BHK Interpretation

(Brower Heyting Kolmogorow)
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A construction of $P \land Q$ consists of a construction of $P$ and a construction of $Q$. 
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- A construction of $P \land Q$ consists of a construction of $P$ and a construction of $Q$.

- A construction of $P \lor Q$ consists of a construction of $P$ or a construction of $Q$. 
BHK Interpretation
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- A construction of $P \wedge Q$ consists of a construction of $P$ and a construction of $Q$.
- A construction of $P \lor Q$ consists of a construction of $P$ or a construction of $Q$.
- A construction of $P \rightarrow Q$ is a method (function) transforming every construction of $P$ into a construction of $Q$. 
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- There is no possible construction for $\bot$. 
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- A construction of $P \land Q$ consists of a construction of $P$ and a construction of $Q$.
- A construction of $P \lor Q$ consists of a construction of $P$ or a construction of $Q$.
- A construction of $P \rightarrow Q$ is a method (function) transforming every construction of $P$ into a construction of $Q$.
- There is no possible construction for $\bot$.
- A construction of $\neg P$ is a method that turns every construction of $P$ into some absurd statement.
Kripke Models
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Def.
A Kripke model is $C = (C, \leq, \Vdash)$, where $C$ is a non-empty set, $\leq$ is a partial order in $C$ and $\Vdash$ ("forces") is a binary relation between elements of $C$ (states or possible worlds) and propositional variables.
A Kripke model is $\mathcal{C} = (C, \leq, \models)$, where $C$ is a non-empty set, $\leq$ is a partial order in $C$ and $\models$ (“forces”) is a binary relation between elements of $C$ (states or possible worlds) and propositional variables, that satisfies the following monotonicity condition: If $c \leq c'$ and $c \models A$ then $c' \models A$ ($c, c' \in C'$)
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possible worlds . . . states of knowledge
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- possible worlds ... states of knowledge
- $\leq$ ... extending states by gaining more knowledge
- $\forces$ ... which atomic formulae are known to be true in a given state
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A Kripke model is $\mathcal{C} = (C, \leq, \models)$, where $C$ is a non-empty set, $\leq$ is a partial order in $C$ and $\models$ (“forces”) is a binary relation between elements of $C$ (states or possible worlds) and propositional variables, that satisfies the following monotonicity condition: If $c \leq c'$ and $c \models A$ then $c' \models A$ $(c, c' \in C)$. Let $\mathcal{C} = (C, \leq, \models)$ be a Kripke model. Then

- $c \models p$ iff $p$ is assigned to the node $c$
- $c \models P \lor Q$ iff $c \models P$ or $c \models Q$
- $c \models P \land Q$ iff $c \models P$ and $c \models Q$
- $c \models P \rightarrow Q$ iff $\forall c' \geq c: \text{if } c' \models P \text{ then } c' \models Q$
- $c \models \neg P$ iff for no $c' \geq c$: $c' \models P$ (for all $c' \geq c : c' \not\models P$)
Kripke Models

Def.
A Kripke model is $\mathbb{C} = (C, \leq, \models)$, where $C$ is a non-empty set, $\leq$ is a partial order in $C$ and $\models$ (“forces”) is a binary relation between elements of $C$ (states or possible worlds) and propositional variables, that satisfies the following monotonicity condition: If $c \leq c'$ and $c \models A$ then $c' \models A$ ($c, c' \in C$). Let $\mathbb{C} = (C, \leq, \models)$ be a Kripke model. Then

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- $c \models P \land Q$ iff $c \models P$ and $c \models Q$
- $c \models P \rightarrow Q$ iff $\forall c' \geq c$: if $c' \models P$ then $c' \models Q$
- $c \models \neg P$ iff for no $c' \geq c$: $c' \models P$ (for all $c' \geq c$: $c' \not\models P$)

We say that $\mathbb{C} \models P$ if $c \models P \ \forall c \in C$
Proof Theoretic Approach

Def.
Deductive system (formal system) $\Sigma = (L, W, R, Ax)$ with
- $L$ is a language
- $W \subseteq L$ is a decidable set of WFF
- $R$ is a finite set of $n$-ary relations on $W$ (inference rules)
- $Ax \subseteq W$ (axioms)
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Let $\Sigma$ be a deductive system. A proof in $\Sigma$ is a finite sequence of WFF $P_1, \ldots, P_n$ s.t. for all $i = 1, \ldots, n$ either $P_i \in Ax$ or $P_i$ is a direct consequence using a rule in $R$ of $P_1, \ldots, P_{i-1}$. 
Intuitive Properties
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\( (\land e_1) A \land B \vdash A \) and \( (\land e_2) A \land B \vdash B \)
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- $(\land e_1) \ A \land B \vdash A$ and $(\land e_2) A \land B \vdash B$
- $(\land i) A, B \vdash A \land B$
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- $(\land i) A, B \vdash A \land B$
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- $(\wedge i) A, B \vdash A \land B$
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- $(\lor e) \text{if } A \vdash C \text{ and } B \vdash C \text{ then } A \lor B \vdash C$
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- $(\lor e)$ if $A \vdash C$ and $B \vdash C$ then $A \lor B \vdash C$
- $(\rightarrow e) A, A \rightarrow B \vdash B$ Modus Ponens
Intuitive Properties

- $(\land e_1) \ A \land B \vdash A$ and $(\land e_2) A \land B \vdash B$
- $(\land i) A, B \vdash A \land B$
- $(\lor i_1) A \vdash A \lor B$ and $(\lor i_2) B \vdash A \lor B$
- $(\lor e) \text{if } A \vdash C$ and $B \vdash C$ then $A \lor B \vdash C$
- $(\to e) A, A \to B \vdash B$ Modus Ponens
- $(\to i) \text{if } A \vdash B$ then $\vdash A \to B$
Intuitive Properties

- \((\wedge e_1)\) \(A \land B \vdash A\) and \((\wedge e_2)\) \(A \land B \vdash B\)
- \((\wedge i)\) \(A, B \vdash A \land B\)
- \((\lor i_1)\) \(A \vdash A \lor B\) and \((\lor i_2)\) \(B \vdash A \lor B\)
- \((\lor e)\) if \(A \vdash C\) and \(B \vdash C\) then \(A \lor B \vdash C\)
- \((\rightarrow e)\) \(A, A \rightarrow B \vdash B\) Modus Ponens
- \((\rightarrow i)\) if \(A \vdash B\) then \(\vdash A \rightarrow B\)
- \((\bot e)\) \(\bot \vdash A\) ex falso quodlibet
Intuitive Properties

- $(\wedge e_1) A \land B \vdash A$ and $(\wedge e_2) A \land B \vdash B$
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- $(\lor i_1) A \vdash A \lor B$ and $(\lor i_2) B \vdash A \lor B$
- $(\lor e) \text{if } A \vdash C$ and $B \vdash C$ then $A \lor B \vdash C$
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- $(\bot e) \bot \vdash A$ ex falso quodlibet
- $(\neg i) \text{if } A \vdash \bot$ then $\vdash \neg A$
Intuitive Properties

- \((\land e_1)\) \(A \land B \vdash A\) and \((\land e_2)\) \(A \land B \vdash B\)
- \((\land i)\) \(A, B \vdash A \land B\)
- \((\lor i_1)\) \(A \vdash A \lor B\) and \((\lor i_2)\) \(B \vdash A \lor B\)
- \((\lor e)\) if \(A \vdash C\) and \(B \vdash C\) then \(A \lor B \vdash C\)
- \((\rightarrow e)\) \(A, A \rightarrow B \vdash B\) Modus Ponens
- \((\rightarrow i)\) if \(A \vdash B\) then \(\vdash A \rightarrow B\)
- \((\bot e)\) \(\bot \vdash A\) ex falso quodlibet
- \((\neg i)\) if \(A \vdash \bot\) then \(\vdash \neg A\)
- \((\neg e)\) \(A, \neg A \vdash \bot\)
Intuitive Properties

- $(\land e_1) \ A \land B \vdash A \text{ and } (\land e_2) A \land B \vdash B$
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- $(\lor e) \text{ if } A \vdash C \text{ and } B \vdash C \text{ then } A \lor B \vdash C$
- $(\rightarrow e) A, A \rightarrow B \vdash B \text{ Modus Ponens}$
- $(\rightarrow i) \text{ if } A \vdash B \text{ then } \vdash A \rightarrow B$
- $(\bot e) \bot \vdash A \text{ ex falso quodlibet}$
- $(\neg i) \text{ if } A \vdash \bot \text{ then } \vdash \neg A$
- $(\neg e) A, \neg A \vdash \bot$
- $(\text{RAA}) \text{ if } \neg A \vdash \bot \text{ then } \vdash A$
Intuitive Properties

- $(\land e_1) A \land B \vdash A$ and $(\land e_2) A \land B \vdash B$
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- $(\lor i_1) A \vdash A \lor B$ and $(\lor i_2) B \vdash A \lor B$
- $(\lor e)$ if $A \vdash C$ and $B \vdash C$ then $A \lor B \vdash C$
- $(\rightarrow e) A, A \rightarrow B \vdash B$ Modus Ponens
- $(\rightarrow i)$ if $A \vdash B$ then $\vdash A \rightarrow B$
- $(\bot e) \bot \vdash A$ ex falso quodlibet
- $(\neg i)$ if $A \vdash \bot$ then $\vdash \neg A$
- $(\neg e) A, \neg A \vdash \bot$
- (RAA) if $\neg A \vdash \bot$ then $\vdash A$
- (CUT) if $\Gamma \vdash A$ and $\Gamma, A \vdash B$ then $\Gamma \vdash B$