Propositional Hilbert-type Systems

‘Classical’ syntax: ⊃, ∧, ∨, ⊥ (or subsets thereof)
Negation is defined by: ¬A \equiv (A \supset \bot)

- **Self Implication**: A \supset A
- **Weakening** (‘positive paradox’): (A \supset (B \supset A))
- **Prefixing**: A \supset (B \supset (C \supset A))
- **Suffixing**: (B \supset A) \supset [(A \supset C) \supset (B \supset C)]
- **Permutation**: [A \supset (B \supset C)] \supset [(B \supset (A \supset C)]
- **Contraction** (Lukasiewicz axiom): (A \supset (A \supset B)) \supset ((A \supset B) \supset A)

- **Ex Falso Quodlibet**: ⊥ \supset A
- **Tertium Non Datur**: A \lor \neg A
- **Elimination of Indirectness**: \neg \neg A \supset A
- **Weak Contraction**: (A \supset (B \supset A)) \supset ((B \supset A) \supset A)
- **Contraposition a**: (A \supset \neg B) \supset (B \supset \neg A)
- **Contraposition b**: (\neg A \supset \neg B) \supset (B \supset A)
- **Linearity**: (A \supset B) \lor (B \supset A)
- **Residuation a**: [(A \supset B) \lor C] \supset [(A \supset C) \lor B]
- **Residuation b**: [(A \supset (B \supset C))] \lor ((A \supset B) \supset C)
- **\land-Commutativity**: (A \land B) \lor (B \land A)
- **\lor-Commutativity**: (A \lor B) \lor (B \lor A)
- **\lor-Commutativity with Negation**: (A \supset (\neg A \supset B))
- **\land-Commutativity with Negation**: (A \lor (\neg A \lor B))
- **Lukasiewicz Axiom**: ((A \supset B) \lor C) \supset ((A \lor B) \supset C)
- **Hajek’s Axiom**: (A \land (A \land B)) \lor (B \land (B \land A))
- **Pre-Linearity**: (A \supset (B \lor C)) \lor ((A \supset B) \lor (A \supset C))
- **\land-Idempotence**: (A \land (A \land A))
- **Transitivity**: (A \supset (B \supset (C \supset (A \supset C))))

(Only) Derivation Rule: \[
\frac{A \supset B}{A} Modus Ponens
\]

**Hilbert-type Systems for Some Important Logics**:

- **Classical Logic CL**: e.g., 2, 5, 13b or: all!
- **Intuitionistic Logic IL**: e.g., 2–10, 15, 16a, 16b or: 2, 25, 6, 7, 8, 9, 20
  - or: all except 11, 12, 13a, 13b, 14, 21, 22, 33
- **Minimal Logic**: 2, 3, 5 or 2, 4, 5 (1 is derivable!)
- **Contraction-free IL — Version IL^-**: e.g., 2–10, 17, 18
- **Contraction-free IL — Version IL^~**: IL^- + 16a, 16b
- **Urquhart’s (original) Logic C**: IL^- + 14
- **Hajek’s ‘Basic Logic’ BL**: 4, 7a, 17, 22, 16a, 23, 10
- **Gödel Logic G (G_\infty)**: BL + 24 or: IL + 14 or: C + 15
- **Łukasiewicz Logic L**: 2, 4, 13b, 21 or: BL + 12
- **Relevance Logic R_{\supset}**: z.B. 1, 3, 5, 15
- **Relevance Logic E_{\supset}**: z.B. 1, 3, 26, 15