Towards a logic for reasoning under vagueness and uncertainty

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Introduction

- The key concepts of vagueness and uncertainty identify two classes of logics:
  - logics for vague reasoning [Pavelka 79], (compositional).
  - plausibility-measure based logics, i.e. possibilistic, probabilistic and belief function logics (non-compositional)

- One important difference between the two classes lies in the presence vs. absence of complete information

- Compare the proposition "The patient is young" with the proposition "The patient will survive next week".

- The former is a fuzzy proposition which is true to some degree.

- While the latter is a crisp (non-fuzzy) proposition which is either (absolutely) true or (absolutely) false; but we do not know which is the case.

- We may have some probability (chance, degree of belief) that the sentence is true; but probability is not a degree of truth.
Introduction

- In real world applications, notions of linguistic vagueness and uncertainty are combined.
- Uncertainty and vagueness are semantically quite different, so it is important to have a unifying formalism for such types of applications.
- We will investigate possible unified formalization for handling both uncertainty and vagueness, and try to find an interpretation for a specific kind of implicative rule within these unified formalization.
- We make a strong distinction between degrees of uncertainty due to a state of incomplete knowledge and intermediary degrees of truth due to the presence of vague predicates.
- Such a formal system can serve as a combined calculus for vagueness and uncertainty.
- We will focus our attention on a particular class of measures: the necessity measures.
Example

Consider the following CADIAG-2 rules, say (R1) and (R2):

(R1): \( \sigma_1 \xrightarrow{0.25} \delta_1 \)

(R2): \( \sigma_2 \& \delta_1 \xrightarrow{0.8} \delta_2 \)

- \( \sigma_1 \): aldolase (serum) highly increased,
- \( \delta_1 \): dermatomyositis,
- \( \sigma_2 \): xerostomia; and
- \( \delta_2 \): arthritis with Sjögren’s syndrome.

Assume to have the assumptions:

- \((\sigma_1, 0.7)\) and \((\sigma_2, 0.8)\).
- Then what about \((\delta_1, ??)\) and \((\delta_2, ??)\)
Certainty rules

- Certainty rules are of the form "The more $x$ is $A$, the more certainly $y$ lies in $B$",
- "The younger a man, the more certainly he is single"
- Interpreting the rule as
  - $\forall u$, if $x = u$, it is at least $\mu_A(u)$-certain that $y$ lies in $B$
- Accordingly,
  \[ S \xrightarrow{d} D \]  
  interpreted as follow:
  "it is at least $v(S) \circ d$-certain that $D$ is true" i.e.,
  \[ v(S) \circ d \leq N(D) \]
  where $\circ$ is a conjunction-like operation.
In 1976, Bellman and Zadeh discussed the notion of truth and suggested that this notion is local rather than absolute: a statement can be true only with respect to another statement held for sure.

The local truth assumption of Bellman and Zadeh, leads to treat partially true statements by reconstructing the true statements underlying them.

As with truth, certainty and possibility are considered by Zadeh as local notions.

Given uncertain statement of the form "$A$ is (at least) $\alpha$-certain$$. Let us remember that a possibility distribution $\pi$ satisfies the above property if $N_\pi(A) \geq \alpha$.

It correspond to the following possibility distribution

$$\pi(u) = \max(1 - \alpha, \mu_A(u)) \tag{3}$$
We point out that the "local" concept of truth was adopted by Boldrin and Sossai in the following articals where the authors considered the (least informative) possibility distribution where the property is true as the semantical meaning of the property, i.e. its truth value.

Definition

An adjointness algebra is an 8-tuple \((L, \leq_L, P, \leq_P, 1_P, \Rightarrow, \&\), \(\supset\)), in which \((L, \leq_L)\), \((P, \leq_P)\) are two posets with a top element \(1_P\) for \(P\).

- The operation (called an implication) \(\Rightarrow: P \times L \rightarrow L\) is antitone in the left argument and monotone in the right argument, and it has \(1 \in P\) as a left identity element.

- The operation (called a conjunction) \(\&: P \times L \rightarrow L\) is monotone in each argument and has \(1_P\) as a left identity element.

- The operation (called a comparator) \(\supset: L \times L \rightarrow P\) is antitone in the left argument and monotone in the right argument, and satisfies
  \[
  \forall y, z \in L : \quad y \supset z = 1_P \iff y \leq_L z. \quad (4)
  \]

- the three operations \(\Rightarrow\), \(\&\) and \(\supset\) are mutually related by the following adjointness condition, \(\forall a \in P, \forall y, z \in L:\)
  \[
  y \leq_L a \Rightarrow z \iff a \& y \leq_L z \iff a \leq_P y \supset z. \quad (5)
  \]
A partially ordered residuated integral monoid (porim) is a special case of adjointness algebras over one poset, in which the conjunction is a monoid operation with unit element $1_P$.

porim takes the form $(P, \leq_P, 1_P, \to, \otimes, \multimap)$

In a porim, both $\to$ and $\multimap$ become simultaneously an implication and a comparator.

If the poset $(P, \leq_P)$ is a lattice, then the algebra $(P, \leq_P, \wedge, \vee, 1_P, \to, \otimes, \multimap)$ is called an integral residuated lattice.

When $\otimes$ is also commutative, this adjointness algebra becomes $(P, \leq_P, 1_P, \otimes, \to)$ (pocrim).
Numerous articles and books are concerned with residuation (= adjointness) in the multiple-valued-logic component of fuzzy logic.

Adjointness lies also at the basis of algebraic logic and substructural logics.

On the one hand, the condition Adjointness is a main tool in building a useful calculus for implications; generating universally valid inequalities.

On the other hand, it is now known that the membership values have different semantics, which frequently coexist in the same application (see, D. Dubois, H. Prade, The three semantics of fuzzy sets, Fuzzy Sets and Systems 90 (1997) 141-150).

It is natural to admit noncommutative, nonassociative conjunctions between truth values of differing semantics. Similar argument restricts the need for the comparator axiom.
A binary operation \( \star \) on \( P \) is said to tie an implication operation \( \Rightarrow : P \times L \rightarrow L \) if the following identity holds

\[
(((x \star y) \Rightarrow c) = (x \Rightarrow (y \Rightarrow c))).
\]  

(6)

- It can be seen as a weakened form of associativity.
- It extends to multiple-valued logic the law of importation from classical logic.
- It holds for several types of implications used in fuzzy logic.

**Theorem**

In adjointness algebras, if \( \Rightarrow \) is tied, (satisfies the EP) and d.l.a., then it is tied by a supremum-preserving (commutative) integral ordered monoid operation on \( P \), called here a tying-conjunction.
Tied adjointness algebras

Definition

A tied adjointness algebra is an algebra

\[ \Lambda = (P, \leq_P, 1_P, L, \leq_L, \Rightarrow, \& , \supset , \otimes , \to) \]

in which, \((L, \leq_L, P, \leq_P, 1_P, \Rightarrow, \& , \supset)\) is an adjointness algebra, \((P, \leq_P, \otimes, \to, 1_P)\) is a commutative porim, and \(\otimes\) ties \(\Rightarrow\); that is,

\[ \forall a, b \in P, \forall z \in L : \]

\[ ((a \otimes b) \Rightarrow z) = (a \Rightarrow (b \Rightarrow z)) \] (7)

Three further identities of interest:

\[ \otimes\text{ ties } \& : ((a \otimes b) \& z) = (a \& (b \& z)) \] (8)

\[ (a \& y \supset z) = (a \to (y \supset z)) \] (9)

\[ (y \supset (a \Rightarrow z)) = (a \to (y \supset z)) \] (10)
Tied adjointness algebras constitute a particularly rich generalization of residuated algebras and exhibits more implications employed frequently in fuzzy logic.

Nevertheless, all the algebraic (in)equality we know to be valid in all residuated algebras remain true for tied algebras, but in forms that distribute roles for the five connectives of the algebra.

Accordingly, a main objective behind tied adjointness algebras is to provide a framework, through which all useful properties of residuated algebras can extend over a much wider scope of logical connectives already in use in fuzzy logic.

This work is founded in the well-established domain of residuated lattices, whereby implications and conjunctions are related by residuation (= adjointness).
This two-lattices approach was started algebraically in


and then adopted by Morsi, Lotfallah and El-Zekey in


Then it was formulated syntactically, within the first order logic


The logic of tied implication build on the work of

We point out that there have been many papers, both theoretical and showing their usefulness in approximate reasoning in the recent past. For instance,

- Mas et al., The law of importation for discrete implications, Info. Sci., Volume 179:24 (December 2009), 4208-4218.
A logic for Tied Implications (AdjTPC)

- A complete syntax has been developed for the semantics based on tied adjointness algebras.
- The semantics of AdjTPC assumes two partially ordered sets $L$ and $P$, whose elements are considered as *truth-values*, as well as the five logical connectives of a tied adjointness algebra.
- Such a formal system can serve as a combined calculus for a pair of two, possibly different, types of uncertainty.
The language of \textbf{AdjTPC} consists of two denumerable sets $WF_L$ and $WF_P$ of formulae, and five logical connectives as follows:

- an \textit{implication} $\Rightarrow$: $WF_P \times WF_L \rightarrow WF_L$,
- a \textit{conjunction} $\&$: $WF_P \times WF_L \rightarrow WF_L$,
- a \textit{comparator} $\supset$: $WF_L \times WF_L \rightarrow WF_P$,
- a \textit{tying-conjunction} $\star$: $WF_P \times WF_P \rightarrow WF_P$,
- an \textit{R-implication} $\rightarrow$: $WF_P \times WF_P \rightarrow WF_P$.

$WF_L$ and $WF_P$ are constructed from two disjoint denumerable subsets $WF_{L_0}$ and $WF_{P_0}$ of atomic formulae, by means of repeated application of the logical connectives.
Axioms of AdjTPC

(A1) ⊢ ξ → (τ → τ)

(A2) ⊢ (ξ → (τ → ζ)) → (((ξ * τ) → ζ)

(A3) ⊢ (((ξ * τ) → η) → (((η → ζ) → (τ → (ξ → ζ)))))

(A4) ⊢ ξ → (β ⊃ β)

(A5) ⊢ (ξ → (β ⊃ γ)) → (((ξ & β) ⊃ γ)

(A6) ⊢ (((ξ & β) ⊃ δ) → (((δ ⊃ γ) → (β ⊃ (ξ ⇒ γ)))))

(A7) ⊢ (β ⊃ (ξ ⇒ γ)) → (ξ → (β ⊃ γ)).

Inference Rule for AdjTPC:

MP: ξ, ξ → τ ⊢ τ (ξ, τ ∈ WFₚ) (Modus Ponens for R-implication).
Note that the theorems of AdjTPC are certain formulae in WF$_P$, but their subformulae are from WF$_P$ ∪ WF$_L$.

Whereas WF$_L$ does not support a notion of absolute truth, and does not contain theorems.

As such, MP always derives one formula in WF$_P$ from two formulae in WF$_P$.

A theory Γ over AdjTPC is a set of formulae in WF$_P$.

An inference (deduction, proof or derivation) Γ ⊢ τ in a theory Γ is defined as usual from the above axioms and MP.

The notation Γ ⊢ α ⊃ β is an abbreviation of Γ ⊢ {α ⊃ β, β ⊃ α}.

The notation Γ ⊢ ξ ←→ τ is an abbreviation of Γ ⊢ {ξ → τ, τ → ξ}. 
Semantics of AdjTPC

An interpretation of AdjTPC is a triple $\mathcal{I} = (\Lambda, \pi_L, \pi_P)$ in which
$\Lambda = (L, \leq_L, P, \leq_P, 1, \Rightarrow', \&', \supset', \star', \rightarrow')$ is a tied adjointness algebra,

\[
\begin{align*}
\pi_L &: WF_L \longrightarrow L \\
\pi_P &: WF_P \longrightarrow P
\end{align*}
\] (11) (12)

valuation functions (or truth functions) of the interpretation;

\[
\begin{align*}
\pi_L (\xi \Rightarrow \gamma) &= \pi_P (\xi) \Rightarrow' \pi_L (\gamma), \\
\pi_L (\xi \& \beta) &= \pi_P (\xi) \&' \pi_L (\beta), \\
\pi_P (\beta \supset \gamma) &= \pi_L (\beta) \supset' \pi_L (\gamma), \\
\pi_P (\xi \star \tau) &= \pi_P (\xi) \star' \pi_P (\tau), \\
\pi_P (\xi \rightarrow \tau) &= \pi_P (\xi) \rightarrow' \pi_P (\tau).
\end{align*}
\] (13) (14) (15) (16) (17)
If $\pi_P(\xi) = 1$ in an interpretation $\mathcal{G}$, $\xi$ is said to be valid (or, true) in $\mathcal{G}$, and we write $\mathcal{G} \models \xi$.

Let $\Gamma$ be a theory over AdjTPC, if $\mathcal{G} \models \lambda$ for all $\lambda \in \Gamma$, we write $\mathcal{G} \models \Gamma$, and we say that $\mathcal{G}$ is a model of $\Gamma$.

If for every interpretation $\mathcal{G}$ such that $\mathcal{G} \models \Gamma$ we have $\mathcal{G} \models \delta$, then we write $\Gamma \models \delta$.

If $\mathcal{G} \models \xi$ for all interpretations $\mathcal{G}$ of AdjTPC, we say that $\xi$ is universally valid (or, a tautology), and we write $\models \xi$.

**Theorem (Completeness)**

Let $\Gamma$ be a theory over AdjTPC. Then for any formula $\xi$ in WF$_P$, $\Gamma \vdash \xi$ if and only if $\Gamma \models \xi$. 
The prelinear tied propositional calculus

In a tied adjointness lattice on \((L, P)\), the meet and join operations on \(P\) will be denoted by \(\land\) and \(\lor\), and on \(L\) by \(\neg\) and \(\top\).

**Definition**

A prelinear tied adjointness algebra is a tied adjointness lattice \(\Lambda = (L, \leq_L, P, \leq_P, 1, \Rightarrow, \& \lor, \ast, \rightarrow, \land, \lor, \neg, \top)\) satisfying the following prelinearity equation for \(\rightarrow\):

\[
\forall a, c \in P : \ (a \rightarrow c) \lor (c \rightarrow a) = 1 \quad (18)
\]

We denote the class of all prelinear tied adjointness algebras by \(L\text{-ADJT}\).

We develop a complete syntax for the semantic domain \(L\text{-ADJT}\). We call it **Propositional Calculus for Prelinear Tied Adjointness Algebras**, abbreviated to \(L\text{-AdjTPC}\).
Example

Let \((P, \star, \rightarrow)\) be a complete residuated chain, let \(U\) be a set, and let \(P^U\) be the product lattice. Define the tied adjointness algebra \(\Lambda = (P^U, P, \Rightarrow, \& , \supseteq , \star, \rightarrow)\) as follows: \(\forall a \in P, \forall \lambda, \mu \in P^U, \forall u \in U:\)

\[
(a \& \lambda)(u) = a \star \lambda(u) \\
(a \Rightarrow \mu)(u) = a \rightarrow \mu(u) \\
\lambda \supseteq \mu = \inf_{w \in U} (\lambda(w) \rightarrow \mu(w))
\]

\(P\) is linear and \(P^U\) is not, \(\Lambda\) is prelinear (i.e. \(\rightarrow\) satisfies the prelinearity but \(\supseteq\) is not).

- Stronger versions of (pre-)linearity are formulated for tied adjointness algebras; by requesting both of the two comparators \(\rightarrow\) and \(\supseteq\) are prelinear.
- Our weak (pre-)linearity version is adequate for the purposes of this paper, and this weak (pre-)linearity accommodates the algebra in the above Example.
L-AdjTPC: The language

- The logic **L-AdjTPC** is an extension of **AdjTPC**.
- The language of **L-AdjTPC** is the language of **AdjTPC**, expanded by the connectives $\land$, $\bar{\land}$, $\lor$.

$\land: WF_P \times WF_P \mapsto WF_P$, is called a *weak conjunction* on $WF_P$.

$\bar{\land}: WF_L \times WF_L \mapsto WF_L$, is called a *conjunction* on $WF_L$.

$\lor: WF_L \times WF_L \mapsto WF_L$, is called a *disjunction* on $WF_L$.

Further definable connectives $\lor$ and $\neg$ (called *disjunctions* and *negation* on $WF_P$) are

$\zeta \lor \tau = (((\zeta \rightarrow \tau) \rightarrow \tau) \land ((\tau \rightarrow \zeta) \rightarrow \zeta)$ \hspace{1cm} (19)

$\neg \zeta = \zeta \rightarrow 0$. \hspace{1cm} (20)
Axioms of L-AdjTPC:

The seven axioms of \textbf{AdjTPC} together with the following axioms:

\begin{align*}
\text{A8: } & \vdash \xi \land \tau \rightarrow \xi. \\
\text{A9: } & \vdash \xi \land \tau \rightarrow \tau \land \xi. \\
\text{A10: } & \vdash (\xi \ast (\xi \rightarrow \tau)) \rightarrow \xi \land \tau. \\
\text{A11: } & \vdash \alpha \overset{\land}{\supset} \beta \supset \alpha \overset{\lor}{\supset} \beta. \\
\text{A12: } & \vdash \gamma \overset{\lor}{\supset} \gamma \supset \gamma \\
\text{A13: } & \vdash \beta \supset \beta \overset{\land}{\supset} \\
\text{A14: } & \vdash (\beta \supset \gamma) \rightarrow (\alpha \overset{\lor}{\supset} \beta \supset \gamma \overset{\lor}{\supset} \alpha). \\
\text{A15: } & \vdash (\beta \supset \gamma) \rightarrow (\alpha \overset{\land}{\supset} \beta \supset \gamma \overset{\land}{\supset} \alpha). \\
\text{A16: } & \vdash (((\xi \rightarrow \tau) \rightarrow \chi) \rightarrow (((\tau \rightarrow \xi) \rightarrow \chi) \rightarrow \chi). \\
\text{A17: } & \vdash 0 \rightarrow \xi. \\
\end{align*}

\textbf{Inference Rule for L-AdjTPC:}

\textbf{MP: } \xi, \xi \rightarrow \tau \vdash \tau \ (\xi, \tau \in WF_P) \ (\text{Modus Ponens for R-implication}).
Our axioms on \((WF_P, \star, \rightarrow, \land, \lor)\), only, are equivalent through MP to the axioms of the Monoidal t-norm Based Logic (MTL), of Esteva and Godo.

This extension is conservative.

In consequence, all theorems and deductions in MTL are perforce theorems and deductions in the residuated part \((WF_P, \star, \rightarrow, \land, \lor)\) of L-AdjTPC (and vice-versa).
Theorem (Representation Theorem)

Each prelinear tied adjointness algebra is a subdirect product of a system of tied adjointness chains.

- Tied adjointness chains are tied adjointness algebras in which their residuated parts only are linear.

Theorem (Completeness)

$L$-$AdjTPC$ is strongly generally complete and strongly chain complete for $L$-$ADJT$. Specifically, let $\Gamma$ be a theory over $L$-$AdjTPC$ and $\tau$ be a formula in WF$_P$. Then the following are equivalent:

(i) $\Gamma \vdash \tau$.

(ii) For each $L$-$ADJT$-model $\mathcal{S} = (\Lambda, \pi_L, \pi_P)$ of $\Gamma$, $\mathcal{S} \models \tau$.

(iii) For each linearly ordered $L$-$ADJT$-model $\mathcal{S} = (\Lambda, \pi_L, \pi_P)$ of $\Gamma$, $\mathcal{S} \models \tau$. 
The logic \textbf{N-AdjTPC}

- We introduce multiple-valued \textbf{modal} logic based on the logic \textbf{L-AdjTPC}.
- The language of \textbf{N-AdjTPC} is the language of \textbf{L-AdjTPC}, expanded by
  - \( \Box : WF_L \leftrightarrow WF_P \), denoting the modality necessity.
- The modal operator is a part of the language.
- Hence mixing of pure propositional formulas and modal formulas, and nested modal operators are allowed.
- Such a formal system can serve as a combined calculus for a pair of two different types of uncertainty (i.e., vagueness and uncertainty measure).
- We will focus our attention on a particular class of measures: the necessity measures.
N-AdjTPC: The Axioms

- Axioms of **N-AdjTPC** are the axioms of **L-AdjTPC** together with the following axioms:

**N1:** \( \vdash \neg \Box \bot \).

**N2:** \( \vdash (\alpha \supset \beta) \rightarrow (\Box \alpha \rightarrow \Box \beta) \).

**N3:** \( \vdash \Box(\xi \Rightarrow \gamma) \leftrightarrow (\xi \rightarrow \Box \gamma) \).

Inference Rule for N-AdjTPC:

**MP:** \( \xi, \xi \rightarrow \tau \vdash \tau \) (\( \xi, \tau \in WF_P \)) (Modus Ponens for R-implication).

The following are provable in **N-AdjTPC**:

1: \( \vdash \Box(\alpha \land \beta) \leftrightarrow (\Box \alpha \land \Box \beta) \).

2: \( \vdash (\xi \times \Box \gamma) \rightarrow \Box(\xi \& \gamma) \).

3: \( \vdash \Box(\Box \gamma \Rightarrow \gamma) \).
The semantics of N-AdjTPC:
We consider a class of necessity Kripke models, denoted by \textbf{N-ADJT}.

\textbf{Definition}

A \textit{weak necessity Kripke model} for \textbf{N-AdjTPC} is a system \( \mathcal{S} = (\Lambda, \pi_P, \pi_L, \mu) \) where:

- \( \Lambda \) is a prelinear tied adjointness algebra.
- \( \mu : L \rightarrow P \) is a necessity over \( L \), i.e. it satisfies: \( \forall x, y \in L \), \( a \in P \)
  - \( \mu(\perp) = 0 \) and \( \mu(\top) = 1 \)
  - \( \mu(x \overline{\wedge} y) = \mu(x) \wedge \mu(y) \)
  - Moreover, \( \mu \) satisfies the following additional property:
    - \( \mu(a \Rightarrow y) = a \rightarrow \mu(y) \)
- \( \pi_L : WF_L \rightarrow L \) and \( \pi_P : WF_P \rightarrow P \) are truth functions of the interpretation as before
- The truth value of modal formulas is defined as: for all \( \beta \in WF_L \)
  \[ \pi_P(\Box \beta) = \mu(\pi_L(\beta)). \]
Completeness of N-AdjTPC logic:

- A necessity Kripke model for N-AdjTPC \( \mathcal{S} = (\Lambda, \pi_P, \pi_L, \mu) \) is linear, when \( \Lambda \) is linear.

**Theorem (Completeness)**

N-AdjTPC is strongly generally complete and strongly chain complete for the class of weak N-ADJT. Specifically, let \( \Gamma \) be a theory over N-AdjTPC and \( \tau \) be a formula in WF\(_P\). Then the following are equivalent:

- (i) \( \Gamma \vdash \tau \).
- (ii) For each weak N-ADJT-necessity Kripke model \( \mathcal{S} = (\Lambda, \pi_P, \pi_L, \mu) \) of \( \Gamma \), \( \mathcal{S} \models \tau \).
- (iii) For each linearly ordered weak N-ADJT-necessity Kripke model \( \mathcal{S} = (\Lambda, \pi_P, \pi_L, \mu) \) of \( \Gamma \), \( \mathcal{S} \models \tau \).
A **strong necessity Kripke model** for **N-AdjTPC** is a system 
\( \mathcal{S} = (W, \Lambda, \pi_P, \pi_{L_0}, \mu) \) where:

- \( W \) is a non-empty set of possible worlds (or possible situations)
- \( \Lambda = (L, \leq_L, P, \leq_P, \Rightarrow, \&, \lor, \neg, \rightarrow, \land, \lor, \neg, \lor) \) is a prelinear tied adjointness algebra, in which
  - \( L = \{ f : W \rightarrow P \} \)
  - \( L_0 = \{ f : W \rightarrow \{0, 1\} \} \subseteq L \)
- \( \mu : L \rightarrow P \) is a necessity over \( L \) as before.
- \( \pi_L : WF_{L_0} \rightarrow L_0 \) and \( \pi_P : WF_P \rightarrow P \) are truth functions
- We define the truth function \( \pi_L : WF_L \rightarrow L \) in a usual way (by extending \( \pi_{L_0} : WF_{L_0} \rightarrow L_0 \) using operations from \( \Lambda \))
- The truth value of modal formulas is defined as before.
Strong necessity Kripke semantics of N-AdjTPC

Theorem

Let $\Gamma$ be a theory over $\textbf{N-AdjTPC}$ and $\tau$ be a formula in $WF_P$. Then

(i) $\Gamma \vdash \tau$.

(ii) For each strong N-ADJT-necessity Kripke model $\mathcal{S} = (W, \Lambda, \pi_P, \pi_{L_0}, \mu)$ of $\Gamma$, $\mathcal{S} \models \tau$.

(iii) For each linearly ordered strong N-ADJT-necessity Kripke model $\mathcal{S} = (W, \Lambda, \pi_P, \pi_{L_0}, \mu)$ of $\Gamma$, $\mathcal{S} \models \tau$.

- **Soundness:** $(i) \Rightarrow (ii) \Rightarrow (iii)$
- **Completeness:** ????

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