Compactness theorem

Note:
Checking satisfiability of finite sets of sentences can be reduced to checking satisfiability of a single sentence.

Can we reduce checking satisfiability for infinite sets of sentences to checking satisfiability for finite sets?

The answer is yes:
Theorem: (Compactness Theorem)
A set of sentences is satisfiable iff all its finite subsets are satisfiable.

[Again, we explore some consequences first]

Corollary: (Overspill principle).
If a set of sentences $\Gamma$ has arbitrary large finite models, then $\Gamma$ also has an infinite model.

Proof:
Consider $\Gamma^* = \Gamma \cup \{I_1, I_2, I_3, \ldots\}$. ($I_k$ as in slide 22.)

Every finite subset of $\Gamma' \subset \Gamma^*$ can only contain finitely many $I_k$.

Since $I_k$ just expresses ‘there is a model of size $\geq k$’ and since $\Gamma$ has a model $M$ of size $\geq k$: $M \models \Gamma'$.

In other words: All finite subsets of $\Gamma^*$ are satisfiable.

Therefore, by compactness, $\Gamma^*$ is satisfiable to.

But $\Gamma^*$ has only infinite models.

Therefore $\Gamma \subset \Gamma^*$ must also have an infinite model. Q.e.d.

Remark: In combination with Löwenheim-Skolem we obtain a denumerable model.

Proposition: (Converse of Vaught’s test fails)
'True arithmetic' $TA$, i.e., the set of sentences $A$ s.t. $\mathbb{N} \models A$, has a model with domain $\omega$ that is not isomorphic to $\mathbb{N}$.

Proof:
Add a new constant $c$ to the language $L^*$.

For every $k \in \omega$ the set of sentences $TA_{>k} = TA \cup \{c > \bar{0}, c > \bar{1}, \ldots, c > \bar{k}\}$ is satisfiable. (Remember: $\bar{i} = 0'\ldots' i$ times successor.)

Therefore, by compactness, also $TA_\omega = TA \cup \{c > \bar{0}, c > \bar{1}, c > \bar{2}, \ldots\}$ is satisfiable.

But models of $TA_\omega$ (restricted to $L^*$, i.e., without $c$) cannot be isomorphic to $\mathbb{N}$.

By Löwenheim-Skolem and the canonical domains lemma models of $TA_\omega$ with domain $\omega$ exist. Q.e.d.

Consequences of compactness

Suppose that the meaning of the following symbols were fixed for all interpretations (just like the meaning of $=$):

<table>
<thead>
<tr>
<th>Symbol</th>
<th>type</th>
<th>meaning</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>constant symbol</td>
<td>the number 0</td>
</tr>
<tr>
<td>$'$</td>
<td>unary function symbol</td>
<td>successor function ($\lambda x[x + 1]$)</td>
</tr>
<tr>
<td>$N$</td>
<td>unary predicate symbol</td>
<td>'is a natural number'</td>
</tr>
</tbody>
</table>

Consequence of the fixed interpretation: For any new constant symbol $c$, the following set of sentences is unsatisfiable:

$$\Gamma = \{N(c)\} \cup \{c \neq \bar{i}: i \in \omega\}$$

But all finite subsets of $\Gamma$ are satisfiable!
Proving Compactness and Löwenheim-Skolem

Idea for showing compactness:
For every set of sentences, where all finite subsets are satisfiable, we construct a term model.

\[\text{DEF:} \quad \text{A term interpretation of a language } L \text{ is an interpretation, in which every element of the domain is denoted by a closed term.}\]

Without identity the closed terms of \(L^+\) can be taken as the domain, where \(L^+ = L + \text{infinitely many new constants}\).

With identity we can take equivalence classes w.r.t. a suitable congruence relation over closed terms of \(L^+\).

Denumerably many new constants suffice for \(L^+\). Hence, also the Löwenheim-Skolem theorem follows.

Splitting the task

Notation: ‘\(L\)-set’ means ‘set of sentences over language \(L\)’

1. The set of satisfiable \(L\)-sets has certain satisfaction properties.
2. The satisfaction properties are of finite character.
3. Satisfaction properties imply closure properties of members.
4. A set has a term model iff it has the closure properties.

3 + 4 implies

5. The satisfaction properties guarantee a term model (\(=\) Model existence lemma)

1 + 2 + 5 implies compactness. [Why?]

Note: It suffices to consider \(\neg, \lor, \exists, =\).

\[\text{DEF: A set } S \text{ of } L\text{-sets has the satisfaction properties means:}\]

\[(S0) \quad \text{If } \Gamma \in S \text{ and } \Gamma_0 \subseteq \Gamma, \text{ then } \Gamma_0 \in S.\]

\[(S1) \quad \text{If } \Gamma \in S, \text{ then not both: } A \in \Gamma \text{ and } \neg A \in \Gamma.\]

\[(S2) \quad \text{If } \Gamma \in S \text{ and } \neg B \in \Gamma, \text{ then } \Gamma \cup \{B\} \in S.\]

\[(S3) \quad \text{If } \Gamma \in S \text{ and } (B \lor C) \in \Gamma, \text{ then } \Gamma \cup \{B\} \in S \text{ or } \Gamma \cup \{C\} \in S.\]

\[(S4) \quad \text{If } \Gamma \in S \text{ and } \neg(B \lor C) \in \Gamma, \text{ then } \Gamma \cup \{\neg B\} \in S \text{ and } \Gamma \cup \{\neg C\} \in S.\]

\[(S5) \quad \text{If } \Gamma \in S \text{ and } \exists x B(x) \in \Gamma, \text{ then } \Gamma \cup \{B(c)\} \in S \text{ for all new } c.\]

\[(S6) \quad \text{If } \Gamma \in S \text{ and } \neg \exists x B(x) \in \Gamma, \text{ then } \Gamma \cup \{\neg B(t)\} \in S \text{ for all closed terms } t.\]

\[(S7) \quad \text{If } \Gamma \in S, \text{ then } \Gamma \cup \{t = t\} \in S \text{ for all closed terms } t.\]

\[(S8) \quad \text{If } \Gamma \in S \text{ and } B(s), s = t \in \Gamma, \text{ then } \Gamma \cup \{B(t)\} \in S.\]

Lemma: (Satisfaction properties lemma = \(1.\))

The set \(Sat\) of all satisfiable \(L\)-sets fulfills (S0) – (S8).

Lemma: (Finite character lemma = \(2.\))

If \(S\) is a set of \(L\)-sets that has the satisfaction properties, then also the set \(S^*\) of all sets \(\Gamma\), such that all finite subsets of \(\Gamma\) are in \(S\) has the satisfaction properties.

Proof: [[ blackboard ]]

Corollary:
The set \(Sat^*\) of sets of sentences, whose finite subsets are satisfiable, has the satisfaction properties (S0) – (S8).

Note:

Compactness means: \(Sat = Sat^*\).

[ Which inclusion is obvious, which is ‘hard’? ]
DEF: A set $\Gamma$ of sentences over $L^+$ is said to have the closure properties if the following holds:

(C1) For no sentence $A$, both: $A \in \Gamma$ and $\neg A \in \Gamma$.
(C2) If $\neg \neg B \in \Gamma$, then $B \in \Gamma$.
(C3) If $(B \lor C) \in \Gamma$, then either $B \in \Gamma$ or $C \in \Gamma$.
(C4) If $\neg (B \lor C) \in \Gamma$, then both: $\neg B \in \Gamma$ and $\neg C \in \Gamma$.
(C5) If $\exists x B(x) \in \Gamma$, then for some closed term $t$ of $L^+$, $B(t) \in \Gamma$.
(C6) If $\neg \exists x B(x) \in \Gamma$, then for every closed term $t$ of $L^+$, $\neg B(t) \in \Gamma$.
(C7) For every closed term $t$ of $L^+$, $t = t \in \Gamma$.
(C8) If $B(s) \in \Gamma$ and $s = t \in \Gamma$, then $B(t) \in \Gamma$.

The following proposition follows, essentially, from the definition of a term model:

Proposition: (Closure properties lemma)
The set of all sentences $\Gamma_M$ over $L^+$ that are true in term model $\mathcal{M}$ has the closure properties (C1) – (C8).

Note: this is the ‘easy direction’ of 4.

However:
We need the following converse!
Lemma: (Term models lemma = interesting part of 4.)
Let $\Gamma$ be a set of sentences with the closure properties.
Then $\Gamma$ has a term model.

Proof of term models lemma without identity and function symbols
By assumption:
– $\Gamma$ fulfills (C1) – (C8).
– The only closed terms are constants.

Remember that the closure properties imply an extension of the language from $L$ to $L^+ = L \cup \{c_0, c_1, c_2, \ldots\}$.

We have to define a term model $\mathcal{M}$ of $\Gamma$:
$|\mathcal{M}|$ consists of the constants $c \in L^+: c^\mathcal{M} = c$.
For all predicate symbols $R$, $R^\mathcal{M}$ is defined by
$R^\mathcal{M}(c_1^\mathcal{M}, \ldots, c_n^\mathcal{M})$ if $R(c_1, \ldots, c_n) \in \Gamma$.
We have to show by induction that
if $B \in \Gamma$ then $\mathcal{M} \models B$.

We show by induction: if $B \in \Gamma$ then $\mathcal{M} \models B$.

Atoms: The definition of truth says:
$\mathcal{M} \models R(c_1, \ldots, c_n)$ iff $R^\mathcal{M}(c_1^\mathcal{M}, \ldots, c_n^\mathcal{M})$.
therefore our definition of $R^\mathcal{M}$ yields q.e.d..

Negated Atoms: If $\neg R(c_1, \ldots, c_n) \in \Gamma$ then, by property (C1),
$R(c_1, \ldots, c_n) \notin \Gamma$.
By the atom case, above, this implies $\mathcal{M} \not\models R(c_1, \ldots, c_n)$.
Consequently $\mathcal{M} \models \neg R(c_1, \ldots, c_n)$, as required.

Connectives: Assume, e.g., $B \lor C \in \Gamma$ then,
by property (C3), $B \in \Gamma$ or $C \in \Gamma$.
The induction hypothesis thus yields $\mathcal{M} \models B$ or $\mathcal{M} \models C$.
Hence the truth definition for disjunction, implies q.e.d. [The other cases are analogous.]
Quantifier cases:
Remember:
The relevant induction hypothesis is: ‘if \( B \in \Gamma \) then \( \mathcal{M} \models B' \) holds for \( B \) that are less complex than \( \exists x B(x) \).
\[ \exists x B(x) \in \Gamma: \]
By property (C5) \( \exists x B(x) \in \Gamma \) implies \( B(c) \in \Gamma \) for some constant \( c \).
Therefore, by induction hypothesis, \( \mathcal{M} \models B(c) \).
Hence also \( \mathcal{M} \models \exists x B(x) \), by the definition of truth.
\[ \neg \exists x B(x) \in \Gamma: \]
By (C6) \( \neg \exists x B(x) \in \Gamma \) implies \( \neg B(c) \in \Gamma \) for all constants \( c \).
Therefore, by induction hypothesis, \( \mathcal{M} \models \neg B(c) \) for all \( c \).
By our definition of \( |\mathcal{M}| \) this implies \( \mathcal{M} \not\models B[m] \), for any \( m \in |\mathcal{M}| \). Hence \( \mathcal{M} \not\models \exists x B(x) \) and thus also \( \mathcal{M} \models \neg \exists x B(x) \).
Q.e.d..

Note:
(E1) – (E3) say that \( \equiv \) is an equivalence relation.
Writing \([t]\) for the equivalence class of terms generated by term \( t \), we may rewrite (E4) and (E5) as follows:
(E4') If \([s_1] = [t_1] \) and \( \ldots \) and \([s_n] = [t_n] \), then
\[ R(s_1, \ldots, s_n) \in \Gamma \text{ iff } R(t_1, \ldots, t_n) \in \Gamma. \]
(E5') If \([s_1] = [t_1] \) and \( \ldots \) and \([s_n] = [t_n] \), then
\[ f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \in \Gamma. \]
The equivalence classes are also called congruence classes.

Term models with identity and function symbols
Lemma: (Congruence lemma)
Let \( \Gamma \) have the closure properties (C1) – (C8).
We write \( s \equiv t \) for ‘\( s = t \in \Gamma \)’. The following hold:

(E1) \( t \equiv t \).
(E2) If \( s \equiv t \), then \( t \equiv s \).
(E3) If \( s \equiv t \) and \( t \equiv r \), then \( s \equiv r \).
(E4) If \( s_1 \equiv t_1 \) and \( \ldots \) and \( s_n \equiv t_n \), then
\[ R(s_1, \ldots, s_n) \in \Gamma \text{ iff } R(t_1, \ldots, t_n) \in \Gamma. \]
(E5) If \( s_1 \equiv t_1 \) and \( \ldots \) and \( s_n \equiv t_n \), then
\[ f(s_1, \ldots, s_n) = f(t_1, \ldots, t_n) \in \Gamma. \]
I.e., \( \equiv \) is a congruence relation of closed terms w.r.t. \( \Gamma \).
Proof: [[ blk board ]]
‘Closing’ satisfiable sets of sentences
Lemma: (Closure lemma = 3.)
Let $S$ be a set of $L$-sets that has the satisfaction properties.
Then all $\Gamma \in S$ can be extended to a set $\Gamma^*$ over $L^+$ having the closure properties (C1) – (C8).

Proof: We augment $\Gamma$ in stages, trying to satisfy stepwise all ‘demands for closure’, raised by sentences that are already in the set.

More exactly, we construct $\Gamma^* = \bigcup_{i \in \omega} \Gamma_i$, where

$$\Gamma = \Gamma_0 \subseteq \Gamma_1 = \Gamma_0 \cup \{A_0\} \subseteq \Gamma_2 = \Gamma_1 \cup \{A_1\} \subseteq \Gamma_3 \ldots$$

Note: Property (C1) will be satisfied because of (S1).

Remember:
The closure lemma 3. and the term models lemma 4. jointly, imply the following

Lemma: (Model existence lemma = 5.)
If $S^*$ is a set of sets of sentences over $L^+$ that has the satisfaction properties, then every $L$-set in $S^*$ has a term model.

By the satisfaction properties lemma 1. and the finite character lemma 2., this concludes the proof of the compactness theorem.