1 Sequent Calculus $LK$ for FOL

It is obtained by adding to $LK$ the following rules for quantifiers

\[ \Gamma, A(y) \vdash \Delta \quad \text{\textit{(3l)}} \quad y \text{ is eigenvariable, i.e. } y \notin \Gamma \cup \Delta \]

\[ \Gamma, \exists x A(x) \vdash \Delta \quad \text{\textit{(3r)}} \]

\[ \Gamma, A(t) \vdash \Delta \quad \text{\textit{(4l)}} \quad y \text{ is eigenvariable, i.e. } y \notin \Gamma \cup \Delta \]

\[ \Gamma, \forall x A(x) \vdash \Delta \quad \text{\textit{(4r)}} \]

Example 1.1.

\[ A(y) \vdash A(y) \]
\[ \vdash \neg A(y), A(y) \quad \text{\textit{(-r)}} \]
\[ \vdash \exists x \neg A(x), A(y) \quad \text{\textit{(3r)}} \text{ we can choose any witness } \implies \text{ choose } y \]
\[ \vdash \neg \forall x A(x), \forall x A(x) \quad \text{\textit{(4r)}} \]
\[ \vdash \neg \forall x A(x) \vdash \exists x \neg A(x) \quad \text{\textit{(-l)}} \]
1.1 Soundness

Proposition 1.1. \( \Gamma \vdash P \implies \Gamma \models P \)

Proof. By induction on the length of the derivation \( \Gamma \vdash P \). E.g.

\((\forall r)\): Let \( P = \forall x P_1 \). By inductive hypothesis \( \Gamma \models P_1[y/x] \), i.e. for each interpretation \((A, \xi^A)\) such that \( (A, \xi^A) \models \Gamma \) then \( (A, \xi^A) \models P_1[y/x] \). Since \( y \notin \Gamma \), \( (A, \xi^A[b/y]) \models \Gamma \forall b \in D_A \) and \( (A, \xi^A[b/y]) \models P_1[y/x] \), \( (A, \xi^A[b/x]) \models P_1 \) and thus \( (A, \xi^A) \models \forall x P_1 \).

\(\square\)

Example 1.2.

\[
\begin{align*}
A(t_1) \vdash A(t_1) & \quad \text{(}\exists r\text{)} & A(t_2) \vdash A(t_2) & \quad \text{(}\exists r\text{)} \\
A(t_1) \vdash \exists x A(x) & \quad \text{(}\exists r\text{)} & A(t_2) \vdash \exists x A(x) & \quad \text{(}\exists r\text{)} \\
\hline
A(t_1) \lor A(t_2) \vdash \exists x A(x) & \quad \text{(}\forall l\text{)}
\end{align*}
\]
1.2 Sequent Calculus $LK'$ for FOL

The first-order version of $LK'$ is obtained by adding to $LK'$ the rules $(\forall r)$, $(\exists l)$ together with

$$
\Gamma, A(t), \forall x A(x) \vdash \Delta \quad (\forall l) \\
\Gamma, \forall x A(x) \vdash \Delta \\
\Gamma \vdash A(t), \exists x A(x), \Delta \\
\Gamma \vdash \exists x A(x), \Delta 
$$

Example 1.3.

$$
A(t_1), A(t_2), A(t_3), \forall x A(x) \vdash A(t_3) \vee B \quad (\forall l) \\
A(t_1), A(t_2), \forall x A(x) \vdash A(t_3) \vee B \quad (\forall r) \\
A(t_1), A(t_2), \forall x A(x) \vdash \forall x (A(x) \vee B) \quad (\forall l) \\
\forall x A(x) \vdash \forall x (A(x) \vee B) \quad (\forall l)
$$

Note that we could generated infinite branches!!!!!!!
1.3 Completeness

\[ \Gamma \vdash P \implies \Gamma \vdash P \]

propositional case: 

\[ \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \downarrow \]

\[ \Gamma \vdash \Delta \]

either leaves are axioms or we can find a counterexample for \( \Gamma \models \Delta \).

**Definition 1.1.** Given an interpretation \((A, \xi_A)\) and a sequent \(\Gamma \vdash \Delta\), a formula \(P\) in the sequent is a *positive constituent* if \(P \in \Delta\) and \(v^{(A, \xi_A)}(P) = 1\) or \(P \in \Gamma\) and \(v^{(A, \xi_A)}(P) = 0\).

**Proposition 1.2.** Given \((A, \xi_A), \Gamma \vdash \Delta\) and a positive constituent \(P\). If \(P\) is \(\neg A\), \(A \star B\) \((\star \in \{\land, \lor, \rightarrow\})\), \(\forall x A\) (in \(\Delta\)) or \(\exists x A\) (in \(\Gamma\)) then in each sequent premise of an introduction rule for \(P\), one of the auxiliary formulas of the rule is a positive constituent.
We apply the following algorithm backward

**Algorithm 1 LK’ proof**

**Require:** sequent $\Gamma \vdash \Delta$

1: the algorithmus tries to apply backward a logical (or quantifier) rule applied to the first compound formula of $\Gamma$ for odd steps, and to $\Delta$ for even steps.

2: the auxiliary formulae generated are added to the end.

The algorithm stops on a branch when a sequent $S$, satisfying the properties below, is found:

- Success ($S$ is an Axiom)

- Failure ($S$ is of the form $\Pi \vdash \Sigma$, where $\Pi, \Sigma$ only contain atomic formulas and no common formula, i.e. $\Pi \cap \Sigma = \emptyset$)

**Proposition 1.3.** If a sequent $S_i$ contains a positive constituent of complexity $n + 1$, it does exist a sequent $S_j$, with $j \geq i$ containing a positive constituent of complexity $n$. 
We show that applying the algorithm above either we find a proof \((\Gamma \vdash \Delta)\) or \(\Gamma \not\models \Delta\).

Two cases can arise:

1. we find a closed tree

2. otherwise

In case 1. \(\Gamma \vdash \Delta\). In case 2. we have to show that \(\Gamma \not\models \Delta\). Two cases can occur: 2.1 there is a failure branch. Then \(\Gamma \not\models \Delta\). 2.2 there is an infinite branch

\[
\vdots
\]
\[
\Gamma_2 \vdash \Delta_2
\]
\[
\Gamma_1 \vdash \Delta_1
\]
\[
\Gamma_0 \vdash \Delta_0
\]

Let \(\mathcal{M}\) be the set of atomic formulas in \(\Gamma_0, \Gamma_1, \ldots\) and \(\mathcal{N}\) the set of atomic formulas in \(\Delta_0 \ldots\).
We have to find an interpretation that is a countermodel for $\Gamma \models \Delta$. This interpretation (canonical interpretation) $(\mathcal{A}, \xi^\mathcal{A})$ is defined as

$$P^{(\mathcal{A}, \xi^\mathcal{A})}(t_1, \ldots, t_n) = 1 \text{ if } f P(t_1, \ldots, t_n) \in \mathcal{M}$$

**Proof.** By contradiction. Assume $\Gamma \models \Delta$. Then it has to contain at least a positive constituent of order $n$. By $n$ applications of Proposition 1.3 we get a contradiction. \qed

### 1.4 Cut-Elimination Theorem

**Corollary 1.1.** The (CUT) rule is redundant in $LK'$ ($LK$ and $LJ$).

- the proof proceeds by induction on the pair $(c, r)$ ... $c$ is the complexity of (CUT), $r$ is the rank of (CUT)
- how to remove applications of (CUT) with cut-formula $\forall x A(x)$ that is principal in both premises of the cut $\implies$
\[(\land l)\quad \frac{\Xi, \Gamma, A, B \vdash \Delta}{\Xi, A \land B, \Gamma \vdash \Delta}\]

\[(\land r)\quad \frac{\Gamma \vdash \Xi, \Delta, A}{\Gamma \vdash \Xi, A \land B, \Delta}\]

\[(\lor l)\quad \frac{\Xi, \Gamma, A \vdash \Delta \quad \Xi, \Gamma, B \vdash \Delta}{\Xi, A \lor B, \Gamma \vdash \Delta}\]

\[(\lor r)\quad \frac{\Gamma \vdash \Xi, \Delta, A, B}{\Gamma \vdash \Xi, A \lor B, \Delta}\]

\[(\to l)\quad \frac{\Xi, \Gamma \vdash \Delta, A \quad \Xi, \Gamma, B \vdash \Delta}{\Xi, A \to B, \Gamma \vdash \Delta}\]

\[(\to r)\quad \frac{\Gamma, A \vdash \Xi, B, \Delta}{\Gamma \vdash \Xi, A \to B, \Delta}\]

\[(\neg l)\quad \frac{\Xi, \Gamma \vdash \Delta, A}{\Xi, \neg A, \Gamma \vdash \Delta}\]

\[(\neg r)\quad \frac{\Gamma, A \vdash \Xi, \Delta}{\Gamma \vdash \Xi, \neg A, \Delta}\]

\[(\forall l)\quad \frac{\Xi, \Gamma, A[t/x] \vdash \Delta}{\Xi, \forall x A, \Gamma \vdash \Delta}\]

\[(\forall r)\quad \frac{\Gamma \vdash \Xi, \Delta, A[y/x]}{\Gamma \vdash \Xi, \forall x A, \Delta}\]

\[(\exists l)\quad \frac{\Xi, \Gamma, A[y/x] \vdash \Delta}{\Xi, \exists x A, \Gamma \vdash \Delta}\]

\[(\exists r)\quad \frac{\Gamma, \vdash \Xi, \Delta, A[t/x]}{\Gamma \vdash \Xi, \exists x A, \Delta}\]