DIPLEAP, Vienna 2010

A fuzzy semantics of the feasible knowledge

Libor Běhounek

Institute of Computer Science
Academy of Sciences of the Czech Republic

Standard epistemic logic

Modality $\mathsf{K}A$ = "the agent knows that A"

The principle of logical rationality of the agent

= the assumption that the agent can make inference steps

$\Rightarrow$ the axiom (K) of propositional epistemic logic:

$$\mathsf{K}A \,\&\, \mathsf{K}(A \to B) \to \mathsf{K}B$$

The axiom is adopted in standard accounts of epistemic logic

Standard epistemic logic = the logic of *logically rational* agents

The logical omniscience paradox

An unwanted consequence of the logical rationality principle:

the agent's knowledge is closed under modus ponens
$\Rightarrow$ under the propositional consequence relation

$\Rightarrow$ the agent knows all propositional tautologies, once
he/she/it knows the axioms of CPC

= an extremely implausible assumption on real-world agents
(consider, eg, a non-trivial tautology with 10^9 variables)

Three kinds of knowledge

Actual knowledge . . . the modality "is *known*"
= knowledge immediately available to the agent
(eg, the contents of its memory)

Potential knowledge . . . the modality "is *knowable*"
= knowledge in principle derivable from the actual knowledge
(by logical inference)

Feasible knowledge . . . the modality "is *realistically* knowable"
= knowledge effectively derivable from the actual knowledge
(taking the agent's physical restrictions into account)

The scope of the logical omniscience paradox

The logical omniscience paradox only affects feasible knowledge:

Actual knowledge is not closed under inference steps
$\Rightarrow$ the axiom (K) is not plausible for actual knowledge

Potential knowledge is indeed closed under logical consequence
$\Rightarrow$ no paradox there

Feasible knowledge, however, seems to be:

- closed under single inference steps
(the agent *can* make them)
- yet not closed under the consequence relation as a whole
(the agent cannot feasibly know all logical truths)

Logical omniscience as an instance of the Sorites

The problem with feasible knowledge is that the agent

- can always make a next step of inference, but
- cannot make an arbitrarily large number of inference steps

= An instance of the sorites paradox for the predicate

$$P(n) \equiv \text{“the agent can make at least } n \text{ inference steps”}$$

Solutions to the logical omniscience paradox

⇒ Every solution to the sorites paradox generates
a solution to the logical omniscience paradox

Eg:

- An epistemicistic solution
  = we cannot know how many steps the agent can make
- A supervaluationistic solution
  = considering all possible bounds on the number of steps
- A degree-theoretical solution
  . . . to be elaborated here

Why a degree-theoretical solution?

There have been many objections against degree-theoretical solutions to the sorites

However, a degree-theoretical solution is particularly suitable to the logical omniscience instances of the sorites, since

- the degrees have a clear interpretation (in terms of costs of the feasible task)
- they can be manipulated by suitable many-valued logics (viz, t-norm logics)
- the (implausible) existence of a sharp breaking point in the number of steps the agent can perform is not presupposed

Resource-aware reasoning about knowledge

What limits the agent's ability to infer knowledge is
the agent's limited resources (time, memory, . . .)

⇒ Resource-aware reasoning about the agent's knowledge needed

Several models of resource-aware reasoning are available
(eg, in dynamic logic)

Fuzzy logics are applicable to resource-aware reasoning, too, capturing moreover the gradual nature of feasibility
(some tasks are more feasible than others)

Connectives in t-norm fuzzy logics

Conjunction $\&$ . . . a left-continuous t-norm
(ie, commutative associative monotone operation with unit 1)

Implication $\to$ . . . its residuum
(ie, the weakest operation validating $A \mathbin{\&} (A \to B) \to B$)

Lattice connectives $\wedge$, $\vee$ . . . the minimum resp. maximum
NB: two conjunctions regularly present in substructural logics

Equivalence $\leftrightarrow$ . . . $A \leftrightarrow B \equiv (A \to B) \mathbin{\&} (B \to A)$
(equivalently, $A \leftrightarrow B \equiv (A \to B) \wedge (B \to A)$)

Negation $\neg$ . . . $\neg A \equiv A \to 0$ (reductio ad absurdum)

Salient examples of t-norm logics

Gödel–Dummett logic (G):

$\|A \,\&\, B\| = \mathrm{min}(\|A\|\,, \|B\|)$

$\|A \to B\| = \|B\|$ if $\|A\| > \|B\|$, otherwise 1

Łukasiewicz logic (Ł):

$\|A \,\&\, B\| = \max(0, \|A\| + \|B\| - 1)$

$\|A \to B\| = 1 - \|A\| + \|B\|$ if $\|A\| > \|B\|$, otherwise 1

Product logic (Π):

$\|A \,\&\, B\| = \|A\| \cdot \|B\|$

$\|A \to B\| = \|B\| \,/\, \|A\|$ if $\|A\| > \|B\|$, otherwise 1

Tautologicity and validity

A is a tautology (wrt a given left-continuous t-norm)
$\equiv$ A evaluates to 1 for all $[0,1]$-evaluations of atoms

Hilbert-style axiomatization is known for each continuous t-norm
and many other left-continuous ones

$\|A \to B\| = 1$ iff $\|A\| \leq \|B\|$ (in all t-norm logics)

$\Rightarrow$ Tautologies of the form $A_1 \,\&\, \ldots \,\&\, A_n \to B$ represent
degree-preserving rules of inference
(the degree of B is at least the &-combination of the degrees A_i)

= inference *salvo gradu* (cf. classical inference *salva veritate*)

Resource-based interpretation of t-norm logics

Interpret degrees as representing costs of tasks
expressed by or associated with formulae, where

1 = for free

0 = a maximal cost (NB: order reversed!)

Connectives of t-norm logics represent some natural operations with costs:

- Conjunction = combination of costs
- Implication = the 'surcharge' for B, given the price of A
  = remaining (in terms of &) cost for B, given the cost for A
- Lattice connectives = max resp. min of the two costs
- Negation = &-remainder to the maximal cost, etc.

Combination of costs in basic t-norm logics

Product logic:

& = addition of costs (via the logarithm)

0 = the infinite cost

Łukasiewicz logic:

& = bounded addition of costs (via a linear function)

0 = the maximal (or unaffordable) cost

Gödel logic:

& = the maximum of costs

natural, eg, in space complexity (erase temporary memory)

Other t-norm logics:

& = certain other ways of cost combination

(eg, additive up to some bound, then maxitive)

Cost-aware reasoning in t-norm logics

Tautologies of the form $A_1 \mathbin{\&} \ldots \mathbin{\&} A_n \to B$ represent cost-preserving rules of inference
(the cost of B is at most the &-combination of the costs of A_i)
= inference *salvis expensis*

Cf. linear logic (which is closely related to t-norm logics) and Girard's example of resource-aware reasoning:

$(F \to M) \mathbin{\&} (F \to C) \to (F \to M \mathbin{\&} C)$ $\quad\dashv$ CPC, $\not\dashv$ LL, Ł, Π

$(F \to M) \mathbin{\&} (F \to C) \to (F \mathbin{\&} F \to M \mathbin{\&} C)$ $\quad\dashv$ LL, Ł, Π

where F = a fiver, M/C = a pack of Marlboro/Camel

Feasibility in t-norm logics

Atomic formulae of t-norm logics can thus be understood as standing under the implicit graded modality *is affordable*, or *is feasible*

The degree of feasibility is inversely proportional (via a suitable normalization function) to the cost of realization (eg, the number of processor cycles)

Logical connectives then express natural operations with costs

Tautologies express degree/cost-preserving rules of inference

Feasible knowledge in t-norm logic

Given the degrees of (the feasibility of) KA and K$(A \to B)$, the degree of KB (inferred by the agent) needs to make allowance for the (small) cost of performing the inference step of modus ponens by the agent (denote it by the atom MP)

The plausible axiom of logical rationality for feasible knowledge in t-norm logics thus becomes:

$$\mathsf{K}A \mathbin{\&} \mathsf{K}(A \to B) \mathbin{\&} \mathsf{MP} \to \mathsf{K}B$$

Logical omniscience in t-norm logics

Since the degree of MP is slightly less than 1
(as the cost of performing modus ponens is small, but non-zero),
it slightly diminishes the degree of the inferred knowledge KB

For longer derivations (of B from $A_1, \dots, A_k$) that require n inference steps, the axiom only yields (where $A^n \equiv A \,\&\, \overset{n}{\ldots} \,\&\, A$)

$$\mathsf{K}A_1 \,\&\, \dots \,\&\, \mathsf{K}A_k \,\&\, \mathsf{MP}^n \to \mathsf{K}B$$

Since & is non-idempotent in t-norm logic, the degree of MP^n
(and so the guaranteed degree of KB) decreases,

- limiting to 0 in product logic (unbounded resources)
- reaching 0 in Łukasiewicz logic (bounded resources)

Elimination of the paradox in t-norm logics

Thus in models over t-norm logics,

- The feasibility of knowledge decreases with long derivations (as it intuitively should)
- The closure of feasible knowledge under logical consequence is only gradual (fading away with the increasing difficulty of derivation),
- Yet the agents are still perfectly logically rational (able to perform each inference step, at appropriate costs)

$\Rightarrow$ No paradox under suitable t-norm logics (Ł or Π)

Other epistemic paradoxes are eliminated in t-norm logics as well (eg, the knowability paradox $A \to \Diamond \mathsf{K} A$)

Other approaches to logical omniscience

Weakening the epistemic part of the logic:

(a) syntactic: restricting the rules on K

(b) semantic: admitting inconsistent epistemic states

Does not work:

- either the logic is so weak as to be useless, or
- the agents remain logically omniscient in some weaker logic

Other approaches to logical omniscience

Explicit indication of the length of derivation:
The single modality K is replaced by modalities K^n for all n
= *is knowable after at most* n *steps of derivation*

Explicit rules for calculating the exponents are given
eg: $\mathsf{K}^n A \,\&\, \mathsf{K}^m(A \to B) \to \mathsf{K}^{m+n+1} B$

Works, but is not elegant:

- Formalism heavy with indices (already in axioms)
- Syntax dependent on particular costs of inference steps
(difficult to adjust to different costs)

Other approaches to logical omniscience

Modeling inference in dynamic logic
= acknowledging that inference is an action:

Introduce an atomic modality (program) for each inference rule
eg, $\langle \mathsf{MP} \rangle$

Use the multimodal dynamic machinery for longer derivations
eg, $\langle \mathsf{MP}; \mathsf{MP}; \mathsf{MP} \rangle$ for a 3-step derivation

Formulate the logical rationality axioms accordingly
eg, $\mathsf{K}A \,\&\, \mathsf{K}(A \to B) \to \langle \mathsf{MP} \rangle \mathsf{K}B$

And derive the theorems on K in dynamic logic

Works, and is in fact closely related to our approach

Relationship between dynamic and t-norm logics

Assign a cost of execution to each atomic program

(unit or otherwise)

The cost of complex programs are expressible by formulae of

(higher-order) t-norm logics

Eg, composition of programs = addition of their costs

= product logic's conjunction

Ie, t-norm logics describe a particular feature of dynamic models

⇒ Dynamic logic gives a fuller account of

how the agent derives feasible knowledge

T-norm logics extract an important part of it (costs = money!)