A curious dialogical logic and its composition problem

Sara L. Uckelman (1) & Jesse Alama (2)

(1) Institute for Logic, Language, and Computation
University of Amsterdam
S.L.Uckelman@uva.nl

(2) Center for Artificial Intelligence
New University of Lisbon
j.alama@fct.unl.pt

DIPLEAP
27 November 2010, Vienna

Dialogical logic

- Introduced in the 1950s and 1960s to give an alternative semantics for intuitionistic logic.
- Based on the existence of winning strategies in finitary open two-person zero-sum games between Proponent and Opponent.
- Players attack and defend formulas asserted by the other according to *particle* and *structural* rules.
- Meaning of connectives is given by their use.
- Recently extended to give new semantics for classical logic, modal logic, free logic, connexive logic, relevance logic, and others.

Set-up

Propositional language: formulas are built from atoms and $\neg$, $\vee$, $\wedge$, and $\rightarrow$.

Set-up

Propositional language: formulas are built from atoms and $\neg$, $\vee$, $\wedge$, and $\rightarrow$.

Three *symbolic attack* expressions: ?, $\wedge_L$, and $\wedge_R$, distinct from all the formulas and connectives.

Set-up

Propositional language: formulas are built from atoms and $\neg$, $\vee$, $\wedge$, and $\rightarrow$.

Three *symbolic attack* expressions: ?, $\wedge_L$, and $\wedge_R$, distinct from all the formulas and connectives.

Two types of rules:

Particle rules Govern how statements can be attacked and defended depending on their main connective.

Structural rules Define what sequences of attacks and defenses count as dialogues.

Particle rules

Assertion	**Attack**	**Response**
$\varphi \wedge \psi$	$\wedge_L$	φ
	$\wedge_R$	ψ
$\varphi \vee \psi$	?	φ or ψ
$\varphi \rightarrow \psi$	φ	ψ
$\neg\varphi$	φ	—

Table: Particle rules for dialogue games

Structural rules

(D10) P may assert an atomic formula only after it has been asserted by O before.

(D11) If p is a P-position, and if at round $n-1$ there are several open attacks made by O, then only the latest of them may be answered at n (and the same with P and O reversed).

(D12) An attack may be answered at most once.

(D13) A P-assertion may be attacked at most once.

(E) O can react only upon the immediately preceding P-statement.

Reference

W. Felscher, "Dialogues, Strategies, and Intuitionistic Provability", *Annals of Pure and Applied Logic* 28 (1985): 217–254.

Winning plays and winning strategies

Play alternates between Proponent and Opponent (starting with Proponent at move 0), and every move (except the initial assertion) is either an attack or a defense against some earlier assertion.

Definition

Given a set S of structural rules, an *S-dialogue* for a formula φ is a dialogue commencing with φ that adheres to the rules of S. Proponent *wins* an S-dialogue if there is a round where Opponent has no legal moves available.

Winning plays and winning strategies

Play alternates between Proponent and Opponent (starting with Proponent at move 0), and every move (except the initial assertion) is either an attack or a defense against some earlier assertion.

Definition

Given a set S of structural rules, an *S-dialogue* for a formula φ is a dialogue commencing with φ that adheres to the rules of S. Proponent *wins* an S-dialogue if there is a round where Opponent has no legal moves available.

Remark: According to this definition, if the dialogue *can* go on, then neither player is said to win; the game proceeds as long as moves are available.

Definition

A player has *winning strategy* for a formula φ if no matter what move the other player makes, the first player has a legal move, and eventually the second player has no legal moves left.

Dialogue games as logical semantics

Definition

For a set S of dialogue rules and a formula φ, the relation $\models_S \varphi$ means that Proponent has an S-winning strategy for φ. If $\not\models_S \varphi$, then we say that φ is S-invalid.

Theorem (Felscher)

- *A formula φ is intuitionistically valid iff $\models_{\mathrm{D}} \varphi$.*
- *A formula φ is intuitionistically valid iff $\models_{\mathrm{D+E}} \varphi$.*

Theorem (Folklore?)

A formula φ is classically valid iff $\models_{\mathrm{D10+D13+E}} \varphi$.

Extending dialogical semantics

- Dialogue games can be extended to give new semantics for paraconsistent, connexive, modal and linear logics.
- These semantics are constructed by maintaining or extending the particle rules and changing the set of structural rules.

Extending dialogical semantics

- Dialogue games can be extended to give new semantics for paraconsistent, connexive, modal and linear logics.
- These semantics are constructed by maintaining or extending the particle rules and changing the set of structural rules.
- In what ways can the structural rules be changed?

Extending dialogical semantics

- Dialogue games can be extended to give new semantics for paraconsistent, connexive, modal and linear logics.
- These semantics are constructed by maintaining or extending the particle rules and changing the set of structural rules.
- In what ways can the structural rules be changed?
    - Remove rules from a rule-set known to give rise to a logic.
    - Create and add new rules.

Extending dialogical semantics

- Dialogue games can be extended to give new semantics for paraconsistent, connexive, modal and linear logics.
- These semantics are constructed by maintaining or extending the particle rules and changing the set of structural rules.
- In what ways can the structural rules be changed?
  - Remove rules from a rule-set known to give rise to a logic.
  - Create and add new rules.
- In what cases will the resulting rule-set actually give you a logic?

What is a logic?

Definition

Given a language $\mathcal{L}$, a *logic* is a set L of $\mathcal{L}$-formulas which is closed under modus ponens (that is, if $\varphi \in \mathrm{L}$ and $\varphi \to \psi \in \mathrm{L}$, then $\psi \in \mathrm{L}$ as well).

Note

Note that we deviate from Tarski's definition of logic by not requiring that L be closed under unrestricted uniform substitution. This is in recognition of the fact that there are a number of well-defined sets of formulas known in the literature which are generally accepted as (non-classical) logics whose dialogical characterizations do not validate unrestricted uniform substitution, such as connexive logic and relevance logic, as well as non-dialogical logics which do not validate uniform substitution, namely connexive logics based on subtraction negation and certain types of strict paraconsistent logics.

The composition problem

Definition

The *composition problem* for a set of dialogue rules S asks whether the set S of formulas φ for which Proponent has a winning strategy in the S-dialogue game commencing with φ is closed under modus ponens.

The composition problem

Definition

The *composition problem* for a set of dialogue rules S asks whether the set S of formulas φ for which Proponent has a winning strategy in the S-dialogue game commencing with φ is closed under modus ponens.

Definition

The *strategy composition problem*, which for a set of dialogue rules S asks, given winning S-strategies for Proponent for formulas φ and one for $\varphi \to \psi$, can we *compose* these strategies into one for ψ?

The composition problem

Definition

The *composition problem* for a set of dialogue rules S asks whether the set S of formulas φ for which Proponent has a winning strategy in the S-dialogue game commencing with φ is closed under modus ponens.

Definition

The *strategy composition problem*, which for a set of dialogue rules S asks, given winning S-strategies for Proponent for formulas φ and one for $\varphi \to \psi$, can we *compose* these strategies into one for ψ?

Note

A positive answer to the strategy composition problem will also be a positive answer to the more general problem, but the reverse is not the case: It may be possible that some set S of formulas is closed under modus ponens, but the winning strategies which generate the set are not composable.

What happens when we remove structural rules?

Recall that:
$\mathrm{D10} + \mathrm{D11} + \mathrm{D12} + \mathrm{D13} = \mathrm{D} = \mathrm{D} + \mathrm{E} = \mathsf{IL}$.
$\mathrm{D10} + \mathrm{D13} + \mathrm{E} = \mathsf{CL}$.

What happens when we remove structural rules?

Recall that:

$\mathrm{D10} + \mathrm{D11} + \mathrm{D12} + \mathrm{D13} = \mathrm{D} = \mathrm{D} + \mathrm{E} = \mathsf{IL}$.

$\mathrm{D10} + \mathrm{D13} + \mathrm{E} = \mathsf{CL}$.

$\mathrm{D10} + \mathrm{D13} = ?$

What happens when we remove structural rules?

Recall that:

$D10 + D11 + D12 + D13 = D = D + E =$ IL.

$D10 + D13 + E =$ CL.

$D10 + D13 = ?$

Definition

Let $N = D10 + D13$. The logic N is the set of formulas for which P has a winning N-strategy.

Properties of winning N-strategies (1)

Theorem

Every branch for an N*-dialogue tree that contains a defensive move by O either terminates at an O-move, or is infinite.*

Corollary

No N*-winning strategy contains a branch where O defends.*

Lemma (Weakening)

If $\vDash_{\mathsf{N}} \psi$*, then* $\vDash_{\mathsf{N}} \varphi \rightarrow \psi$*, for all formulas* φ*.*

Properties of winning N-strategies (2)

Lemma

No atomic formula is N*-valid.*

Corollary

N *is consistent.*

Theorem (13)

If $\vDash_{\mathsf{N}} \neg\varphi$, *then* φ *is a negation* $\neg\psi$ *and* $\vDash_{\mathsf{N}} \psi$.

Characterization of implication

Theorem (Characterization of implication)

Every N*-valid implication* $\varphi \to \psi$ *satisfies one of the following three conditions: (1)* φ *is atomic, (2)* φ *is negated, (3)* $\vDash_{\mathsf{N}} \psi$.

Proof.

Case (3) is just a restatement of the Weakening Lemma. Suppose now that φ is not atomic and ψ is not an N-validity. Proceed by cases:

- If φ is an implication $\alpha \to \beta$, then the N-dialogue tree opens with O attacking the initial statement by asserting $\alpha \to \beta$. In any N-winning strategy for $(\alpha \to \beta) \to \psi$, Proponent cannot attack O's assertion of $\alpha \to \beta$. Thus, any winning strategy s must choose, for P's response to O's initial attack, to defend by asserting the consequent ψ of the entire formula, and no branch of s can attack the antecedent implication $\varphi \to \psi$. By renumbering the reference labels for nodes of s below the P's assertion of ψ in the obvious way (renumber k to $k-2$), we obtain a winning strategy for ψ, contradicting our assumption.
- Likewise, φ cannot be a disjunction, nor could it be a conjunction, for similar reasons: In any N-winning strategy s for $(\alpha \vee \beta) \to \psi$ (or for $(\alpha \wedge \beta) \to \psi$), Proponent never attacks $\alpha \vee \beta$ (respectively, $\alpha \wedge \beta$), so we can recover from s a winning strategy for ψ, contradicting our assumption.

The only possibility left is that φ is a negation. □

A positive solution to the composition problem for N

From the characterization of implication it is straightforward to prove the following:

Theorem (Composition)

If $\vDash_{\mathsf{N}} \varphi$ *and* $\vDash_{\mathsf{N}} \varphi \to \psi$, *then* $\vDash_{\mathsf{N}} \psi$.

Properties of N (1)

Theorem

N $\subset$ CL.

Proof.

Every $\mathrm{D10} + \mathrm{D13}$-strategy is also a $\mathrm{D10} + \mathrm{D13} + \mathrm{E}$-strategy. That the inclusion is strict follows from the fact that $\not\models_{\mathrm{N}} (((p \to q) \to p) \to p)$ (Peirce's law), which is classically valid. $\square$

As a corollary, N is not a connexive logic.

Properties of N (2)

Lemma

N $\not\subseteq$ IL *and* IL $\not\subseteq$ N.

Proof.

For the first claim, $\models_{\mathsf{N}} p \vee \neg p$. For the second claim, $\models_{\mathsf{IL}} (\neg p \vee \neg q) \rightarrow \neg(p \wedge q)$, which, by Theorem 13 is not N-valid, since $\not\models_{\mathrm{N}} \neg(p \wedge q)$. □

It follows from this that N is not a relevance logic, since these lie below IL. Further, since N is neither sub-intuitionistic nor super-intuitionistic, but is sub-classical, it lies in an interesting and as yet under-investigated part of the lattice of propositional logics.

Conclusion

- We made a simple and intuitive modification of the usual rules for classical dialogue games, and obtained a set N of dialogically valid formulas for which we proved a positive answer for its composition problem, thus allowing us to call N a logic.
- The positive solution was proved directly through semantic means, rather than detouring through a cut-free proof system. (No proof theory exists yet for N — but see the next talk!)
- The logic N has a number curious features which arise from the fact that if Opponent can defend once, he can always defend again.
- Philosophical questions: What kind of interpretation can N be given? If the meaning of logical connectives is given by their use (that is, how they can be attacked and defended), why is it that keeping the particle rules fixed and changing the structural rules results in such a wildly different logic? Do logics like N undermine Lorenzen's "meaning as use" interpretation of dialogical logic?