# Cut-Elimination in a Sequents-of-Relations Calculus for Gödel Logic*

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## Abstract

*In [5] the analytic calculus $\mathbf{R}\mathrm{G}_\infty$ for Gödel logic has been introduced. $\mathbf{R}\mathrm{G}_\infty$ operates on "sequents of relations". We show constructively how to eliminate cuts from $\mathbf{R}\mathrm{G}_\infty$-derivations. The version of the cut rule we consider allows to derive other forms of cut as well as a rule corresponding to the "communication rule" of Avron's hypersequent calculus for $\mathrm{G}_\infty$. Moreover, we give an explicit description of all the axioms of $\mathbf{R}\mathrm{G}_\infty$ and prove their completeness.*

## 1. Introduction

Gödel logic $\mathrm{G}_\infty$ — also called (Gödel-)Dummett logic, since Dummett [6] presented the first axiomatization matching Gödel's matrix characterization — is one of the most important many-valued logic. It naturally turns up in a number of different contexts. Already in the 1930s Gödel [9] used it in investigations of intuitionistic logic; later, Dunn and Meyer [7] pointed out its relevance for relevance logic; Visser [13] employed it in investigations of the provability logic of Heyting arithmetic; and eventually, it was recognized as one of the most useful species of fuzzy logic (see [10]). In contrast to other fuzzy logics, convincing analytic proof systems have been presented for Gödel logic. In particular, we here investigate the calculus $\mathbf{R}\mathrm{G}_\infty$, introduced in [5], which is based on so-called "sequents of relations". In $\mathbf{R}\mathrm{G}_\infty$ all rules are local, have at most two premises, introduce at most one connective at a time and are invertible. These properties render this calculus particularly apt for (human and automated) proof search. Alternative analytic systems for $\mathrm{G}_\infty$ can be found, e.g., in [11, 1, 2, 3, 8, 4]. In particular, the axioms (basic hypersequents) introduced in [4] are closely related to the axioms of $\mathbf{R}\mathrm{G}_\infty$.

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Soundness and completeness of $\mathbf{R}\mathrm{G}_\infty$ were already proved in [5] (in a more general setting). It also was shown there that certain forms of cut are admissible and therefore (semantically) redundant. However, a central topic, namely (constructive, stepwise) elimination of cuts from proofs was left open. Cut-elimination for a particularly useful form of cut is the main result of this paper. The cut we consider allows to derive other forms of cut as well as a rule corresponding to the "communication rule" of Avron's hypersequent calculus for $\mathrm{G}_\infty$ [2]. Another new contribution of this paper concerns the axioms for $\mathbf{R}\mathrm{G}_\infty$. Their effective construction is a non-trivial problem. In [5] the set of axioms was only presented in an indirect form and without proof. Here we give an explicit description of all axioms and prove their completeness.

## 2. Gödel logic

The language we use for Gödel logics is based on the binary *connectives* $\wedge, \vee$, and $\supset$ and the *truth constants* $0$ and $1$; $\neg A$ abbreviates $A \supset 0$.

An *interpretation* $\mathcal{I}$ is a mapping from propositional variables into a set of *truth values* $V$. In the case of infinite-valued Gödel logic, $V$ is the real interval $[0,1]$. Finite valued Gödel logics are obtained by taking as $V$ finite subsets of $[0,1]$ containing 0 and 1. An interpretation $\mathcal{I}$ extends to an *evaluation* $val_\mathcal{I}$ by stipulating $val_\mathcal{I}(p) = \mathcal{I}(p)$, for propositional variables (atomic formulas) $p$, $val_\mathcal{I}(0) = 0$, $val_\mathcal{I}(1) = 1$, and

$$val_\mathcal{I}(A \supset B) = \begin{cases} 1 & \text{if } val_\mathcal{I}(A) \leq val_\mathcal{I}(B) \\ val_\mathcal{I}(B) & \text{otherwise,} \end{cases}$$

$$val_\mathcal{I}(A \wedge B) = \min(val_\mathcal{I}(A), val_\mathcal{I}(B)),$$
$$val_\mathcal{I}(A \vee B) = \max(val_\mathcal{I}(A), val_\mathcal{I}(B)).$$

## 3. The calculus $\mathbf{R}\mathrm{G}_\infty$

In [5] the construction of the calculus $\mathbf{R}\mathrm{G}_\infty$ for Gödel logic was sketched to illustrate a more general framework for the derivation of analytic calculi for certain types of many-valued logics (projective logics). This framework is based on “sequents of relations”.

In $\mathbf{R}\mathrm{G}_\infty$ a *sequent* is a finite set of *components* of the form $A < B$ or $A \leq B$ for arbitrary formulas $A, B$.

**Remark 3.1** *In [5] sequents were defined as* sequences *of components. However, it is easy to see that it suffices to consider* sets *instead of sequences (or multi-sets). This allows to drop the external rules of permutation and contraction from* $\mathbf{R}\mathrm{G}_\infty$.

Sequent calculi of relations are closely related to hypersequent calculi (see, e.g., [2, 3]). We denote sequents (of relations) as

$$A_1 \lhd_1 B_1 \mid \ldots \mid A_n \lhd_n B_n \ ,$$

where the sign $\lhd_i$ ($1 \leq i \leq n$) is either $<$ or $\leq$ and plays a role similar to the sequent arrow in traditional sequent calculi.

A sequent is called *atomic* if all $A_i$, $B_i$ are atomic formulas.

The separation sign “|” is interpreted as disjunction (at the meta-level). More formally, a component $A \lhd B$ is *satisfied* by an interpretation $\mathcal{I}$ if $val_\mathcal{I}(A) \lhd val_\mathcal{I}(B)$ (for $\lhd \in \{<, \leq\}$). A sequent $\Sigma$ is satisfied by $\mathcal{I}$ if $\mathcal{I}$ satisfies at least one of its components. $\Sigma$ is *valid* if it is satisfied by all interpretations.

The logical rules of $\mathbf{R}\mathrm{G}_\infty$ — i.e., the rules for introducing connectives at any place of a sequent — are easily computed given the semantics of $\mathrm{G}_\infty$, as described in [5]. For convience, we restate the complete set of rules.

For disjunction and conjunction we have:

$$\frac{C \lhd A \mid \mathcal{H} \qquad C \lhd B \mid \mathcal{H}}{C \lhd (A \wedge B) \mid \mathcal{H}}\ (\wedge\!:\lhd\!:r) \qquad \frac{A \lhd C \mid B \lhd C \mid \mathcal{H}}{(A \wedge B) \lhd C \mid \mathcal{H}}\ (\wedge\!:\lhd\!:l)$$

$$\frac{C \lhd A \mid C \lhd B \mid \mathcal{H}}{C \lhd (A \vee B) \mid \mathcal{H}}\ (\vee\!:\lhd\!:r) \qquad \frac{A \lhd C \mid \mathcal{H} \qquad B \lhd C \mid \mathcal{H}}{(A \vee B) \lhd C \mid \mathcal{H}}\ (\vee\!:\lhd\!:l)$$

where $\lhd$ stands for either $<$ or $\leq$, uniformly in each rule. The rules $(\supset\!:<\!:r)$, $(\supset\!:<\!:l)$, $(\supset\!:\leq\!:r)$ and $(\supset\!:\leq\!:l)$ for implication are, respectively:

$$\frac{A \leq B \mid C < B \mid \mathcal{H} \qquad C < 1 \mid \mathcal{H}}{C < (A \supset B) \mid \mathcal{H}} \qquad \frac{B < A \mid \mathcal{H} \qquad B < C \mid \mathcal{H}}{(A \supset B) < C \mid \mathcal{H}}$$

$$\frac{A \leq B \mid C \leq B \mid \mathcal{H}}{C \leq (A \supset B) \mid \mathcal{H}} \qquad \frac{1 \leq C \mid B < A \mid \mathcal{H} \qquad B \leq C \mid \mathcal{H}}{(A \supset B) \leq C \mid \mathcal{H}}$$

The indicated compound formula in the lower sequent of each rule is called *principal formula*.

We also need (external) weakening:

$$\frac{\mathcal{H}}{A \lhd B \mid \mathcal{H}}\ (\mathrm{EW})$$

The following version of the cut rule is part of $\mathbf{R}\mathrm{G}_\infty$ here, too:

$$\frac{\mathcal{H} \mid A \leq B \qquad \mathcal{H} \mid B < A}{\mathcal{H}}\ (\mathrm{cut}_{</\geq})$$

$A$ and $B$ are called *cut-formulas*; and the indicated components are referred to as *cut-components*.

So far we have not stated any axioms for $\mathbf{R}\mathrm{G}_\infty$. In fact, the computation of a complete set of axioms, for which cuts can be eliminated, is not trivial. In [5] the following was stated (without proof): Axioms of $\mathbf{R}\mathrm{G}_\infty$ are all sequents that contain a sequent

$$A_1 \lhd_1 A_2 \mid A_2 \lhd_2 A_3 \mid \ldots \mid A_k \lhd_k A_1$$

for $k \geq 1$, where $\lhd_i \in \{<, \leq\}$ for all $1 \leq i \leq k$, but $\lhd_i \equiv \leq$ for at least one $i$. In addition, all sequents that are obtained from the above ones by deleting components of form

$$A < 0, \qquad 1 < A, \qquad \text{or} \qquad 1 \leq 0$$

are axioms.

We present a more explicit description of the set of axioms of $\mathbf{R}\mathrm{G}_\infty$, that corresponds to the original set, up to external weakening.

(a) $A_1 \lhd_n A_n \mid \ldots \mid A_3 \lhd_2 A_2 \mid A_2 \leq A_1$, where $\lhd_i \in \{<, \leq\}$ and the case $n = 1$ is defined as $A_1 \leq A_1$,

(b) $A_n \leq A_{n-1} \mid A_{n-1} < A_{n-2} \mid \ldots \mid A_1 < 1$, where the case $n = 1$ is defined as $A_1 \leq 1$,

(c) $0 < A_n \mid \ldots \mid A_3 < A_2 \mid A_2 \leq A_1$, where the case $n = 1$ is defined as $0 \leq A_1$,

(d) $0 < A_1 \mid A_1 < A_2 \mid \ldots \mid A_n < 1$, where the case $n = 0$ is defined as $0 < 1$.

We call sequents of type (a), (b), (c), and (d), *cycles*, 1-*chains*, 0-*chains*, and 0-1-*chains*, respectively.

It is easy to check that all of the above axioms are valid in $\mathrm{G}_\infty$. However, to guarantee completeness we also have to show the converse: namely, that all valid atomic sequents are obtained from these axioms using external weakening only. For this purpose it is better to consider the dual form of the axioms. I.e., we make use of the fact that $val_\mathcal{I}(A) < val_\mathcal{I}(B)$ iff $\neg[val_\mathcal{I}(B) \leq val_\mathcal{I}(A)]$, and thus may consider conjunctions of components instead of disjunctions.

**Definition 3.2** *A set of components is called* dual to axioms *if it does not contain any subset of one of the following forms:*

*(a)* (anti-cycle) $\{A_1 < A_2, A_2 \lhd_2 A_3, \ldots, A_n \lhd_n A_1\}$, *where* $\lhd_i \in \{<, \leq\}$ *and the case* $n = 1$ *is defined as* $\{A_1 < A_1\}$,

*(b)* (anti-1-chain) $\{1 \leq A_1, \ldots, A_{n-2} \leq A_{n-1}, A_{n-1} < A_n\}$, *where the case $n = 1$ is defined as $\{1 < A_1\}$,*

*(c)* (anti-0-chain) $\{A_1 < A_2, A_2 \leq A_3, \ldots, A_n \leq 0\}$, *where the case $n = 1$ is defined as $\{A_1 < 0\}$,*

*(d)* (anti-0-1-chain) $\{1 \leq A_1, A_2 \leq A_3, \ldots, A_n \leq 0\}$, *where the case $n = 0$ is defined as $\{1 \leq 0\}$ .*

It suffices to prove the following:

**Theorem 3.3** *Let $\Gamma$ be a finite set of components $A \lhd B$, $\lhd \in \{<, \leq\}$, where $A$ and $B$ are either propositional variables or truth constants. If $\Gamma$ is dual to axioms then $\Gamma$ is satisfiable; i.e., there exists an interpretation that satisfies all components of $\Gamma$.*

To prove Theorem 3.3 we extend any $\Gamma$ that is dual to axioms to a "maximal" set $\Gamma^*$ that is still dual to axioms. Let us write $B \in [A] \iff \{A \leq B, B \leq A\} \subseteq \Gamma^*$. It will follow from Propostion 3.4 and Lemma 3.5, below, that this is an equivalence relation and that the set of equivalence classes $\overline{\Gamma^*} = \{[A] : A \text{ occurs in } \Gamma^*\}$ is totally ordered with respect to $[A] < [B] \iff A < B$. The minimal element of the ordering is $[0]$ and its maximal element is $[1]$ (if $0$ and $1$ occur in $\Gamma$). The ordering thus allows to match equivalence classes with truth values in a way that induces an interpretation satisfying $\Gamma^*$ and therefore also $\Gamma$.

We first add $A \leq B$ to $\Gamma$ whenever $A < B \in \Gamma$. This is justified by the following simple observation:

**Proposition 3.4** *If $\Gamma$ is dual to axioms then $\Gamma \cup \{A \leq B : A < B \in \Gamma\}$ is dual to axioms, too.*

The existence of $\overline{\Gamma^*}$ follows from the following:

**Lemma 3.5** *If $\Gamma$ is dual to axioms then either $\Gamma \cup \{A < B\}$ or $\Gamma \cup \{B \leq A\}$ is dual to axioms, too.*

**Proof:** The proof proceeds by case distinctions:

(1) $\Gamma \cup \{A < B\}$ contains an anti-cycle. Then either already $\Gamma$ contains an anti-cycle or $\{B \leq U_1, \ldots, U_n \leq A\} \subseteq \Gamma$. From this it follows that $\Gamma \cup \{B \leq A\}$ is dual to axioms iff $\Gamma$ is dual to axioms.

(2) $\Gamma \cup \{B \leq A\}$ contains an anti-cycle. Then either already $\Gamma$ contains an anti-cycle or $\{A \leq U_1, \ldots, U_k < U_{k+1}, \ldots U_n \leq B\} \subseteq \Gamma$. From this it follows that $\Gamma \cup \{A < B\}$ is dual to axioms iff $\Gamma$ is dual to axioms.

(3) Neither $\Gamma \cup \{A < B\}$ nor $\Gamma \cup \{B \leq A\}$ contains an anti-cycle.

(3.1) $\Gamma \cup \{A < B\}$ contains an anti-1-chain W.l.o.g., the anti-1-chain is not already contained in $\Gamma$. Therefore (**a**): $\{1 \leq V_1, \ldots, V_{n-1} \leq A\} \subseteq \Gamma$.

(3.1.1) $\Gamma \cup \{B \leq A\}$ contains an anti-1-chain that is not already contained in $\Gamma$. Therefore $\{1 \leq U_1, \ldots, U_{k-1} \leq B\} \subseteq \Gamma$ and $\{A \leq U_{k+1}, \ldots, U_{k+m-1} < U_{k+m}\} \subseteq \Gamma$. The latter subset can be combined with **(a)** to an anti-1-chain in $\Gamma$.

(3.1.2) $\Gamma \cup \{B \leq A\}$ contains an anti-0-chain that is not already contained in $\Gamma$. Therefore $\{U_1 < U_2, \ldots, U_{k-1} \leq B\} \subseteq \Gamma$ and $\{A \leq U_{k+1}, \ldots, U_{k+m} \leq 0\} \subseteq \Gamma$. The latter subset can be combined with **(a)** to an anti-0-1-chain in $\Gamma$.

(3.1.3) $\Gamma \cup \{B \leq A\}$ contains an anti-0-1-chain that is not already contained in $\Gamma$. Therefore $\{1 \leq U_1, \ldots, U_{k-1} \leq B\} \subseteq \Gamma$ and $\{A \leq U_{k+1}, \ldots, U_{k+m} \leq 0\} \subseteq \Gamma$. The latter subset can be combined with **(a)** to an anti-0-1-chain in $\Gamma$.

(3.2) $\Gamma \cup \{B \leq A\}$ contains an anti-1-chain that is not already contained in $\Gamma$. Therefore **(b1)**: $\{1 \leq V_1, \ldots, V_{k-1} \leq B\} \subseteq \Gamma$ and **(b2)**: $\{A \leq V_{k+1}, \ldots, V_{k+m-1} < V_{k+m}\} \subseteq \Gamma$.

(3.2.1) $\Gamma \cup \{A < B\}$ contains an anti-1-chain. This case was already settled in (3.1.1).

(3.2.2) $\Gamma \cup \{A < B\}$ contains an anti-0-chain that is not already contained in $\Gamma$. Then $\{B \leq U_2, \ldots, U_n \leq 0\} \subseteq \Gamma$. This subset can be combined with **(b1)** to an anti-0-1-chain in $\Gamma$.

(3.3) Neither $\Gamma \cup \{A < B\}$ nor $\Gamma \cup \{A \leq B\}$ contain an anti-1-chain.

(3.3.1) $\Gamma \cup \{A < B\}$ contains an anti-0-chain that is not already contained in $\Gamma$. Then **(c)** $\{B \leq V_2, \ldots, V_n \leq 0\} \subseteq \Gamma$.

(3.3.1.1) $\Gamma \cup \{B \leq A\}$ contains an anti-0-chain that is not already contained in $\Gamma$. Therefore $\{U_1 < U_2, \ldots, U_{k-1} \leq B\} \subseteq \Gamma$ and $\{A \leq U_{k+1}, \ldots, U_{k+m} \leq 0\} \subseteq \Gamma$. The first subset can be combined with **(c)** to an anti-0-chain in $\Gamma$.

(3.3.1.2) $\Gamma \cup \{B \leq A\}$ contains an anti-0-1-chain that is not already contained in $\Gamma$. Therefore $\{1 < U_2, \ldots, U_{k-1} \leq B\} \subseteq \Gamma$ and $\{A \leq U_{k+1}, \ldots, U_{k+m} \leq 0\} \subseteq \Gamma$. The first subset can be combined with **(c)** to an anti-0-1-chain in $\Gamma$.

Finally observe that if $\Gamma \cup \{A < B\}$ contains an anti-0-1-chain then this anti-0-1-chain is already contained in $\Gamma$. It is easy to check that this settles all remaining cases. $\square$

**Remark 3.6** *To obtain a calculus for* $n$*-valued Gödel logic one only has to add to* $\mathbf{R}\mathrm{G}_\infty$ *the axiom*

$$A_1 \lhd_1 A_2 \mid A_2 \lhd_2 A_3 \mid \ldots \mid A_l \lhd_l A_{l+1}$$

*where* $\lhd_i \equiv \leq$ *for at least* $n$ $i$*, with* $i \in \{1, \ldots l+1\}$.

**Remark 3.7** *As pointed out already in [5], different forms of cuts are admissible in* $\mathbf{R}\mathrm{G}_\infty$*. Focusing on* $(cut_{</\geq})$ *is motivated by the fact that it allows to simulate other forms of cut straightforwardly. E.g., the following* transitivity-cut

$$\frac{A < B \mid \mathcal{H} \qquad B < C \mid \mathcal{H}}{A < C \mid \mathcal{H}}\ (tr\text{-}cut_<)$$

*can be derived from a 3-component-cycle by applying* $(cut_{</\geq})$ *twice in the following way:*

$$\cfrac{\cfrac{C \leq B \mid B \leq A \mid A < C \qquad B < C \mid \mathcal{H}}{B \leq A \mid A < C \mid \mathcal{H}} \qquad A < B \mid \mathcal{H}}{A < C \mid \mathcal{H}}$$

*Similar admissible rules involving* $\leq$ *instead of* $<$ *can be treated analogously. Most interestingly,* $(cut_{</\geq})$ *also allows to derive a version of Avron's* communication rule. *Recall that this rule was introduced in [2] to define a hypersequent calculus for* $\mathrm{G}_\infty$ *based on Gentzen's sequent calculus for intuitionistic logic. Indeed, consider the rule*

$$\frac{A \leq B \mid \mathcal{H} \qquad C \leq D \mid \mathcal{H}}{A \leq D \mid C \leq B \mid \mathcal{H}}\ (comm.)$$

*It can be derived from a 4-component-cycle by applying* $(cut_{</\geq})$ *twice in the following way:*

$$\cfrac{C \leq D \mid \mathcal{H} \qquad \cfrac{A \leq B \mid \mathcal{H} \qquad B < A \mid A \leq D \mid D < C \mid C \leq B}{A \leq D \mid D < C \mid C \leq B \mid \mathcal{H}}}{A \leq D \mid C \leq B \mid \mathcal{H}}$$

*A different type of admissible rule, related to cut, is the so-called Takeuti-Titani rule (see [12]), which expresses the density of the set of truth values:*

$$\frac{F \leq p \mid p \leq G \mid \mathcal{H}}{F \leq G \mid \mathcal{H}}\ (tt)$$

*where* $p$ *is a propositional variable not occurring in the upper sequent. It is interesting to observe that* $(tt)$ *cannot be derived in* $\mathbf{R}\mathrm{G}_\infty$ *since — in contrast to* $(cut_{</\geq})$ *and the other rules of* $\mathbf{R}\mathrm{G}_\infty$ *—* $(tt)$ *is not strongly sound, e.g., in finite valued Gödel logics.*

## 4. Cut-Elimination

**Theorem 4.1** *Every derivation of a sequent* $\mathcal{H}$ *in* $\mathbf{R}\mathrm{G}_\infty$ *can be stepwise transformed into a cut-free derivation of* $\mathcal{H}$.

The proof of Theorem 4.1 consists of four parts:

1. Replacement of compound axioms by atomic ones (Lemma 4.2).
2. Reduction of cuts involving compound formulas (Lemmas 4.4 and 4.5).
3. Moving atomic cuts up to atomic sequents (Lemma 4.6).
4. Elimination of cuts involving only axioms (Lemma 4.7).

**Lemma 4.2** *In* $\mathbf{R}\mathrm{G}_\infty$ *non-atomic axioms are derivable from atomic axioms.*

**Proof:** By induction on the structure of formulas. The induction step is immediate in the case of conjunctive and disjunctive formulas. Let us consider implicative formulas. As an example, observe that a non-atomic cycle of the form

$$A_1 \lhd_n A_n \mid \ldots \mid A_i \leq P \mid P < A_{i-1} \mid \ldots \mid A_2 \leq A_1$$

where $P = B \supset C$, can be derived from two cycles involving $B$ and $C$, as follows:

$$\cfrac{\cfrac{B \leq C \mid C < B \quad A_1 \lhd_n A_n \mid \ldots \mid A_i \leq C \mid C < A_{i-1} \mid \ldots \mid A_2 \leq A_1}{A_1 \lhd_n A_n \mid \ldots \mid B \leq C \mid A_i \leq C \mid P < A_{i-1} \mid \ldots \mid A_2 \leq A_1}}{A_1 \lhd_n A_n \mid \ldots \mid A_i \leq P \mid P < A_{i-1} \mid \ldots \mid A_2 \leq A_1}$$

$\square$

A derivation $d$ (in $\mathbf{R}\mathrm{G}_\infty$) is considered, as usual, as an upward rooted tree of sequents generated from subtrees by applying inference rules. This allows for the following definitions:

**Definition 4.3** *The* length $|d|$ *of* $d$ *is the maximal number of sequents occurring on any branch of* $d$.

*The* complexity of a cut *is the number of connectives occurring in a cut-component of it plus* 1*. A cut of complexity* 1 *is called* atomic.

*By* $\rho(d)$ *we denote the maximal complexity of cuts in* $d$.

If $d$ is a derivation of $\mathcal{H}$ we write $d \vdash \mathcal{H}$.

**Lemma 4.4 (Inversion Lemma)** *If* $d$ *is a derivation in* $\mathbf{R}\mathrm{G}_\infty$ *of* $A \circ B \lhd C \mid \mathcal{H}$ *or* $C \lhd A \circ B \mid \mathcal{H}$*, where* $\circ \in \{\wedge, \vee, \supset\}$ *and* $\lhd \in \{<, \leq\}$*, then one can find a derivation* $d_1$ *of a sequent that is the instance of the premise (or derivations* $d_1$ *and* $d_2$ *of sequents that are the instances of the premises) of the rule for introducing* $A \circ B$ *such that* $\rho(d_i) \leq \rho(d)$*, for* $i = 1, 2$.

**Proof:** By Lemma 4.2 we may assume that all the axioms in $d$ are atomic. The proof proceeds by induction on $|d|$. Cases are distinguished according to the form of the indicated component of the last sequent. As an example we illustrate the case $d \vdash (A \supset B) \leq C \mid \mathcal{H}$ in detail.

Let $d \vdash (A \supset B) \leq C \mid \mathcal{H}$ then we have to find a derivation $d_1$ of $1 \leq C \mid B < A \mid \mathcal{H}$ and a derivation $d_2$ of $B \leq C \mid \mathcal{H}$ were $\rho(d_i) \leq \rho(d)$, for $i = 1, 2$. Let $R$ be the last inference in $d$. Three possibilities arise:

(1) $R$ is a logical inference.

(1.1) The indicated occurrence of $A \supset B$ is the principal formula of $R$. Then $d_1$ and $d_2$ are obtained as the two immediate sub-derivations of $d$.

(1.2) The principal formula of $R$ is not the indicated occurrence of $A \supset B$. Suppose, e.g., that $C \equiv C_1 \wedge C_2$ and $d$ ends in a rule $(\wedge\colon \leq\colon r)$ as follows

$$\frac{(A \supset B) \leq C_1 \mid \mathcal{H} \qquad (A \supset B) \leq C_2 \mid \mathcal{H}}{(A \supset B) \leq (C_1 \wedge C_2) \mid \mathcal{H}}$$

By the induction hypothesis, we obtain the four proofs $e_i \vdash 1 \leq C_i \mid B < A \mid \mathcal{H}$ and $f_i \vdash B \leq C_i \mid \mathcal{H}$ for $i = 1, 2$, with the required properties. Clearly, $(\wedge\colon \leq\colon r)$ can be applied to $e_1$ and $f_1$ ($e_2$ and $f_2$) to obtain $d_1(d_2)$.

The case in which the principal formula of $R$ occurs in $\mathcal{H}$ is handled analogously.

(2) $R$ is $(EW)$.

(2.1) $R$ introduces the indicated component. Then $d_1$ is obtained by adding the components $1 \leq C$ and $B < A$ to the premise of $R$, using $(EW)$ twice. Similarly for $d_2$.

(2.2) $R$ introduces a component of $\mathcal{H}$. We apply the induction hypothesis to the premise of $R$. $d_1$ and $d_2$ are then obtained by applying $(EW)$.

(3) $R$ is $(\mathrm{cut}_{</\geq})$. Analogous to case (1.2).

The cases for the connectives are similar. □

**Lemma 4.5 (Reduction Lemma)** *Let $d \vdash \mathcal{H}$ be a derivation in $\mathbf{R}\mathrm{G}_\infty$ ending in a cut of maximal complexity with a cut-formula $A \circ B$, ($\circ \in \{\wedge, \vee, \supset\}$). Then one can find a derivation $d'$ of $\mathcal{H}$ where this cut is replaced by cuts involving as cut formulas only $A$, $B$, $0$, or $1$; thus in $d'$ the number of cuts with complexity $\rho(d)$ is strictly smaller than in $d$; moreover $\rho(d') \leq \rho(d)$.*

**Proof:** The proof proceeds by cases, according to the form of the cut-component. We illustrate one case in detail. Suppose $d$ ends with

$$\frac{\mathcal{H} \mid C < (A \supset B) \qquad \mathcal{H} \mid (A \supset B) \leq C}{\mathcal{H}}\ (\mathrm{cut}_{</\geq})$$

then we apply the Inversion Lemma to obtain derivations $d_1 \vdash A \leq B \mid C < B \mid \mathcal{H}$, $d_2 \vdash C < 1 \mid \mathcal{H}$, $d_3 \vdash 1 \leq C \mid B < A \mid \mathcal{H}$, and $d_4 \vdash B \leq C \mid \mathcal{H}$, where $\rho(d_i) \leq \rho(d)$, $1 \leq i \leq 4$. These can be joint to the required derivation $d'$ of $\mathcal{H}$ as follows:

$$\dfrac{B \leq C \mid \mathcal{H} \qquad \dfrac{A \leq B \mid C < B \mid \mathcal{H} \qquad \dfrac{C < 1 \mid \mathcal{H} \quad 1 \leq C \mid B < A \mid \mathcal{H}}{B < A \mid \mathcal{H}}}{C < B \mid \mathcal{H}}}{\mathcal{H}}$$

The other cases are similar. □

**Lemma 4.6** *Let $d \vdash \mathcal{H}$ be a derivation in $\mathbf{R}\mathrm{G}_\infty$ from atomic axioms whose last inference is an atomic cut. Then one can find a derivation $d'$ of $\mathcal{H}$, with $\rho(d') \leq \rho(d)$, where this cut is replaced by cuts applied to atomic sequents.*

**Proof:** The proof proceeds by induction on the number of connectives in $\mathcal{H}$ and applying Lemma 4.4. □

**Lemma 4.7** *The conclusion of every cut between two axioms contains an axiom.*

**Proof:** The proof proceeds by cases according to the types of axioms involved. Let $\Sigma_1$ and $\Sigma_2$ be the two premises of the cut and $\Pi$ its conclusion. The following table gives the type of axiom contained in $\Pi$ for all 16 cases:

| $\Sigma_1 \backslash \Sigma_2$ | c | 0 | 1 | 0-1 |
|---|---|---|---|---|
| c | c | 0 | 1 | 0, 1, or 0-1 |
| 0 | 0 | 0 | 0-1 | 0-1 |
| 1 | 1 | 0-1 | 1 | 0-1 |
| 0-1 | 0, 1, or 0-1 | 0-1 | 0-1 | – |

We present two cases in detail.

- The entry c in the c-column of the c row is to be read as follows: If both $\Sigma_1$ and $\Sigma_2$ are cycles then $\Pi$ contains a cycle.

  We illustrate one subcase. Let

  $$\Sigma_1 = A_1 \lhd_n A_n \mid \ldots \mid \underline{A_k \lhd_{k-1} A_{k-1}} \mid \ldots \mid A_2 \leq A_1$$

  where $\lhd_i \in \{<, \leq\}$ and

  $$\Sigma_2 = B_1 \lhd'_m B_m \mid \ldots \mid B_3 \lhd'_2 B_2 \mid \underline{B_2 \leq B_1},$$

  where $\lhd'_i \in \{<, \leq\}$. If $A_k \equiv B_1$ and $A_{k-1} \equiv B_2$ and $\lhd_{k-1} \equiv <$ then we can cut upon the underlined components and obtain $\Pi = A_1 \lhd_n A_n \mid \ldots \mid A_{k+1} \lhd A_k \mid A_{k-1} \lhd_{k-1} A_{k-2} \mid \ldots \mid A_2 \leq A_1 \mid B_1 \lhd'_m B_m \mid \ldots \mid B_3 \lhd'_2 B_2$.
  $\Pi$ is easily recognized as a cycle by ordering its components according to the following sequence of its left hand formulas: $A_1, A_n, A_{n-1}, \ldots, A_k[\equiv B_1], B_m, B_{m-1}, \ldots B_3, B_2\ [\equiv A_{k-1}], A_{k-2} \ldots, A_2$.

- The entry 0-1 in the 0-column of the 1-row says that if $\Sigma_1$ is a 0-chain and $\Sigma_2$ a 1-chain then $\Pi$ contains a 0-1-chain. E.g., let

  $$\Sigma_1 = 0 < A_n \mid \ldots \mid A_3 < A_2 \mid \underline{A_2 \leq A_1}$$

  and

  $$\Sigma_2 = B_m \leq B_{m-1} \mid \ldots \mid \underline{B_k < B_{k-1}} \mid \ldots \mid B_1 < 1.$$

  If $A_2 \equiv B_{k-1}$ and $A_1 \equiv B_k$ then we can cut upon the underlined components. The conclusion $\Pi$ contains the sequent

  $$0 < A_n \mid \ldots \mid A_3 < A_2 \mid A_2 < B_{k-2} \mid \ldots \mid B_1 < 1$$

  which is a 0-1-chain.

- The entry "$-$" in the last column of the last row asserts that there can be no cut between two 0-1-chains. $\square$

**Proof of Theorem 4.1**
Let $d \vdash \mathcal{H}$. The transformation of $d$ into a cut-free derivation from atomic axioms proceeds in 4 steps:

1. Apply Lemma 4.2 to obtain $d' \vdash \mathcal{H}$, where all axioms in $d'$ are atomic.
2. Apply the Reduction Lemma (Lemma 4.5) to a subderivation of $d'$ that ends with a cut of maximal complexity. Repeat this step until all cuts are atomic.
3. Apply Lemma 4.6 to obtain a derivation $d^a$ in which cuts are only applied to atomic sequents.
4. Observe that $(EW)$ and $(\text{cut}_{</\geq})$ are the only inference rules that can occur in a sub-derivation of $d^a$ that ends in a cut. Let $d^*$ be such a sub-derivation of maximal length. Lemma 4.7 implies that the last sequent of $d^*$ contains an axiom. We therefore can replace $d^*$ by this axiom, possibly followed by applications of $(EW)$. This is repeated until all cuts have been removed. $\square$

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