Some further properties of the dialogical logic N

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Contents

- Review of dialogue games
- The curious dialogical logic N
- Challenge: find a tractable metatheory for N
- Some validity-preserving rules
- Failure of uniform substitution; isolation of some (kinds of) validity-preserving substitutions

Lorenzen dialogue games

- Dialogue games have two players, **P**roponent and **O**pponent. Play alternates; **P** begins. Players *attack* or *defend* statements that have already been played.
  - *Particle rules* say what kinds of moves are available based on the structure of formulas;
  - *Structural rules* govern the overall shape of the game.
- A player loses when he can make no further move that adheres to the particle and structural rules.

Particle Rules

Formula	Attack	Defense
$\alpha \wedge \beta$	?L	α
	?R	β
$\alpha \vee \beta$	?	α or β
$\neg\alpha$	α	—
$\alpha \to \beta$	α	β
$\exists x \alpha$	?	$\varphi(c)$
$\forall x \alpha$	c	$\varphi(c)$

Procedural Rules for Classical Logic

- Proponent may assert an atomic formula only after Opponent has asserted it;
- An assertion made by Proponent may be attacked at most once.
- (E) Opponent must respond to Proponent's immediately previous move.

Call this set of rules CL.

Theorem: **P** has a winning strategy for the CL-dialogue commencing with φ iff φ is classically valid (a tautology).

A new dialogical logic: N

- Let N be the set of (propositional) formulas for which **P** has a winning strategy a dialogue game that adheres to to the rules $\mathrm{CL} - \{\mathrm{E}\}$.
- N is thus characterized by dialogue games, and we know of no other characterization of N.
- The *composition problem* for dialogical logic: to characterize the properties of (sets of) dialogue rules such that they give rise to a logic, which, for us, means: closed under modus ponens.
- N is an early experiment motivated by the composition problem.

Challenge: Can we find a tractable metatheory for N,
such as an axiomatization or a set of nice ‘proof rules’?

Some curiosities about N

- Although N is closed under modus ponens, it nonetheless fails to validate modus ponens, considered as a formula: $(\varphi \wedge (\varphi \rightarrow \psi)) \rightarrow \psi$. (No instance of this scheme is N-valid.) An alternative 'implicational' reading of modus ponens as a formula,

$$\varphi \rightarrow ((\varphi \rightarrow \psi) \rightarrow \psi),$$

  is likewise not N-valid (in general).

- If $\vDash \varphi \wedge \psi$, then $\vDash \varphi$; but the formula $(\varphi \wedge \psi) \rightarrow \varphi$ is not N-valid.

Despite these curiosities, some positive, familiar results are available.

Notation: '$\vDash \varphi$' means that φ is valid in N.

‘Axioms’ for N

$$
\begin{array}{cc}
p \vee \neg p & \neg p \vee \neg\neg p \\
(p \to q) \vee (p \to \neg q) & (p \to q) \vee (q \to p) \\
\neg\neg p \to p & p \to \neg\neg p \\
p \to (p \vee q) & p \to (p \wedge p) \\
\neg p \to (p \to q) & \neg(p \vee \neg p) \to q
\end{array}
$$

One can verify by calculation that these are N-valid.

(Don’t do the calculation by hand! Use **http://dialogical-logic.info**, our sandbox for exploring dialogues and dialogical logic.)

These formulas are sufficiently simple that they are plausible candidates for ‘axioms’ of N.

Some easy N-validity-preserving transformations

- $\models \psi$ implies $\models \varphi \to \psi$ (weakening)
- $\models \varphi$ iff $\models \neg\neg\varphi$ (double negation)
- $\models \varphi \wedge \psi$ iff $\models \varphi$ and $\models \psi$ (conjunction)

The first can be proved by simply noting that when playing the game for $\varphi \to \psi$, **O** must begin the game by asserting φ; **P** can just ignore this information and continue using his winning strategy for ψ.

The second follows from a characterization of N-valid negations: if $\models \neg\varphi$, then φ is a negation, $\neg\psi$, and $\models \psi$.

The third is simple consequence of the particle rules.

Contraposition

Theorem: If $\vDash \varphi \to \psi$, then $\vDash \neg\psi \to \neg\varphi$.

Proof: Every winning strategy for $\varphi \to \psi$ begins thus

0	**P**	$\varphi \to \psi$	*(initial move)*
1	**O**	φ	[A,0]

with **O** asserting φ.

The winning strategy continues beyond move 1, since in this dialogue **P** loses. But we can't say exactly *how* it continues.

In any event, **P** can continue the game after **O**'s assertion of in such a way that **P** can always bring the game to an end, with **P** winning, no matter how **O** plays.

Contraposition (continued)

Now consider the following opening of an N-dialogue for $\neg\psi \to \neg\varphi$ (we will show how to extend this to a winning strategy):

0	**P**	$\neg\psi \to \neg\varphi$	*(initial move)*
1	**O**	$\neg\psi$	[A,0]
2	**P**	$\neg\varphi$	[D,1]
3	**O**	φ	[A,3]

O's move at step 3 is forced: the initial move cannot be re-attacked because **P**'s statements cannot be attacked more than once.

Now **P** continues the game according to the winning strategy for $\varphi \to \psi$.

(Such 'gluing' of the earlier winning strategy onto this initial sequence of moves does indeed produce a winning strategy because **O** can respond to none of **P**'s assertions made in these first four moves.)

Exchange

Theorem: if $\vDash \varphi \to \psi \to \theta$, then $\vDash \psi \to \varphi \to \theta$.

Proof: Suppose we have a winning strategy for $\varphi \to \psi \to \theta$. We will use it later.

To define a winning strategy for $\psi \to \varphi \to \theta$, consider the following opening of an N-dialogue for $\psi \to \varphi \to \theta$:

0	**P**	$\psi \to (\varphi \to \theta)$	*(initial move)*
1	**O**	ψ	[A,0]
2	**P**	$\varphi \to \theta$	[D,1]
3	**O**	φ	[A,2]

Exchange (continued)

Any winning strategy for $\varphi \to \psi \to \theta$ (of which one exists by assumption) opens thus:

0	**P**	$\varphi \to (\psi \to \theta)$	*(initial move)*
1	**O**	φ	[A,0]
2	**P**	$\psi \to \theta$	[D,1]
3	**O**	ψ	[A,2]

This is the same as the first four moves considered on the previous slide, but with φ and ψ swapped.

Glue this winning strategy—stripping off first this initial segment of length four—onto the bottom of the initial sequence of length four that we just considered. The result of this gluing is a winning strategy for $\psi \to \varphi \to \theta$.

Uniform substitution

- Curiously, unrestricted uniform substitution is not valid for N:

  Example: Although $p \to p$ is valid in N, the instance

  $$(p \wedge p) \to (p \wedge p)$$

  is *not* valid in N. One can show this by hand, or appeal to the characterization of N's valid implications. There are many more examples.

- Thus, if one requires of a logic that it validate unrestricted uniform substitution, then N is not a logic.

- Nonetheless, there *are* forms of substitution that do preserve validity (even thought unrestricted uniform substitution in general does not).

Valid substitution: atoms for atoms

Theorem: Uniformly substituting an atom q for an atom p in an N-valid formula preserves N-validity.

Proof: extending the substitution $s\colon p \mapsto q$ from formulas to dialogues, we obtain a mapping s' from dialogues to dialogues that preserves all moves. (One needs to verify this for all particle and structural rules.)

Hence s' preserves **P**-wins and **P**-losses.

It follows that extending s' from dialogues to (extensive form) dialogue trees, we likewise preserve winning strategies.

Valid substitution: double negating atoms

Theorem: If $\vDash \varphi$, then $\vDash \varphi[p := \neg\neg p]$

Proof: Define the following operation on a dialogue d: for each occurrence of p in φ, replace the uses of this occurrence of p as an assertion in d as follows:

$$
\begin{array}{cccc}
\vdots & & & \\
m & \mathbf{P}/\mathbf{O} & p & [\mathrm{A}/\mathrm{D},k] \\
\vdots & & &
\end{array}
\quad \Rightarrow \quad
\begin{array}{cccc}
\vdots & & & \\
m & \mathbf{P}/\mathbf{O} & \neg\neg p & [\mathrm{A}/\mathrm{D},k] \\
m+1 & \mathbf{O}/\mathbf{P} & \neg p & [\mathrm{A},m] \\
m+2 & \mathbf{P}/\mathbf{O} & p & [\mathrm{A},m+1] \\
\vdots & & &
\end{array}
$$

Valid substitution: double negating atoms (continued)

Claim: this operation, applied for each occurrence of p in φ, sends winning strategies for φ to winning strategies for $\varphi[p := \neg\neg p]$.

(The operation maps dialogues to dialogues, but it can be extended to map dialogue trees to dialogue trees, which is the kind of mapping under discussion here.)

The key fact used in the proof is that, in N, defenses by **O** never occur in a winning strategy (because once they do occur, **O** always has the option of repeating them, whence **P** cannot ensure that the game ends at all).

Conjectured validity-preserving substitutions

We believe that the following two kinds of substitutions preserve N-validity:

- Substituting $\neg p$ for p, and
- Substituting a validity φ for p.

(We have proofs for neither of these.)

Summary

- We defined the the logic N and motivated it as an experiment in dialogue games bearing on the *composition problem*.
- We exhibited some of N's curious features.
- We showed that for N we nonetheless do have some nice ‘axioms’ and that N is closed under certain familiar rules of inference.

Open problems:

- We still don't know whether N is axiomatizable.
- We lack a semantics for N different from that given by dialogue games.
- We don't know to what extent we can ‘salvage’ uniform substitution.

References

- W. Felscher, '**Dialogues, strategies, and intuitionistic provability**', *Annals of Pure and Applied Logic* **28** (1985), pp. 217–254.
- J. Alama and S. L. Uckelman, '**A curious dialogical logic and its composition problem**', preprint.