Implications as rules

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Philosophical / foundational perspective

If we want to explain the meaning of implication, we need an elementary – pre-logical – notion at the structural level.

Similar to pairing for conjunction, and schematic reasoning for universal quantification.

For that we propose the notion of a ‘rule’.

Proposal: At the logical level, an implication $A \to B$ expresses a rule $\frac{A}{B}$.

The deductive meaning of implication

Implications-as-rules corresponds to the deductive meaning of implication:

A deduction $\begin{array}{c} A \\ \vdots \\ B \end{array}$ justifies the rule $\dfrac{A}{B}$

and conversely,

the rule $\dfrac{A}{B}$ justifies a deduction $\begin{array}{c} A \\ \vdots \\ B \end{array}$.

Rules as primitives

No specific syntactic notion of deduction is presupposed.

Rules are primitive entities are rooted in everyday practice.

Rules are prior to deductions. Rule following lies at the basis of cognitive activities.

Implications as rules in natural deduction: *modus ponens*

Modus ponens can be viewed as rule application:

Read $\dfrac{A \to B \quad A}{B}$ as $A \to B \; \dfrac{A}{B}$

The implications-as-rules view is inherent in natural deduction.

Implication introduction = establishing a rule,

modus ponens = applying a rule.

Implications in the sequent calculus

This differs from the view of implication in the sequent calculus.

Gentzen’s implication left schema:

$$\frac{\Gamma \vdash A \quad \Delta, B \vdash C}{\Gamma, \Delta, A \to B \vdash C}$$

This schema, which is based on a different intuition, also underlies the dialogical interpretation.

As an alternative schema, I propose:

$$\frac{\Gamma \vdash A}{\Gamma, A \to B \vdash B}$$

This expresses the notion of implications-as-rules in the sequent calculus.

Implications-as-rules from the database perspective: *resolution*

Suppose the implication $A \to B$ is available in our database.

Then the goal B can be reduced to the goal A.

More generally: Given a database (or logic program)

$$\begin{cases} A_1 \to B \\ \quad\vdots \\ A_n \to B \end{cases}$$

then the goal B can be reduced to any of the goals A_i.

This reduction is called ‘resolution’.

Reasoning with respect to a database of implications means reading them as rules.

Summary

The understanding of implications as rules is

- philosophically fundamental
- psychologically elementary
- supported by the natural deduction view of reasoning
- supported by the database perspective

Assuming an implication means: Putting it into a (virtual) database of rules, from which it can be

- applied in forward reasoning: modus ponens
- applied in backward reasoning: resolution

Dialogues

A dialogue for $a \to (b \wedge a)$

$$\text{positions}\left\{\begin{array}{lll} 0. & \underbrace{P\ \ a\to(b\wedge a) & } \\ \end{array}\right.$$

$$\text{positions}\left\{\begin{array}{ll} 0. & \begin{array}{ll} P & a\to(b\wedge a) \end{array} \\ 1. & \begin{array}{ll} O & a \end{array} \quad [0, \text{attack}] \\ 2. & \begin{array}{ll} P & b\wedge a \end{array} \quad [1, \text{defense}] \\ 3. & \begin{array}{ll} O & \wedge_2 \end{array} \quad [2, \text{attack}] \\ 4. & \begin{array}{ll} P & a \end{array} \quad [3, \text{defense}] \end{array}\right.$$

moves

Argumentation forms

X and Y, where $X \neq Y$, are variables for P and O.

implication $\to$:	assertion:	$X\, A\to B$	
	attack:	$Y\, A$	
	defense:	$X\, B$	
conjunction $\wedge$:	assertion:	$X\, A_1\wedge A_2$	
	attack:	$Y\, \wedge_i$	(Y chooses $i = 1$ or $i = 2$)
	defense:	$X\, A_i$	

Dialogues

Dialogue (1)

A *dialogue* is a sequence of moves

(i) made alternatingly by P and O

(ii) according to the argumentation forms,

(iii) and P makes the first move.

Dialogue (2)

(D) P may assert an atomic formula only if it has been asserted by O before.

(E) O can only react on the immediately preceding P-move.

(plus some other conditions)

A dialogue beginning with $P\,A$ is called *dialogue for the formula A*.

Asymmetry between proponent P and opponent O due to (D) and (E).

Strategies

P wins a dialogue for a formula A if

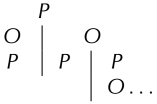
(i) the dialogue is finite,

(ii) begins with the move $P\,A$ and

(iii) ends with a move of P such that O cannot make another move.

Strategy

A *dialogue tree* contains all possible dialogues for A as paths.

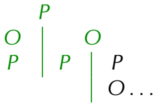


A *strategy* for a formula A is a subtree S of the dialogue tree for A such that

(i) S does not branch at even positions (i.e. at P-moves),

(ii) S has as many nodes at odd positions as there are possible moves for O,

(iii) all branches of S are dialogues for A won by P.



Strategies

Example, strategy for $(a{\to}b){\to}((b{\to}c){\to}(a{\to}c))$

0.	P	$(a{\to}b){\to}((b{\to}c){\to}(a{\to}c))$	
1.	O	$a{\to}b$	[0, attack]
2.	P	$(b{\to}c){\to}(a{\to}c)$	[1, defense]
3.	O	$b{\to}c$	[2, attack]
4.	P	$a{\to}c$	[3, defense]
5.	O	a	[4, attack]
6.	P	a	[1, attack]
7.	O	b	[6, defense]
8.	P	b	[3, attack]
9.	O	c	[8, defense]
10.	P	c	[5, defense]

A strategy for A is a proof of A.

Implications as Rules: Argumentation Forms

assertion:	$O\ A{\to}B$	
attack:	*no attack*	
defense:	*(no defense)*	

assertion:	$O\ A_1{\wedge}A_2$	
attack:	$P\ \wedge_i$	$(i = 1$ or $2)$
defense:	$O\ A_i$	

assertion:		$P\ A{\to}B$	
question:		$O\ ?$	
choice:	$P\ \lvert A{\to}B \rvert$	$P\ C$	only if $O\ C{\to}(A{\to}B)$ before
attack:	$O\ A$		
defense:	$P\ B$		

assertion:		$P\ A_1{\wedge}A_2$	
question:		$O\ ?$	
choice:	$P\ \lvert A_1{\wedge}A_2 \rvert$	$P\ C$	only if $O\ C{\to}(A_1{\wedge}A_2)$ before
attack:	$O\ \wedge_i$ $(i = 1$ or $2)$		
defense:	$P\ A_i$		

P/*O*-symmetry of argumentation forms is given up.

Implications as Rules: Dialogues and Strategies

Dialogues

(D') P may assert an atomic formula without O having asserted it before.

(E) O can only react on the immediately preceding P-move.

(F) O can question a formula A if and only if
(i) A has not yet been asserted by O, or
(ii) A has already been attacked by P.

(Strategies defined as before.)

Corresponds to sequent calculus with alternative schema

$$\frac{\Gamma \vdash A}{\Gamma, A \to B \vdash B}$$

Yields 'dialogical' interpretation of implications-as-rules concept.

Implications as Rules: Example

0.	P	$(a \to b) \to ((b \to c) \to (a \to c))$		
1.	O	?	question	
2.	P	$\lvert (a \to b) \to ((b \to c) \to (a \to c)) \rvert$	choice	
3.	O	$a \to b$	attack	assuming rule $b \leftarrow a$
4.	P	$(b \to c) \to (a \to c)$	defense	

Implications as Rules: Example

0.	$P\ (a\to b)\to((b\to c)\to(a\to c))$		
1.	O ?	question	
2.	$P\ \lvert(a\to b)\to((b\to c)\to(a\to c))\rvert$	choice	
3.	$O\ a\to b$	attack	(assuming rule $b\leftarrow a$)
4.	$P\ (b\to c)\to(a\to c)$	defense	
5.	O ?	question	
6.	$P\ \lvert(b\to c)\to(a\to c)\rvert$	choice	
7.	$O\ b\to c$	attack	assuming rule $c\leftarrow b$
8.	$P\ a\to c$	defense	

Implications as Rules: Example

0.	P	$(a \to b) \to ((b \to c) \to (a \to c))$		
1.	O	?	question	
2.	P	$\lvert (a \to b) \to ((b \to c) \to (a \to c)) \rvert$	choice	
3.	O	$a \to b$	attack	(assuming rule $b \leftarrow a$)
4.	P	$(b \to c) \to (a \to c)$	defense	
5.	O	?	question	
6.	P	$\lvert (b \to c) \to (a \to c) \rvert$	choice	
7.	O	$b \to c$	attack	(assuming rule $c \leftarrow b$)
8.	P	$a \to c$	defense	
9.	O	?	question	
10.	P	$\lvert a \to c \rvert$	choice	
11.	O	a	attack	
12.	P	c	defense	

Implications as Rules: Example

0.	$P\ (a\to b)\to((b\to c)\to(a\to c))$		
1.	$O\ ?$	question	
2.	$P\ \lvert(a\to b)\to((b\to c)\to(a\to c))\rvert$	choice	
3.	$O\ a\to b$	attack	(assuming rule $b\leftarrow a$)
4.	$P\ (b\to c)\to(a\to c)$	defense	
5.	$O\ ?$	question	
6.	$P\ \lvert(b\to c)\to(a\to c)\rvert$	choice	
7.	$O\ b\to c$	attack	(assuming rule $c\leftarrow b$)
8.	$P\ a\to c$	defense	
9.	$O\ ?$	question	
10.	$P\ \lvert a\to c\rvert$	choice	
11.	$O\ a$	attack	
12.	$P\ c$	defense	
13.	$O\ ?$	question	
14.	$P\ b$	choice	using rule $c\leftarrow b$

Implications as Rules: Example

0.	P	$(a{\to}b){\to}((b{\to}c){\to}(a{\to}c))$		
1.	O	?	question	
2.	P	$\lvert(a{\to}b){\to}((b{\to}c){\to}(a{\to}c))\rvert$	choice	
3.	O	$a{\to}b$	attack	(assuming rule $b{\leftarrow}a$)
4.	P	$(b{\to}c){\to}(a{\to}c)$	defense	
5.	O	?	question	
6.	P	$\lvert(b{\to}c){\to}(a{\to}c)\rvert$	choice	
7.	O	$b{\to}c$	attack	(assuming rule $c{\leftarrow}b$)
8.	P	$a{\to}c$	defense	
9.	O	?	question	
10.	P	$\lvert a{\to}c\rvert$	choice	
11.	O	a	attack	
12.	P	c	defense	
13.	O	?	question	
14.	P	b	choice	(using rule $c{\leftarrow}b$)
15.	O	?	question	
16.	P	a	choice	using rule $b{\leftarrow}a$

Implications as Rules: Example

0.	P	$(a \to b) \to ((b \to c) \to (a \to c))$		
1.	O	?	question	
2.	P	$\|(a \to b) \to ((b \to c) \to (a \to c))\|$	choice	
3.	O	$a \to b$	attack	(assuming rule $b \leftarrow a$)
4.	P	$(b \to c) \to (a \to c)$	defense	
5.	O	?	question	
6.	P	$\|(b \to c) \to (a \to c)\|$	choice	
7.	O	$b \to c$	attack	(assuming rule $c \leftarrow b$)
8.	P	$a \to c$	defense	
9.	O	?	question	
10.	P	$\|a \to c\|$	choice	
11.	O	a	attack	
12.	P	c	defense	
13.	O	?	question	
14.	P	b	choice	(using rule $c \leftarrow b$)
15.	O	?	question	
16.	P	a	choice	(using rule $b \leftarrow a$)

O cannot question $P\, a$ due to (F): a asserted by O before and not attacked by P.
Dialogue is won by P and is a strategy for $(a \to b) \to ((b \to c) \to (a \to c))$.

Implications as Rules and Cut

Argumentation form for Cut:	assertion:	$O\ A$	(or O ?, ...)
	attack:	$P\ \ B$	
	defense:	$O\ B$	

0.			$P\ \ a{\to}((a{\to}(b{\wedge}c)){\to}b)$		
1.			O ?	[0, question]	
2.			$P\ \ \lvert a{\to}((a{\to}(b{\wedge}c)){\to}b)\rvert$	[1, choice]	
3.			$O\ a$	[2, attack]	
4.			$P\ \ (a{\to}(b{\wedge}c)){\to}b$	[3, defense]	
5.			O ?	[4, question]	
6.			$P\ \ \lvert(a{\to}(b{\wedge}c)){\to}b\rvert$	[5, choice]	
7.			$O\ a{\to}(b{\wedge}c)$	[6, attack]	(assuming rule $(b{\wedge}c){\leftarrow}a$)
8.			$P\ \ b{\wedge}c$	[Cut]	
9.	$O\ b{\wedge}c$	[Cut]	O ?	[8, question]	
10.	$P\ \ \wedge_1$	[9, attack]	$P\ \ a$	[9, choice]	(using rule $(b{\wedge}c){\leftarrow}a$)
11.	$O\ b$	[10, defense]			
12.	$P\ \ b$	[7, defense]			