On the Undecidability of Some Sub-classical First-order Logics*

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Abstract. A general criterion for the undecidabily of sub-classical first-order logics and important fragments thereof is established. It is applied, among others, to Urquart's (original version of) **C** and the closely related logic $\mathbf{C}^*$. In addition, hypersequent systems for (first-order) **C** and $\mathbf{C}^*$ are introduced and shown to enjoy cut-elimination.

1 Introduction

A wide range of non-classical logics can be viewed as sub-classical. By this we mean that they are based on (a subset of) the signature of classical logic and can be extended to classical logic. Intuitionistic logic, fragments of linear logic, Łukasiewicz logic, Gödel/Dummett logic (in fact, all intermediate logics, and most substructural or many-valued logics), are just a few prominent examples.

(Un)decidability issues are a fundamental topic of Logic in the context of Theoretical Computer Science. In the literature there are results on the decision problem for *particular* first-order sub-classical logics. For instance, in [14, 11, 10] the decision problem for some substructural logics, that is logics that when formulated as a Gentzen system do not have all the structural rules, has been investigated. It was proved that removing the weakening rule from standard Gentzen-style formulations of classical or intuitionistic first-order logic gives a logic that is still undecidable. On the other hand, removing the contraction rule from a sequent calculus for classical or intuitionistic logic results in a decidable logic. However it is clear that the lack of contraction in itself is *not sufficient* for the decidability of a sub-classical logic, as witnessed, e.g., by Łukasiewicz's and other important "fuzzy logics" (see [8]).

In this paper we provide a sufficient and quite general criterion for the undecidability of first-order sub-classical logics, with or without function symbols (in fact: already fragments thereof). Moreover, we continue

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the proof theoretical investigation—initiated in [3]—of the logics **C** (introduced in by A. Urquhart in [16][1], see also [13]), and the related $\mathbf{C}^*$. These are underlying the most important formalizations of fuzzy logic, namely Gödel, Łukasiewicz and Product logic (see, e.g., [8]). **C** and $\mathbf{C}^*$ lack contraction, i.e., $[A \supset (A \supset B)] \supset (A \supset B)$ does not hold. Moreover, they share the property that their truth values are linearly ordered, that is they satisfy the linearity axiom $(A \supset B) \vee (B \supset A)$.

In [3] we introduced *propositional* cut-free hypersequent calculi for **C** and $\mathbf{C}^*$. This allowed us to establish that derivability in these logics is decidable. In this paper we consider the *first-order* version of these logics and define analytic (cut-free) calculi for them. The analyticity of the logics **C** and $\mathbf{C}^*$ is contrasted with their undecidability, which is a corollary to the general undecidability result mentioned above. It should also be seen in light of the fact that their purely intuitionistic counterparts $\mathbf{I}^-$ and $\mathbf{I}^{-*}$—obtained by dropping the linearity axiom from **C** and $\mathbf{C}^*$, respectively—are decidable.

2 Basic definitions

We investigate logics that are based on the language of classical first-order logic without equality. More precisely, we use the binary *connectives* $\vee$, $\wedge$, $\supset$ and the *truth constant* $\bot$. Negation is defined, as usual, by: $\neg A := (A \supset \bot)$.

Remark 1. Treating negation as a defined connective is just a matter of convenience. Explicit axioms and rules for negation could easily be added in all our cases. However—as witnessed by intuitionistic logic—the other connectives and the quantifiers will not be interdefinable in general.

Object variables are denoted by x, y, ... The *quantifiers* $\forall$ (for all) and $\exists$ (exists) refer to these variables. Moreover, there is an infinite supply of n-ary *predicate symbols* for every $n \geq 0$. We consider both, logics with and without *function symbols.* In the former case an infinite supply of n-ary function symbols and constants is assumed for every $n \geq 0$. Constants are considered as 0-ary function symbols. *Terms* and *Formulas* are inductively defined in the usual way. Classical first-order logic with function symbols is denoted as $\mathbf{CL}f$, without function symbols as **CL**.

Here, we assume logics to be specified by Hilbert-style systems. *Axioms* are always considered as schemata. (I.e., all instances are axioms,

[1] Recently A. Urquhart discovered that his axiom system for **C** as given in [16] is incomplete with respect to an intended semantics [17]. Because of its proof theoretic interest we—like others, e.g. [13]—will deal with the original system **C** here.

too.) *Rules* of derivation are written in the form

$$\vdash A_1, \ldots, \vdash A_n \Longrightarrow \vdash B.$$

A *logic* $\mathcal{L}$ is identified with the set of provable formulas. We write $\vdash_{\mathcal{L}} F$ for $F \in \mathcal{L}$.

Definition 1. *A logic $\mathcal{L}$ (with or without function symbols) is called* sub-classical *if it can be extended to classical logic. More formally: if $\vdash_{\mathcal{L}} F$ implies $\vdash_{\mathbf{CL}^f} F$.*

Definition 2. *A rule $\vdash A_1, \ldots, \vdash A_n \Longrightarrow \vdash B$ is* admissible *in logic $\mathcal{L}$ if $\vdash_{\mathcal{L}} B'$ whenever $\vdash_{\mathcal{L}} A'_i$ for all $1 \leq i \leq n$, where B' and A'_i are corresponding instances of the schematic formulas of the rule.*

We write $\alpha[F/p]$ for the formula that arises by instantiating the formula variable p of the schema $\alpha[p]$.

3 Two general undecidability results

We establish sufficient conditions that allow to embed (undecidable fragments of) classical logic into sub-classical logics. As a central notion for the "simulation of two-valued-ness" in non-classical contexts we introduce the following:

Definition 3. *A schema of form $\alpha[p] \vee \overline{\alpha}[p]$ with exactly one formula variable p is called a* duality principle *of a logic $\mathcal{L}$ if for all formulas F: $\vdash_{\mathcal{L}} \alpha[F/p] \vee \overline{\alpha}[F/p]$ but there exists a formula G such that $\nvdash_{\mathbf{CL}} \alpha[G/p]$ and $\nvdash_{\mathbf{CL}} \overline{\alpha}[G/p]$.*

For the undecidabilty proof below we will assume admissibility[2] of the following rules for disjunction:

$\vdash A \Longrightarrow \vdash A \vee B$	(weak)	$\vdash A \vee (B \vee C) \Longrightarrow \vdash (A \vee B) \vee C$	(assoc-l)
$\vdash A \vee B \Longrightarrow \vdash B \vee A$	(comm)	$\vdash (A \vee B) \vee C \Longrightarrow \vdash A \vee (B \vee C)$	(assoc-r)
$\vdash A \vee A \Longrightarrow \vdash A$	(idem)		

The following rule expresses a weak form of distributivity:

$$\vdash A \vee C,\ \vdash B \vee C \Longrightarrow \vdash (A \wedge B) \vee C \qquad \text{(distr)}$$

A "minimal" rule for existential quantification is:

$$\vdash A(t) \Longrightarrow \vdash \exists x\colon A(x) \qquad \text{(}\exists\text{-in')}$$

where t is any term.

[2] Remember that admissibility of rules is a much weaker condition than the derivability of corresponding formulas. E.g., the admissibility of idem in $\mathcal{L}$, in general, does *not* imply $\vdash_{\mathcal{L}} (A \vee A) \supset A$.

Definition 4. *We call a rule* $\vdash A_1, \ldots, \vdash A_n \Longrightarrow \vdash B$ $\vee$-normal *in a logic* $\mathcal{L}$ *if its admissibility in* $\mathcal{L}$ *implies the admissibilty of* $\vdash A_1 \vee C, \ldots, \vdash A_n \vee C \Longrightarrow \vdash B \vee C$

Theorem 1. *Any sub-classical logic* $\mathcal{L}$ *with function symbols that has a duality principle and in which the rules* weak, assoc-l, assoc-r, comm, idem, distr, $\exists$-in' *are admissible and* $\vee$*-normal is undecidable.*

Proof. The proof proceeds as follows: we consider any classical formula F in conjunctive normal form and apply Herbrand's theorem. By using the duality principle we can translate the resulting Herbrand instance F_{Her} into a formula $\delta(F_{Her})$ such that $\vdash_{\mathbf{CL}^f} F_{Her}$ if and only if $\vdash_{\mathcal{L}} \delta(F_{Her})$. This relation transfers to F and $\delta(F)$. Hence follows that $\mathcal{L}$ is undecidable.

It is well known that validity (provability) of formulas of form

$$F = \exists \boldsymbol{x}\colon \bigvee_{1 \le i \le m} \bigwedge_{1 \le j \le n_i} L_{i,j}(\boldsymbol{x})$$

where the $L_{i,j}$ are literals and $\boldsymbol{x}$ is a vector of variables, is undecidable in classical logic with function symbols.

Assume that $\vdash_{\mathbf{CL}^f} F$. By Herbrand's theorem there exist vectors of ground terms $\boldsymbol{t}^{(1)}, \ldots \boldsymbol{t}^{(k)}$ such that also the Herbrand instance

$$F_{Her} = \bigvee_{1 \le \ell \le k} \bigvee_{1 \le i \le m} \bigwedge_{1 \le j \le n_i} L_{i,j}(\boldsymbol{t}^{(\ell)})$$

is provable.

A disjunction $\bigvee_{1 \le i \le (km)} L_i$ is called a *path* of F_{Her} if each of the literals L_i occurs in exactly one of the "clauses" $\bigwedge_{1 \le j \le m_i} L_{i,j}(\boldsymbol{t}^{(\ell)})$ of F_{Her}. A pair of literals of the form $(P(\boldsymbol{t}), \neg P(\boldsymbol{t}))$ is called *complementary*. It is easy to check that $\vdash_{\mathbf{CL}^f} F_{Her}$ iff every path π of F_{Her} contains a complementary pair $(P(\boldsymbol{t}), \neg P(\boldsymbol{t}))$, i.e. if both $P(\boldsymbol{t})$ and $\neg P(\boldsymbol{t})$ occur as disjuncts in π.

Let $\alpha[p] \vee \overline{\alpha}[p]$ be a duality principle of $\mathcal{L}$. Consider the following translation $\delta(.)$ of literals:

$$\begin{aligned} \delta(P) &= \alpha[P/p] \\ \delta(\neg P) &= \overline{\alpha}[P/p] \end{aligned}$$

The translation extends homomorphically to arbitrary formulas.

We first show that the classical provability of F_{Her} implies that $\vdash_{\mathcal{L}} \delta(F_{Her})$. Let $(P(\boldsymbol{t}), \neg P(\boldsymbol{t}))$ be the complementary pair of a path π of F_{Her}. By definition we have $\delta(P(\boldsymbol{t}) \vee \neg P(\boldsymbol{t})) = \delta(P(\boldsymbol{t})) \vee \delta(\neg P(\boldsymbol{t})) = \alpha[P(\boldsymbol{t})/p] \vee \overline{\alpha}[P(\boldsymbol{t})/p]$, and therefore $\vdash_{\mathcal{L}} \delta(P(\boldsymbol{t}) \vee \neg P(\boldsymbol{t}))$. Using rule weak we can add all the other literals of π disjunctively to $\delta(P(\boldsymbol{t}) \vee \neg P(\boldsymbol{t}))$ in $\mathcal{L}$. By the admissibility of the rules assoc-l, assoc-r, comm and the fact that they are $\vee$-normal in $\mathcal{L}$ we can reorder the disjuncts as needed. After having derived (in $\mathcal{L}$) the formulas corresponding to every path of F_{Her}, we join them conjunctively using rule distr. We thus have

$$\vdash_{\mathcal{L}} \delta(F_{Her})$$

Now since rule $\exists$-in' is admissible and $\vee$-normal we can re-introduce the existential quantifiers in front of the inner disjunction. I.e. we obtain

$$\vdash_{\mathcal{L}} \bigvee_{1\leq \ell \leq k} \exists \boldsymbol{x}\colon \bigvee_{1\leq i\leq m} \bigwedge_{1\leq j\leq n_i} \delta(L_{i,j}(\boldsymbol{x})).$$

Observe that we used the same variables as in the original formula F. Therefore all k disjuncts are identical. Since rule idem is admissible and $\vee$-normal we can contract this to

$$\vdash_{\mathcal{L}} \delta(F).$$

We thus have proved: $\vdash_{\mathbf{CL}f} F$ implies $\vdash_{\mathcal{L}} \delta(F)$.

For the other direction, observe that $\vdash_{\mathcal{L}} \delta(F)$ implies $\vdash_{\mathbf{CL}f} \delta(F)$ since $\mathcal{L}$ is sub-classical. It remains to check that $\delta(F)$ is equivalent to F in classical logic with respect to provability. For this observe that, since p is the only formula variable of the duality principle, $\alpha[p]$ is classically equivalent either to p or to $\neg p$. (It cannot be constantly equivalent to $\top$ by $\not\vdash_{\mathbf{CL}} \alpha[G/p]$, for some G. It also cannot be constantly equivalent to $\bot$ since all instances of $\alpha[p] \vee \overline{\alpha}[p]$ are provable in $\mathcal{L}$ and therefore also in classical logic.) The same holds for $\overline{\alpha}[p]$. Since all instances of $\alpha[p] \vee \overline{\alpha}[p]$ are provable in $\mathcal{L}$ and therefore also in classical logic, we conclude that

either: $\alpha[p] \equiv p$ and $\overline{\alpha}[p] \equiv \neg p$,
or: $\alpha[p] \equiv \neg p$ and $\overline{\alpha}[p] \equiv p$.

where $\equiv$ denotes classical equivalence. Both ways the duality principle is classically equivalent to $p \vee \neg p$. Summarizing, $\vdash_{\mathcal{L}} \delta(F)$ implies $\vdash_{\mathbf{CL}f} F$, qed.

¿From the proof of Theorem 1 we immediately obtain:

Corollary 1. *The existential fragments of logics fullfilling the conditions stated in the previous theorem are undecidable.*

To obtain a useful criterion for undecidability of logics without function symbols we need to consider also the universal quantifier. Consider the standard rule for universal quantification:

$$A(x) \vdash \forall x\colon A(x) \qquad (\forall\text{-in}')$$

Remark 2. We require that, if rule $\forall$-in' is $\vee$-normal, the *eigenvariable condition* is obeyed in its application. I.e., in

$$A(x) \vee C \vdash (\forall x\colon A(x)) \vee C \qquad (\forall\text{-in}')$$

x must not occur in C.

If, in addition to the assumptions of Theorem 1 we require $\forall$-in' to be admissible and $\vee$-normal (including the eigenvariable condition) we obtain undecidablity for logics without function symbols.

Theorem 2. *Any sub-classical logic $\mathcal{L}$ that has a duality principle and in which the rules* weak, assoc-l, assoc-r, comm, idem, distr, $\exists$-in', *as well as* $\forall$-in' *are admissible and $\vee$-normal is undecidable, even without function symbols.*

Proof. The proof is analogous to that of Theorem 1. We start with formulas in prenex conjuntive normal form:

$$F = \mathsf{Q}x_1 \ldots \mathsf{Q}x_n\colon \bigvee_{1 \le i \le m} \bigwedge_{1 \le j \le n_i} L_{i,j}(\boldsymbol{x}),$$

where $\mathsf{Q} \in \{\exists, \forall\}$ and $\boldsymbol{x} = x_1, \ldots, x_n$.

Since we have no function symbols we apply Herbrand's theorem in its orginal form (i.e., without Skolemizing). In the Herbrand instance

$$F_{Her} = \bigvee_{1 \le \ell \le k} \bigvee_{1 \le i \le m} \bigwedge_{1 \le j \le n_i} L_{i,j}(\boldsymbol{y}^{(\ell)})$$

the variables x_i are replaced by variables $y_i^{(\ell)}$ in such a way that introducing the quantifiers in a proof of F given F_{Her}, can be done without violating the eigenvariable condition. Again the proof can be transferred to the logic $\mathcal{L}$ using the duality principle and the fact that the stated rules are admissible and $\vee$-normal. (Note that it is essential that $\forall$-in' is applied in its $\vee$-normal variant, obeying the eigenvariable condition.) The rest of the proof proceeds exactly as in Theorem 1. I.e., we have $\vdash_{\mathbf{CL}} F$ if only if $\vdash_{\mathcal{L}} \delta(F)$.

4 First-order extensions of Urquhart's (original) C

In [3] we investigated some propositional logics underlying the most important formalizations of fuzzy logic [8]. Here we focus attention to the extensions of these logics to the *first-order level.*

Our logics are defined by their Hilbert-style axiomatizations. We refer to the following list of axioms (cf. [3]):

Ax1: $A \supset (B \supset A)$	*Res1* : $[(A \wedge B) \supset C] \supset [A \supset (B \supset C)]$
Ax2: $(A \supset B) \supset [(C \supset A) \supset (C \supset B)]$	*Res2* : $[(A \supset (B \supset C)] \supset [(A \wedge B) \supset C]$
Ax3: $[A \supset (C \supset B)] \supset [(C \supset (A \supset B)]$	*Lin* : $(A \supset B) \vee (B \supset A)$
Ax4: $(A \wedge B) \supset A$	*Com* : $[A \wedge (A \supset B)] \supset [B \wedge (B \supset A)]$
Ax5: $(A \wedge B) \supset B$	*Abs* : $\perp \supset A$
Ax6: $A \supset [C \supset (A \wedge C)]$	*Contr* : $[A \supset (A \supset B)] \supset (A \supset B)$
Ax7: $A \supset (A \vee B)$	*Dneg* : $\neg\neg A \supset A$
Ax8: $B \supset (A \vee B)$	
Ax9: $(A \supset B) \supset [(C \supset B) \supset [(A \vee C) \supset B]]$	

Taking *Modus Ponens* as the only rule of derivation we define the propositional fragments of the logics $\mathbf{I}^-$, $\mathbf{I}^{-*}$, $\mathbf{C}$, and $\mathbf{C}^*$ by the following sets of axioms[3]:

$$\begin{array}{ll} \mathbf{I}^- \;: \{Ax1, \ldots, Ax9, Abs\} & \mathbf{C} \;: \mathbf{I}^- \cup \{Lin\} \\ \mathbf{I}^{-*} : \mathbf{I}^- \cup \{Res1, Res2\} & \mathbf{C}^* : \mathbf{I}^{-*} \cup \{Lin\} \end{array}$$

The first-order logics are given by adding to the corresponding propositional systems the axioms:

$$Ax\forall : \forall x\colon A(x) \supset A(t) \qquad\qquad Ax\exists : A(t) \supset \exists x\colon A(x)$$

and the rules

$$\vdash B \supset A(x) \Longrightarrow \vdash B \supset \forall x\colon A(x) \quad (\forall\text{-in}) \qquad \vdash A(x) \supset B \Longrightarrow \vdash \exists x\colon A(x) \supset B \quad (\exists\text{-in})$$

The quantifier axioms and rules are subject to the usual variable conditions: t is free for x and x does not ocur free in B.

$\mathbf{C}$ coincides with Urquhart's "basic" many-valued logic as defined in [16] (see also [13]). $\mathbf{I}^{-*}$ extends to intuitionistic logic $\mathbf{I}$ by adding *Contr* (contraction). By adding to $\mathbf{I}^{-*}$ the axiom *Dneg* (involutivity of negation) we obtain the affine multiplicative fragment of linear logic [6] **aMLL** (extended with the additive disjunction). Hájek's Basic Logic **BL** [8] corresponds to the logic obtained by adding axiom *Com* to $\mathbf{C}^*$ (as proved in [3]). If we add to **BL** axiom *Dneg* we get Łukasiewicz logic [12]; while by extending **BL** with the axioms $\neg\neg A \supset ((B \wedge A \supset C \wedge A) \supset (B \supset C))$ and $A \wedge \neg A \supset \perp$ we obtain Product logic [9]. $\mathbf{C}$, $\mathbf{C}^*$, and **BL** all turn into Gödel logic $\mathbf{G}_\infty$ [7, 5] if we add axiom *Contr*.

In [3, 1] analytic—i.e., cut-free Gentzen-style—calculi for the *propositional versions* of the logics $\mathbf{I}^-$, $\mathbf{I}^{-*}$, $\mathbf{C}$, $\mathbf{C}^*$, and $\mathbf{G}_\infty$ have been introduced. These calculi are called $\mathbf{LJ}^-$, $\mathbf{LJ}^{-*}$, **HC**, $\mathbf{HC}^*$ and **GLC**, respectively. As a consequence one obtains that the derivability problem for these logics is

[3] In fact, *Res1* is redundant in presence of *Ax1, Ax2, Ax3*.

decidable and is at most in PSPACE. **HC**, **HC***and **GLC** are based on *hypersequents*—a simple and natural generalization of Gentzen sequents to *multisets* of sequents (see [2] for an overview).

The axioms and rules of the calculus **HC** are as follows:

Axioms:

$$A \vdash A \qquad \bot \vdash A$$

Cut Rule:

$$\frac{G \mid \Gamma_1 \vdash A \quad G' \mid A, \Gamma_2 \vdash B}{G \mid G' \mid \Gamma_1, \Gamma_2 \vdash B} \ (\mathrm{cut})$$

Structural Rules:

$$\frac{G \mid \Gamma \vdash C}{G \mid \Gamma, A \vdash C} \ (\mathrm{W}) \qquad \frac{G \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta \mid \Gamma' \vdash \Delta'} \ (\mathrm{EW})$$

$$\frac{G \mid \Gamma \vdash \Delta \mid \Gamma \vdash \Delta}{G \mid \Gamma \vdash \Delta} \ (EC) \qquad \frac{G \mid \Pi_1, \Gamma_1 \vdash A \quad G' \mid \Pi_2, \Gamma_2 \vdash B}{G \mid G' \mid \Pi_1, \Pi_2 \vdash A \mid \Gamma_1, \Gamma_2 \vdash B} \ (\mathrm{Comm})$$

Logical Rules:

$$\frac{G \mid \Gamma, A \vdash B}{G \mid \Gamma \vdash A \supset B} \ (\supset\text{-right}) \qquad \frac{G \mid \Gamma_1 \vdash A \quad G' \mid B, \Gamma_2 \vdash C}{G \mid G' \mid \Gamma_1, \Gamma_2, A \supset B \vdash C} \ (\supset\text{-left})$$

$$\frac{G \mid \Gamma_1 \vdash A \quad G' \mid \Gamma_2 \vdash B}{G \mid G' \mid \Gamma_1, \Gamma_2 \vdash A \wedge B} \ (\wedge\text{-right}) \qquad \frac{G \mid \Gamma, A_i \vdash C}{G \mid \Gamma, A_1 \wedge A_2 \vdash C} \ (\wedge_i\text{-left})$$

$$\frac{G \mid \Gamma \vdash A_i}{G \mid \Gamma \vdash A_1 \vee A_2} \ (\vee_i\text{-right}) \qquad \frac{G \mid \Gamma, A \vdash C \quad G' \mid \Gamma, B \vdash C}{G \mid G' \mid \Gamma, A \vee B \vdash C} \ (\vee\text{-left})$$

The calculus **HC*** is obtained by substituting in the above calculus the ($\wedge_i$−left) rule with the following one:

$$\frac{G \mid \Gamma, A, B \vdash C}{G \mid \Gamma, A \wedge B \vdash C} \ (\wedge\text{-left})$$

The calculus **GLC** (see [1]) is obtained by simply adding the contraction rule to **HC** or **HC***. The *first-order* calculi **HC**, **HC***, and **GLC** are obtained by adding to the corresponding calculi, the following rules:

$$\frac{G \mid \Gamma \vdash F(a)}{G \mid \Gamma \vdash \forall x F(x)} \ (\forall\text{-right}) \qquad \frac{G \mid \Gamma, F(t) \vdash C}{G \mid \Gamma, \forall x F(x) \vdash C} \ (\forall\text{-left})$$

$$\frac{G \mid \Gamma \vdash F(t)}{G \mid \Gamma \vdash \exists x F(x)} \ (\exists\text{-right}) \qquad \frac{G \mid \Gamma, F(a) \vdash C}{G \mid \Gamma, \exists x F(x) \vdash C} \ (\exists\text{-left})$$

where—in the rules ($\forall$-right) and ($\exists$-left)—a does not occur in the lower hypersequent.

The correctness and completeness results for our hypersequent calculi (with respect to the Hilbert style systems) are easily extended from the propositional level (see [3]) to the first-order level.

Theorem 3. *Provability of "$\vdash F$" in* **HC**, **HC***, *and* **GLC** *coincides with provability of "F" in* **C**, **C***, *and* $\mathbf{G}_\infty$, *respectively.*

Corollary 2. *(1) The existential fragments and of the logics* **C**, **C***, *and* $\mathbf{G}_\infty$ *with function symbols are undecidable. (2) The prenex fragments of these logics without function symbols are undecidable.*

Proof. It is easy to check that the rules weak, assoc-l, assoc-r, comm, idem, distr, $\exists$-in' are both: admissible and $\vee$-normal, in the mentioned logics.

The schema $(p \supset \neg p) \vee (\neg p \supset p)$, which is an instance of the linearity axiom is a duality principle for **C**, **C***, and $\mathbf{G}_\infty$. Indeed, by using Avron's Communication rule (Comm) it can be derived in the corresponding calculi (see [1, 3]) but neither $p \supset \neg p$ nor $\neg p \supset p$ is provable in classical logic. Therefore the statements (1) and (2) follow from Theorems 1 and Theorem 2, respectively.

This should be contrasted with the fact that decision procedures for $\mathbf{I}^-$ and $\mathbf{I}^{-*}$ can directly be extracted from corresponding cut-free calculi as proved in [14].

Finally, we remark that our general undecidability results also apply to (other) important fuzzy logics:

Corollary 3. *Also for* **BL**, *Lukasiewicz and Product logic the existential fragment with function symbols and the prenex fragments without function symbols are undecidable.*

5 Analyticity and a decidable subclass

Proposition 1. *The first-order calculi* $\mathbf{LJ}^-$, $\mathbf{LJ}^{-*}$ *and* **LJ** *admit cut-elimination.*

Proof. (Sketch) For the sequent calculus **LJ** for intuitionistic logic, this is well-known. An inspection of the classical proof shows that the absence of contraction does not affect cut-elimination.

In [3] ([1]) it was proved that the hypersequent calculi **HC** and **HC*** (respectively, **GLC**) admit cut-elimination. In order to extend these results to the first-order level we need the following "eigenvariable" lemma. (The straightforward proof is omitted for space reasons.)

Lemma 1. *Let $\mathcal{P}(a)$ be a proof in the first-order calculus* **HC** *(***HC****,* **GLC***) of the hypersequent S that contains the variable a. If, throughout the proof, we replace a by a term t, containing only variables that do not occur in $\mathcal{P}(a)$, we obtain a proof $\mathcal{P}(t)$ ending with $S[t/a]$.*

Theorem 4. *First-order* $\mathbf{HC}, \mathbf{HC}^*$ *and* $\mathbf{GLC}$ *admit cut-elimination.*

Proof. It is enough to show that if $\mathcal{P}$ is a proof of a hypersequent H containing only one cut rule which occurs as the last inference of $\mathcal{P}$, then H is derivable without the cut rule.

Cut-elimination for hypersequent calculi works essentially in the same way as for the corresponding sequent calculi. As was noticed in [4], a simple way to make the inductive argument work in the presence of the (EC) rule, is to consider the number of applications of this rule in a given derivation as an independent parameter. The proof will proceed by induction on lexicographically-ordered triple (r, c, h), where r is the number of the applications of the (EC) rule in the proofs of the premises of the cut rule, c is the complexity of the cut formula, and h is the sum of the length of the proofs of the premises of the cut rule.

It suffices to consider the following cases according to which inference rule is being applied just before the application of the cut rule:

1. either $G \mid \Gamma \vdash A$ or $G' \mid \Gamma', A \vdash B$ is an initial hypersequent;
2. either $G \mid \Gamma \vdash A$ or $G' \mid \Gamma', A \vdash B$ is obtained by a structural rule;
3. both $G \mid \Gamma \vdash A$ and $G' \mid \Gamma', A \vdash B$ are lower sequents of some logical rules such that the principal formulas of both rules are just the the cut formulas;
4. either $G \mid \Gamma \vdash A$ or $G' \mid \Gamma', A \vdash B$ is a lower sequent of a logical rule whose principal formula is not the cut formula.

We will give here a proof for some relevant cases. Suppose that the last inference in the proof of one premise of the cut is the (EC) rule, and the proof ends as follows:

$$\dfrac{\dfrac{G \mid \Gamma \vdash A \mid \Gamma \vdash A}{G \mid \Gamma \vdash A}\,(\mathrm{EC}) \qquad G' \mid A, \Gamma' \vdash B}{G \mid G' \mid \Gamma', \Gamma \vdash B}\,(\mathrm{cut})$$

Let r' be the number of applications of the (EC) rule in the above proof. This proof can be replaced by

$$\dfrac{\dfrac{\dfrac{G \mid \Gamma \vdash A \mid \Gamma \vdash A \quad G' \mid A, \Gamma' \vdash B}{G \mid G' \mid \Gamma, \Gamma' \vdash B \mid \Gamma \vdash A}\,(\mathrm{cut}) \qquad G' \mid A, \Gamma' \vdash B}{G \mid G' \mid G' \mid \Gamma', \Gamma \vdash B \mid \Gamma', \Gamma \vdash B}\,(\mathrm{cut})}{G \mid G' \mid \Gamma', \Gamma \vdash B}\,(\mathrm{EC})$$

which contains two cuts with $r' - 1$ applications of the (EC) rule. Then, they can be eliminated by induction hypothesis.

Cases concerning the remaining structural rules and the logical ones are treated as in the corresponding proofs of the cut-elimination theorem for the hypersequent calculi $\mathbf{HC}, \mathbf{HC}^*$ and $\mathbf{GLC}$ (see [3]). In the following we show how to eliminate a cut involving the rules for quantifiers.

Suppose that the last inference in the proof of one premise of the cut is the ($\exists$-left) rule, and the proof, say $\mathcal{P}$, ends as follows:

$$\dfrac{\dfrac{G \mid B(a), \Gamma \vdash A}{G \mid \exists x B(x), \Gamma \vdash A}\,(\exists\text{-left}) \qquad G' \mid \Sigma, D(a), A \vdash C}{G \mid G' \mid \exists x B(x), \Gamma, \Sigma, D(a) \vdash C}\,(\text{cut})$$

Let $\mathcal{P}(a)$ be the proof ending with $G' \mid \Sigma, D(a), A \vdash C$. By Lemma 1, by replacing a by b throughout $\mathcal{P}(a)$ one gets a proof ending with $G'[b/a] \mid \Sigma[b/a], D(a), A \vdash C$ Thus the proof $\mathcal{P}$ can be replaced by

$$\dfrac{\dfrac{G \mid B(a), \Gamma \vdash A \qquad G'[b/a] \mid \Sigma[b/a], D(b), A \vdash C}{G \mid G'[b/a] \mid B(a), \Gamma, \Sigma[b/a], D(b) \vdash C}\,(\text{cut})}{G \mid G'[b/a] \mid \exists x B(x), \Gamma, \Sigma[b/a], D(b) \vdash C}\,(\exists\text{-left})$$

in which the cut is shifted upward, so it can be eliminated by induction hypothesis.

Suppose that both the premises of the cut rule are lower sequents of the rules for the quantifier $\exists$ and that the cut formula is the principal formula of both rules, that is

$$\dfrac{\dfrac{G \mid \Gamma \vdash B(t)}{G \mid \Gamma \vdash \exists x B(x)}\,(\exists\text{-right}) \qquad \dfrac{G' \mid \Sigma, B(a) \vdash C}{G' \mid \Sigma, \exists x B(x) \vdash C}\,(\exists\text{-left})}{G \mid G' \mid \Gamma, \Sigma \vdash C}\,(\text{cut})$$

Let $\mathcal{P}(a)$ and $\mathcal{Q}(a)$ be the proofs ending with $G \mid \Gamma \vdash B(a)$ and $G' \mid \Sigma, B(t) \vdash C$, respectively. By repeatedly applying Lemma 1 one can replace the previous proof by the following one:

$$\dfrac{\Gamma \vdash B(t) \qquad \Sigma, B(t) \vdash C}{\Gamma, \Sigma \vdash C}\,(\text{cut})$$

that contains a cut whose complexity of the cut formula is smaller than that one of the cut in the previous proof. Then it can be eliminated by induction hypothesis.

Cases involving the quantifier $\forall$ can be treated similarly.

claim that rule—does concerned.

As an easy consequence of cut-elimination we obtain the decidabilty of the fragments of $\mathbf{C}$, $\mathbf{C}^*$, and $\mathbf{G}_\infty$, respectively, where quantifiers occur only strongly:

Corollary 4. *Let H consist of sequents $\Gamma_i \vdash C_i$, where the formulas of Γ_i are purely existential and C_i is purely universal. The derivability of H in* **HC**, **HC***, *as well as* **GLC** *is decidable.*

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