Multi-modal logics

- Syntax:
  more than one (non-dual) modal operators. We will consider only unary modal operators, here.

- Semantics:
  interpretations (and frames) with more than one accessibility relation over one and the same set of states/worlds:
  \( \mathcal{M} = \langle W, R_1, \ldots, R_n, V \rangle \) bzw. \( \mathcal{F} = \langle W, R_1, \ldots, R_n \rangle \)

Each accessibility relation \( R_i \) determines a modality (e.g., denoted by \( \Box_i \)) plus the corresponding dual modality (\( \Diamond_i \))

\[
v_{\mathcal{M}}(\Box_i F, w) = \begin{cases} 
1 & \text{if } \forall u \ wR_i u \Rightarrow v_{\mathcal{M}}(F, u) = 1 \\
0 & \text{otherwise}
\end{cases}
\]
Multi-modal logics (ctd.)

Note:
Not only the properties of the (isolated) accessibility relations are important, but also particular relations between these relations are used to model intended semantic facts.

▶ Axioms expressing relations between modalities:
E.g., $\mathcal{F} \models □_2 A \supset □_1 A$
whenever in $\mathcal{F} = \langle W, R_1, R_2 \rangle$ we have:
$R_1$ is a sub-relation of $R_2$ (ie., $R_1 \subseteq R_2$).

▶ The inverse task is even more important:
to find axioms that characterize given semantic connections between relations.
$R_1$ is a subrelation of $R_2$ in a multi-relation frame $\mathcal{F}$,
whenever $\mathcal{F} \models □_2 A \supset □_1 A$

Exercise 32:
Prove the two mentioned facts.
An example of a multi-modal logic: Temporal logic

Consider the bi-modal frame $\mathcal{F} = \langle W, R_P, R_F \rangle$ intended to model a temporal structure (e.g., to serve as basis to formalize the dynamic behaviour of a system, via corresponding states): $R_P$ and $R_F$ are the relations ‘earlier’ and ‘later’ respectively.

(Think of adequate properties of $R_P$ and $R_F$ . . . )

The corresponding induced modalities express:

- $\lbrack P \rbrack$: in the past always — $\langle P \rangle$: sometimes in the past
- $\lbrack F \rbrack$: in the future always — $\langle F \rangle$: sometime in the future

Characterizing the relation between future and past time:

$\mathcal{F} \models A \supset [P] \langle F \rangle A \iff \forall s \forall t (s R_P t \Rightarrow t R_F s)$

$\mathcal{F} \models A \supset [F] \langle P \rangle A \iff \forall s \forall t (s R_F t \Rightarrow t R_P s)$

**Exercise 33:**
Prove these facts.
Reasoning about knowledge: multi-modal epistemic logic

Finite set of agents: 1, . . . , \( n \)

For each agent \( i \) there is a modal operator \( K_i \) with the following intended interpretation:

\( K_i A \ldots \) ‘Agent \( i \) knows that \( A \)’

Example (from the Watergate scandal):

Dean doesn’t know whether Nixon knows that Dean knows that Nixon knows that McCord broke into O’Brien’s Watergate office.

\[
\begin{align*}
K_1 A & \ldots \ ‘Dean knows that A’ \\
K_2 A & \ldots \ ‘Nixon knows that A’ \\
p & \ldots \ ‘McCord broke into O’Brien’s office’
\end{align*}
\]

therefore:

\[ \neg K_1 A \land \neg K_1 \neg A \ldots \) ‘Dean doesn’t know whether A’ \]

The above sentence is thus expressed by:

\[ \neg K_1 K_2 K_1 K_2 p \land \neg K_1 \neg K_2 K_1 K_2 p \]
Principles of modelling knowledge

Every agent should know whatever is true (already) for purely logical reasons. (‘Agents are perfect reasoners’) Consequently the following hold:

- Distribution axiom (– internalized *modus ponens*):
  \[(1) \ (K_i A \land K_i (A \supset B)) \supset K_i B \text{ is valid}\]

- Knowledge generalization rule:
  \[(2) \text{ If } A \text{ is valid then also } K_i A \text{ is valid}\]

Note:
These two principles imply that Kripke interpretations \(\mathcal{M} = \langle W, R_1, \ldots, R_n, V \rangle\) are adequate for modeling knowledge of perfect reasoners.

Exercise 34:
Prove that (1) is equivalent to axiom \((K)\) (with \(\Box\) replaced by \(K_i\)).
Principles of modelling knowledge (ctd.)

The following specific logical principles arise from the intended concept of knowledge:

- **knowledge axiom**: $K_i A \supset A$
  
  ‘only facts can be known’

- **positive introspection**: $K_i A \supset K_i K_i A$
  
  ‘an agent knows that she knows what she knows’

- **negative introspection**: $\neg K_i A \supset K_i \neg K_i A$
  
  ‘an agent knows that she doesn’t know what she doesn’t know’

**Note** the following relation to our axiom list ($A1 – A10$):

The knowledge axiom is $A1$ — aka. (T).

Positive introspection is $A4$ — aka. (4)

Negative introspection is equivalent to $A5$ — aka. (5)

**Exercise 35:**

Prove: $\mathcal{F} \models A5$ iff $\mathcal{F} \models$ negative introspection.