Comments on *Reasoning About Uncertainty of Fuzzy Events: An Overview* by Tommaso Flaminio, Lluís Godo, and Enrico Marchioni

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The presented contribution deals with formal logical models capturing both *vagueness* and *uncertainty*. The vagueness facet of such models is represented by fuzzy events (many-valued events) which are measured by degrees of uncertainty (belief). Therefore, any logic for fuzzy events under uncertainty must have a sufficient expressive power to model inference on the side of the fuzzy events as well as to reproduce reasoning with degrees of uncertainty associated with those events. In particular, the logic chosen should be versatile enough to include in its language virtually any of the approaches to uncertainty processing, such as probability theory, possibility/necessity, Dempster-Shafer theory etc. The meaning of fuzzy events and their associated uncertainty degrees is usually rendered through de Finetti coherence criterion.

The authors provide a solid motivation for their approach and a historical summary of the similar investigations in Section 1. Discussing two components (vagueness and uncertainty) of the reasoning, they single out a crucial difference between *degrees of truth* and *degrees of belief:* intermediate truth degrees are compositional, whereas degrees of belief are not. For example, the truth degree of a conjunction $\varphi \wedge \psi$ is a two-place function depending only on the truth degrees of φ and ψ. On the other hand, the probability of $\varphi \wedge \psi$ is not, in general, a two-place function of the two corresponding probabilities of φ and ψ. The conclusion: since the role of the two degrees is essentially different, a variant of *modal logic* should be employed for expressing the degree of belief of φ as the degree of truth of the modal formula $P\varphi$ whose intended meaning is "φ is plausible". This direction was first pursued by Hájek and Harmancová.

Section 2 contains the necessary background on the logical apparatus used. The authors introduce the logic MTL of Esteva and Godo together with its algebraic semantics based on MTL-algebras. After discussing their basic properties and several types of completeness criteria, they discuss a few expansions of MTL:

- expansions with the involutive negation,
- expansions with rational truth constants,
- Łukasiewicz logic.

The importance of the last mentioned expansion cannot be overestimated in the studied context. Indeed, measure theory of fuzzy events based on *Łukasiewicz clans (tribes)* has already become a sound platform for developing several deep mathematical problems ranging from generalized measure theory and MV-probability (Butnariu, Navara, Riečan) over piecewise-linear homeomorphisms and geometry of unimodular triangulations (Mundici, Panti) to conditioning and de Finetti-type theorems (Fedel, Flaminio, Keimel, Montagna). This fact is not surprising since the Łukasiewicz disjunction has a straightforward arithmetic meaning. On the contrary, Gödel logic, which is another expansion of MTL, provides no connectives enabling to model the standard addition of reals.

Important classes of uncertainty measures are discussed in Section 3. The authors divide the content into the case of classical events and the case of fuzzy events, where the former case naturally motivates a very general definition of an uncertainty measure introduced in the latter case. Specifically, a *generalized fuzzy measure* is defined to be a monotone normalized $[0,1]$-valued function on an MTL-algebra. This general concept is meant to cover special classes of measures, which are included in the section: possibility/necessity measures and finitely additive probabilities/states (the Dempster-Shafer's belief functions are not considered in this paper). It is known that every state is just an integral (a sum) of fuzzy events with respect to the classical probability measure. Along the same lines, the authors develop representation of necessities by generalized Sugeno integrals over fuzzy events with respect to classical necessity measures. This result is of chief importance in justifying the usual "min-preserving" definition since it shows that generalized necessities are extensions of classical necessities.

The core of the authors' contribution lies in Sections 4–5, where they present fuzzy modal logics for the previously mentioned classes of generalized fuzzy measures. Only expansions of MTL *compatible* with a given subclass of fuzzy measures are admissible, that is, the axioms of the given class of measures must be expressible by means of functions definable in the expansion of MTL. For example, Gödel logic is not compatible with the class of probabilities since probabilistic degrees of belief cannot be reproduced faithfully in this logic. The fundamental logical framework introduced by the authors consists of logic $\mathscr{L}_1$ (an expansion of MTL representing fuzzy events) and logic $\mathscr{L}_2$ (an expansion of MTL compatible with a specific class of uncertainty measures). Consequently, one can distinguish formulas from $\mathscr{L}_1$ (denoting fuzzy events) and modal formulas inductively defined over the formulas in $\mathscr{L}_1$ by considering the modality $\mathscr{M}$. Thus expression like $\mathscr{M}(\varphi \vee \psi)$ reads as "φ or ψ are plausible". Neither formulas such as $\varphi \wedge \mathscr{M}\psi$ nor nested modalities are allowed in the language. The semantics based on the set of possible worlds is adopted. The authors proceed with discussion of modal logic for generalized plausibility measures, logics for (representable) generalized possibility/necessity, logics for generalized probability, and their corresponding expansions with rational truth constants. The axioms and inference rules are introduced together with several completeness results (hyperreal completeness as well, in particular). Needless to say, these results witness that the program of formalizing uncertainty reasoning under vagueness is both feasible and successful.

Section 6 is devoted to a generalization of the well-known de Finetti coherence problem. In the framework of fuzzy events and their (rational) uncertainty assessments, this problem can be stated in the following way: given finitely-many formulas and their $[0,1]$-valued rational assessment, is there an uncertainty measure extending the assessment to the Lindenbaum-Tarski algebra generated by the variables occurring in the formulas? This definition of a coherence problem is fairly general in allowing the uncertainty measure to belong to any class of uncertainty measures. It is shown that the coherence of a rational assessment is equivalent to consistency (in a relevant expansion of rational Łukasiewicz logic) of a certain modal theory naturally associated with the given set of fuzzy events. Finally, Section 7 concludes with a summary of state-of-the-art approaches to the topic, mentioning many relevant papers and books devoted to vagueness/uncertainty formalization.

The steadily expanding area of reasoning under both uncertainty and vagueness attracts researchers from a number of scientific fields: mathematics, logic, philosophy etc. The authors of this paper, who count among the experts in logic, made their exposition accessible to anyone with background knowledge in modern mathematical many-valued (fuzzy) logics. The view towards different uncertainty theories respects contemporary trends in uncertainty processing. In my opinion, the presented contribution is a valuable piece in the ever growing mosaic of logical formalizations of uncertainty reasoning with vague information.

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