Expressiveness — Size and number of models

Remember:
A model of a sentence \( A \) [set of sentences \( \Gamma \)] is an interpretation which makes \( A \) [all \( A \in \Gamma \)] true.

DEF: The size of a model \( M \) is the cardinality of \( |M| \).
DEF: The spectrum of a sentence \( A \) collects the sizes of all finite models of \( A \). More formally:
\[
\text{spec}(A) = \{ \text{card}(|M|) : M \models A, |M| \text{ finite} \}
\]

We study the following questions:
- Which model sizes can be 'enforced' by sentences?
- When should two models be considered 'different'?
- How many different models can a sentence have?

Remarks on terminology

Following [BBJ], we use 'denumerable' for 'of countably infinite cardinality', and 'enumerable' for 'of finite or countably infinite cardinality' (= countable).

Consequently 'non-enumerable' means: uncountable.

A sentence \( A \) is
- (a) finitely satisfiable, or
- (b) denumerably satisfiable, or
- (c) enumerably satisfiable,
respectively, if there is a model \( M \) of \( A \), where
- (a) \( \text{card}(|M|) < \omega \), or
- (b) \( \text{card}(|M|) = \omega \), or
- (c) \( \text{card}(|M|) \leq \omega \),
respectively.

Expressiveness of the empty language

It is easy to express 'there are at least \( n \) distinct individuals':
\[
I_n = \exists x_1 \ldots \exists x_n \bigwedge_{1 \leq i < j \leq n} x_i \neq x_j
\]

More formally: \( \text{spec}(I_n) = \{ i \mid i \geq n \} \).
\( J_n = \neg I_{n+1} \) expresses 'there are at most \( n \) distinct individuals'.
Alternatively we may take
\[
J'_n = \exists x_1 \ldots \exists x_n \forall y \bigvee_{1 \leq i \leq n} y = x_i
\]

What is expressed by the following sentences?

What are the corresponding spectra?
- \( I_4 \lor I_6 \), \( I_4 \land J_m \), \( I_4 \lor J_n \)
- \( K_n = I_n \land J_n \), \( K_2 \lor K_6 \), \( K_2 \land K_6 \)
- \( I_n \rightarrow I_m \), \( I_n \rightarrow J_m \), \( I_m \rightarrow J_n \)

Remarks on the 'spectrum problem' [*Advanced*]

Already 50 years ago G. Asser asked:
Are the spectra (of first-order sentences) closed under complement?
Over the empty language the answer is 'yes'.
(Easy, part of exercise 13)
The full problem is still unsolved.
For cognoscenti:
It is equivalent to \( NP_1 =? coNP_1 \)
and therefore to \( NEXP =? coNEXP \).
Important connections to structural complexity theory:
Many Results on generalizations of first-order logic
(fragments of second-order logic, transitive closure operator, . . . )
Enforcing infinite models
Sentences over the empty language may have infinite models, but all satisfiable sentences are already finitely satisfiable. I.e.: we cannot 'enforce' infinity over the empty language.
For this we need an (at least) binary predicate (or function symbols):

\[ \forall x \exists y R(x, y) \]
\[ \forall x \forall y R(x, y) \rightarrow \neg R(y, x) \]
\[ \forall x \forall y \forall z ((R(x, y) \land R(y, z)) \rightarrow R(x, z)) \]

We show that \( F u \land AS \land Tr \) has only infinite models.
I.e., it is satisfiable, but not finitely satisfiable.

How many models?
How many models has \( \forall x \forall y x = y \)?
How many 'different' models?
What about \( \forall x \forall y P(x, y) \)?
In which sense are the following two interpretations 'the same'?
\[ M_1 = (\{2, 10\}; \leq) \quad \text{and} \quad M_2 = (\{abb, ab\}; 'is substring of') \]

DEF: Interpretations \( P \) and \( Q \) are isomorphic iff there is a bijection \( \xi \) (called isomorphism) between \( |P| \) and \( |Q| \) s.t. for every \( n \)-ary predicate \( R \) and all \( p_1, \ldots, p_n \in |P| \)
\[ (I1) \ R^P(p_1, \ldots, p_n) \iff R^Q(\xi(p_1), \ldots, \xi(p_n)), \]
\[ (I2) \ \xi(c^P) = c^Q \text{ for every constant } c, \text{ and} \]
\[ (I3) \ \xi(f^P(p_1, \ldots, p_n)) = f^Q(\xi(p_1), \ldots, \xi(p_n)) \text{ for every } n \text{-ary function symbol } f \]