (A \to B) \vee (B \to A) | \Rightarrow (A \to B) \vee (B \to A)}\,{\scriptstyle (\vee,\mathrm{r})}}{\Rightarrow (A \to B) \vee (B \to A)}\,{\scriptstyle (\mathrm{EC})}$$

Theorem 3.15 $\mathbf{HC}^{\#}$ *is sound and complete for* $\mathbf{C}^{\#}$.

Proof: (*Soundness*) In the light of Theorem 3.12 we must only take care of the external structural rules and of the (com) rule. $(A \vee A) \to A$ corresponds to (EC). $A \to (A \vee B)$ corresponds to (EW). These formulas are already derivable in $\mathbf{I}^{-\#}$, and hence also in $\mathbf{C}^{\#}$. For the (com) rule, see [3].
(*Completeness*) From Theorem 3.12 and Lemma 3.14.

Corollary 3.16 $\mathbf{C}^{\#}$ *is decidable.*

Corollary 3.17 *The logics $\mathbf{C}^{\#}$ and $\mathbf{C}$ are incomparable versions of Gödel logic without contraction.*

Proof: The proof follows by soundness and completeness of the $\mathbf{HC}$ calculus with respect to $\mathbf{C}$ (see [3]), together with Theorem 3.15 and Corollary 3.10.

4 $\mathbf{C}_{new}$ coincides with $\mathbf{C}^{\#}$

In this section we prove that Urquhart's $\mathbf{C}_{new}$ logic coincides with the contraction-free fragment of Gödel logic $\mathbf{C}^{\#}$.

Proposition 4.1 $\mathbf{C}_{new} = \mathbf{C}^{\#} \cup \{U2\}$.

Proof: It is immediate to see that $\mathbf{C}_{new} \subseteq \mathbf{C}^{\#} \cup \{U2\}$. The converse directly follows from the derivability of axioms $Ax1$ in $\mathbf{C}_{new}$. Hence, $\mathbf{C}_{new} = \mathbf{C}^{\#} \cup \{U2\}$.

Proposition 4.2 *For $k \geq 2$, the family of axioms $U2$ can be derived in $\mathbf{HC}^{\#}$.*

Proof: For all $n \geq 0$, let us denote with nP the (possibly empty) set of formulas $P, \ldots, P$ (n times).

Then, for each $k \geq 2$, we have the following derivation of axiom $U2$ in $\mathbf{HC}^{\#}$:

$$\dfrac{\begin{array}{c}\dfrac{\dfrac{\vdots\ \ (*)\ \ \vdots \qquad\qquad \vdots\ \ (*)\ \ \vdots}{(A^k \to C) \wedge (B^k \to C), k(A \vee B) \Rightarrow C}\,{\scriptstyle (\vee,l)}}{}\,{\scriptstyle (\to,r)} \\ \vdots \qquad\qquad \vdots \end{array}}{\Rightarrow ((A^k \to C) \wedge (B^k \to C)) \to ((A \vee B)^k \to C)}\,{\scriptstyle (\to,r)}$$

where, all the (2^k) upper sequents in $(*)$ obtained applying the $(\vee, l)$ rule as many times as possible, have the form

$$(A^k \to C) \wedge (B^k \to C), nA, mB \Rightarrow C \text{ with } n + m = k$$

All these sequents are provable in $\mathbf{HC}^{\#}$. To see this we shall argue by cases as follows:

- If $n = k$ we proceed as follows:

$$\dfrac{\dfrac{A \Rightarrow A \qquad \dfrac{\begin{array}{c}\dfrac{C \Rightarrow C \quad A \Rightarrow A}{} \\ \vdots \qquad \vdots\end{array}}{A^{k-1} \to C, (k-1)A \Rightarrow C}\,{\scriptstyle (\to,l)}}{A^k \to C, kA \Rightarrow C}\,{\scriptstyle (\to,l)}}{(A^k \to C) \wedge (B^k \to C), kA \Rightarrow C}\,{\scriptstyle (\wedge',l)}$$

- The case $m = k$ is similar.

- If $0 < m \leq n < k$, let us denote with $[P, C, n, m]$ the sequent $P \to C, nA, mB \Rightarrow C$. We can write

$$\dfrac{A \Rightarrow A \qquad \dfrac{\dfrac{\vdots \quad (\star) \quad \vdots \qquad\qquad \vdots \quad (\star\star) \quad \vdots}{[A^m, C, 0, m] \mid [B^n, C, n, 0] \mid [B^n, C, n, 0]}\,{\scriptstyle (\rightarrow, l)}}{[A^m, C, 0, m] \mid [B^n, C, n, 0]}\,{\scriptstyle (EC)} \qquad B \Rightarrow B}{\vdots \qquad\qquad \vdots}\,{\scriptstyle (\rightarrow, l)}$$

$$\dfrac{\dfrac{A \Rightarrow A \qquad \overline{[A^{k-1}, C, n-1, m] \mid [B^{k-1}, C, n, m-1]} \qquad B \Rightarrow B}{[A^k, C, n, m] \mid [B^k, C, n, m]}\,{\scriptstyle (\rightarrow, l)}}{(A^k \rightarrow C) \wedge (B^k \rightarrow C), nA, mB \Rightarrow C}\,{\scriptstyle (EC)+(\wedge', l)}$$

where $(\star)$ and $(\star\star)$ are respectively given by:

$$\dfrac{\dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{B \Rightarrow A | A \Rightarrow B}\,{\scriptstyle (com)} \qquad \dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{\vdots \qquad \vdots}\,{\scriptstyle (com)}}{B \Rightarrow A | \, [B^{n-1}, C, n-1, 0]}\,{\scriptstyle (\rightarrow, l)}}{B \Rightarrow A | B \Rightarrow A | \, [B^n, C, n, 0]}\,{\scriptstyle (\rightarrow, l)}}{B \Rightarrow A | \, [B^n, C, n, 0]}\,{\scriptstyle (EC)}$$

and

$$\dfrac{\dfrac{\dfrac{\vdots \quad (\star) \quad \vdots}{B \Rightarrow A | \, [B^n, C, n, 0]} \qquad \dfrac{\dfrac{A \Rightarrow A \quad B \Rightarrow B}{\vdots \qquad \vdots}\,{\scriptstyle (com)}}{[A^{m-2}, C, 0, m-2] \mid [B^n, C, n, 0]}}{[A^{m-1}, C, 0, m-1] \mid [B^n, C, n, 0] \mid [B^n, C, n, 0]}\,{\scriptstyle (\rightarrow, l)}}{[A^{m-1}, C, 0, m-1] \mid [B^n, C, n, 0]}\,{\scriptstyle (EC)}$$

- The case $0 < n < m < k$ is similar.

This concludes the proof.

Theorem 4.3 $\mathbf{C}_{new}$ *coincides with* $\mathbf{C}^{\#}$.

Proof: By the previous propositions.

Corollary 4.4 $\mathbf{C}_{new}$ *and* $\mathbf{C}$ *are distinct versions of Gödel logic without contraction.*

Proof: By the previous theorem and Corollary 3.17.

Corollary 4.5 $\mathbf{C}_{new}$ *is decidable.*

Proof: Follows by Corollary 3.16 and Theorem 4.3.

Corollary 4.6 *All inclusions of logics in Figure 1 are proper.*

References

[1] A. Avron. Hypersequents, logical consequence and intermediate logics for concurrency. *Annals of Mathematics and Artificial Intelligence*, 4:225–248, 1991.

[2] A. Avron. The method of hypersequents in the proof theory of propositional nonclassical logics. In *Logic: from Foundations to Applications, European Logic Colloquium*, pages 1–32. Oxford Science Publications. Clarendon Press. Oxford, 1996.

[3] M. Baaz, A. Ciabattoni, C. Fermüller, and H. Veith. Proof theory of fuzzy logics: Urquart's C and related logics. In *Mathematical Foundations of Computer Science (MFCS98)*, Lectures Notes in Computer Science, vol. 1450, pages 203–212, 1998.

[4] C. C. Chang. Algebraic analysis of many-valued logics. *Trans. of the Amer. Math. Soc.*, 88:467–490, 1958.

[5] A. Ciabattoni. *Proof-theoretic techniques in many-valued logics*. PhD thesis, 1999. University of Milan, Italy.

[6] J. Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–101, 1987.

[7] P. Hájek. *Metamathematics of Fuzzy Logic*. Kluwer, 1998.

[8] J. Lukasiewicz. Philosophische bemerkungen zu mehrwertigen systemen der aussagenlogik. *Comptes Rendus de la Societé des Science et de Lettres de Varsovie*, pages 51–77, 1930.

[9] J. Mendéz and F. Salto. Urquhart's C with intuitionistic negation: Dummett's LC without the contraction axiom. *Notre Dame J. of Formal Logic*, 36(3):407–413, 1995.

[10] J. Mendéz and F. Salto. A natural negation complementation of urquhart's many-valued logic C. *J. of Philosophical Logic*, 27:75–84, 1998.

[11] H. Ono. Proof-theoretic methods in nonclassical logic – an introduction. In M. Takahashi, M. Okada, and M. Dezani-Ciancaglini, editors, *Theories of Types and Proofs*, pages 207–254. Mathematical Society of Japan, 1998.

[12] H. Ono and Y. Komori. Logics without the contraction rule. *J. of Symbolic Logic*, 50:169–201, 1985.

[13] A. Prijatelj. Bounded contraction and gentzen style formulation of Lukasiewicz logics. *Studia Logica*, 57:437–456, 1996.

[14] A. Troelstra and H. Schwichtenberg. *Basic Proof Theory*. Cambridge University Press, 1996.

[15] A. Urquhart. Many-valued logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic, Vol. III: Alternatives in Classical Logic*, pages 71–116. Reidel, Dordrecht, 1986.

[16] A. Urquhart. Basic many-valued logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic: Alternatives in Classical Logic*. Reidel, Dordrecht, To appear.