Models of higher-order vagueness in FTT

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Outline

Indeterminacy phenomenon

Complementary facets of indeterminacy
(uncertainty in broader sense)

- **Uncertainty**
- **Vagueness**

Uncertainty

- Lack of knowledge about occurrence of an event accompanying some experiment (process, test, etc.) that is expected to take place
  (no uncertainty after the experiment passed off)
- Abstract uncertainty — character of events is irrelevant (they can be both crisply as well as vaguely delineated)
- Mathematical model (quantified characterization): probability theory
  numerical measure of the likelihood that a particular event will occur
- Other mathematical theories: possibility theory, belief measures and others

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Vagueness

- Raises when trying to group together objects that have a certain property φ: actualized grouping of objects
  $$X = \{o \mid o \text{ is an object having the property } \varphi\}$$
- *X cannot (generally) be taken as a set*
  borderline elements o for which it is unclear whether they have the property φ
  At least some typical objects (prototypes)
- imperceptible gradual change of a property from its presence to its non-presence (continuity)

Common real events are both uncertain as well as vague

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Global characterization of fuzzy logic

Mathematization of vagueness by introduction of degrees (of truth) taken from some scale

Special many-valued logic whose aim is to provide means that can be used for modeling of various aspects of the vagueness phenomenon via the use of degrees (of truth)

- Well established sound formal system
  Fuzzy logic is in its maturity state
- Capable of explanation and modeling of phenomena in which vagueness plays a significant role
- A lot of well justified applications

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Truth degrees

Truth degrees provide technical means for dealing with vagueness

They have little use alone!

I love you in a degree **0.954867283** ???!
Specific truth values are assigned only in the model!

I love you **very much**
It is important to compare truth values

fuzzy sets, their shape: $A : U \longrightarrow L$

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1. in narrow sense (FLn)
   propositional, predicate

   higher order fuzzy type theory
2. in broader sense (FLb)
   Extension of FLn
   - *Mathematical model of the meaning of some expressions of natural language*
     trichotomous evaluative linguistic expressions, generalized quantifiers
   - *Model of human way of reasoning*

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Fuzzy type theory – syntax

Higher order fuzzy logic

Types: o (truth values), ϵ (elements), $\beta\alpha$

Formulas have types: $A_\alpha \in Form_\alpha$, $A_\beta \equiv B_\beta$, $\lambda x_\alpha\, C_\beta$, $\mathbf{\Delta}_{oo}$

Formulas of type o are propositions

Description operators $\iota_{o(oo)}, \iota_{\epsilon(o\epsilon)}, \iota_{\alpha(o\alpha)}$

Formula $\dagger$ of type o — the most indefinite truth value for which $\vdash \neg\dagger \equiv \dagger$

Fuzzy type theory – semantics

Frame $\mathcal{M}$

$$\mathcal{M} = \langle (M_\alpha, =_\alpha)_{\alpha \in \mathit{Types}}, \mathcal{L}_\Delta \rangle$$

$\mathcal{L}_\Delta$: *IMTL$_\Delta$-algebra, standard Łukasiewicz Δ-algebra, BL-algebra, ŁΠ-algebra*

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Fuzzy type theory — semantics

Fuzzy equality

$$\vdash x_\alpha \equiv x_\alpha \qquad \text{(reflexivity)}$$

$$\vdash (x_\alpha \equiv y_\alpha) \equiv (y_\alpha \equiv x_\alpha) \qquad \text{(symmetry)}$$

$$\vdash (x_\alpha \equiv y_\alpha) \,\&\, (y_\alpha \equiv z_\alpha) \Rightarrow (x_\alpha \equiv z_\alpha) \qquad \text{(transitivity)}$$

Example

$=_\alpha : M_\alpha \times M_\alpha \longrightarrow L$

$$[x =_\alpha y] = 0 \vee (1 - |x - y|)$$

Interpretation of formulas: $\mathcal{M}(A_{\beta\alpha}) \in M_\beta^{M_\alpha}$, preserving the fuzzy equality

Claim

All essential properties of vague predicates are formally expressible in FTT and so, they have a many-valued model

Special crisp formulas

$\mathbf{\Delta}_{oo}$: interpreted by $\Delta = \begin{cases} \mathbf{1} & \text{if } a = 1, \\ 0 & \text{otherwise.} \end{cases}$ for linearly ordered L

$\mathbf{\Delta}$ corresponds to D-operator

$$\hat{\Upsilon}_{oo} \equiv \lambda z_o \cdot \neg\mathbf{\Delta}(z_o \vee \neg z_o)$$
$$\mathcal{M}(\hat{\Upsilon} z_o) = \mathbf{1} \quad \text{iff} \quad \mathbf{1} > z_0 > \mathbf{0}$$

$\hat{\Upsilon}$ corresponds to I-operator (indefinitely)

Sorites paradox in fuzzy predicate logic I

Theory of natural numbers in FLn — extended by a new predicate $\mathbb{FN}(n)$: *"n is small"; "n is feasible"; "n is finite"*

$$\mathbb{FN}(0), \mathbb{FN}(0) \Rightarrow \mathbb{FN}(1), \mathbb{FN}(1), \dots, \\ \dots, \mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1), \mathbb{FN}(n+1), \dots$$

Axioms:

- *"there is a small number"* — valid
- *"if n is small then n + 1 is also small"* — practically valid
- *"there is a number not being small"* — valid

No contradiction!
(impossible in classical logic)

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Sorites paradox in fuzzy predicate logic II

Evaluated syntax

Theorem

T be a fuzzy theory in which all Peano axioms are accepted in the degree 1. Let $1 \geq \varepsilon > 0$ *and* $\mathbb{FN} \notin J(T)$ *be a new predicate. Then*

$$T^+ = T \cup \{{}^{1}/\mathbb{FN}(0),\; {}^{1-\varepsilon}/(\forall n)(\mathbb{FN}(n) \Rightarrow \mathbb{FN}(n+1)),\; {}^{1}/(\exists n)\neg\mathbb{FN}(n)\}$$

is a conservative extension of T

Higher-order vagueness

Higher-order vagueness — vagueness has no "end", there are no sharp boundaries in any respect.

No sharp boundary between positive and negative cases, nor between borderline cases and other sharp boundaries.

$A(x)$ — vague property. Then there are values of x s.t. "$A(x)$ is borderline" is itself borderline.

In other words, the property "to be borderline" is also vague.

We can iterate — there are borderline borderline cases as well as borderline cases of "definitely $A(x)$", "definitely $\neg A(x)$", etc.

HOV stems from vagueness of the property "to be borderline", there are elements "more or less borderline".

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Fuzzy logic model of HOV

Elements having a given property A that are “typically borderline” should be distinguishable from elements that are borderline only “a little” — close to prototypical examples of A or not A.

How can we recognize a borderline element in the fuzzy logic model? What does it actually mean that a given property is vague?

In fuzzy logic, this should be done in accordance with the assigned truth degrees.

For simplicity, we will consider only first-order properties of type $A_{o\alpha}$

Property “to be vague”:

$$Vag := \lambda u_{o\alpha}\, (\forall t_o)(\exists y_\alpha)\Delta(u_{o\alpha} y_\alpha \equiv t_o).$$

$Vag\, A_{o\alpha}$ simply says that $A_{(o\alpha_n)}$ is vague if its interpretation is a surjective function to the set of all truth values.

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Fuzzy logic model of HOV

Justification of definition of *Vag* — very principle of fuzzy logic — to model vagueness via assignment of truth values from a (infinite) scale to elements so that $\mathcal{M}_p(A_{o\alpha}(x_\alpha)) = a \in L$ expresses a truth value of the statement "an element $m_\alpha = p(x_\alpha)$ has the property A".

The truth values $a \neq \mathbf{0}, \mathbf{1}$ characterize borderline cases. Since there should be no truth value gaps, a vague property must principally attain any truth value from the scale (algebra) L.

If the degree of $\mathcal{M}_p(A_{o\alpha}(x_\alpha))$ is close to 0 or 1, then $p(x_\alpha)$ is more definite, i.e., less borderline. The closer it is to the neutral value $\dagger$ the more it is borderline.

We introduce a formula $Brd_{o\alpha(o\alpha)}$ expressing that an element x_α of type α is a borderline case of a property $A_{o\alpha}$. We will do it iteratively for arbitrary order.

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First, we define recursively special formulas:

$$C_0 := \lambda z_o \, z_o, \tag{1}$$

$$C_1 := \lambda z_o \, (C_0 \, z_o \equiv \dagger)^2, \tag{2}$$

$$\dots \quad \dots\dots\dots\dots\dots\dots\dots\dots$$

$$C_n := \lambda z_o \, (C_{n-1} \, z_o \equiv \dagger)^2. \tag{3}$$

Lemma

If $\vdash Vag\, u_{o\alpha}$ *then* $\vdash Vag(C_n u_{o\alpha})$ *for all* n.

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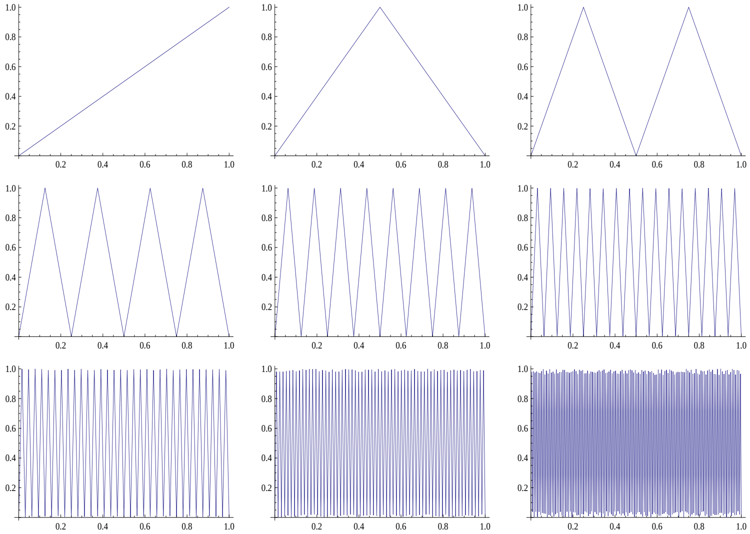
C functions



Fuzzy logic model of HOV

Lemma

(a) $\vdash \hat{\Upsilon}(z_o \equiv \dagger)^2 \Rightarrow \hat{\Upsilon} z_o$.

(b) $\vdash \hat{\Upsilon} z_o \wedge \hat{\Upsilon}(z_o \equiv \dagger)^2 \equiv \hat{\Upsilon}(z_o \equiv \dagger)^2$.

(c) $\vdash (z_o \equiv \dagger)^2 \Rightarrow \hat{\Upsilon} z_o$.

Predicate "to be borderline" on the level n :

$$Brd^{(1)} := \lambda u_{o\alpha}\, \lambda x_\alpha \cdot \hat{\Upsilon}(C_0(u_{o\alpha} x_\alpha)) \wedge (C_0(u_{o\alpha} x_\alpha) \equiv \dagger)^2$$

$$Brd^{(2)} := \lambda u_{o\alpha}\, \lambda x_\alpha \cdot \hat{\Upsilon}(Brd^{(1)}\, u_{o\alpha} x_\alpha) \wedge (Brd^{(1)}\, u_{o\alpha} x_\alpha \equiv \dagger)^2$$

$$Brd^{(n)} := \lambda u_{o\alpha}\, \lambda x_\alpha \cdot \hat{\Upsilon}(Brd^{(n-1)}\, u_{o\alpha} x_\alpha) \wedge (Brd^{(n-1)}\, u_{o\alpha} x_\alpha \equiv \dagger)^2$$

Idea: all elements x for which Ax is true in a degree different from 0, 1 are borderline. The closer this degree is to 0.5 the more indefinite this element is; if it is 0.5 then it is "typically borderline", i.e. neither close to definitely Ax, nor close to definitely not Ax. Similarly for any n.

Fuzzy logic model of HOV

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$$Brd^{(2)} := \lambda u_{o\alpha}\, \lambda x_\alpha \cdot \hat{\Upsilon}(Brd^{(1)}\, u_{o\alpha} x_\alpha) \wedge (Brd^{(1)}\, u_{o\alpha} x_\alpha \equiv \dagger)^2$$

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Idea: all elements x for which Ax is true in a degree different from 0, 1 are borderline. The closer this degree is to 0.5 the more indefinite this element is; if it is 0.5 then it is "typically borderline", i.e. neither close to definitely Ax, nor close to definitely not Ax. Similarly for any n.

Fuzzy logic model of HOV

Lemma

(a) $\vdash \hat{\Upsilon}(z_o \equiv \dagger)^2 \Rightarrow \hat{\Upsilon} z_o$.

(b) $\vdash \hat{\Upsilon} z_o \wedge \hat{\Upsilon}(z_o \equiv \dagger)^2 \equiv \hat{\Upsilon}(z_o \equiv \dagger)^2$.

(c) $\vdash (z_o \equiv \dagger)^2 \Rightarrow \hat{\Upsilon} z_o$.

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Lemma

$$\vdash Brd^{(n)}\, u_{o\alpha} x_\alpha \equiv \hat{\Upsilon}(C_{n-1}(u_{o\alpha} x_\alpha)) \wedge C_n(u_{o\alpha} x_\alpha)$$

for all $n \geq 1$.

Theorem

(a) *If* $\vdash Vag\, u_{o\alpha}$ *then* $\vdash Vag(Brd^{(n)}\, u_{o\alpha})$ *for all* $n \geq 1$.

(b) *To every* $A_{o\alpha}$ *(vague property) and every* $n \geq 1$:

- (i) $\vdash (\exists x_\alpha)\, Brd^{(n)}\, A_{o\alpha} x_\alpha$,
- (ii) $\vdash (\exists x_\alpha) \hat{\Upsilon}(Brd^{(n)}\, A_{o\alpha} x_\alpha)$.

This theorem states that the property "to be borderline" is vague. Moreover, the case (b) states that there is an element that is "borderline borderline . . . borderline", and also an element that is "definitely borderline . . . borderline".

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Using IF-THEN rules

$Brd_{o\alpha}$: x_α is a borderline case of $A_{o\alpha}$
$Brd\, x_\alpha$: "borderliness" of x_α, $\mathcal{M}(Brd\, x_\alpha) \in L$

Apply fuzzy/linguistic IF-THEN rules (sophisticated theory in FL)
1^{st} order:

$$\text{IF } (Ax \equiv \dagger)^2 \text{ is } Bi \text{ THEN } Brd\, x \text{ is } Bi$$
$$\text{IF } (Ax \equiv \dagger)^2 \text{ is } Sm \text{ THEN } Brd\, x \text{ is } Sm$$

2^{nd} order:

$$\text{IF } Bi(Brd\, x) \text{ is } Sm \text{ THEN } Brd^{(2)}\, x \text{ is } Bi$$
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Simplification:

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Then: n^{th} order:

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Hereditary borderline element x: $\vdash \hat{\Upsilon} Brd^{(n)}\,x$ for all n

Using IF-THEN rules

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Conclusions

- Fuzzy logic is a group of powerful formal tools

  Higher-order fuzzy logic (FTT) provides means for characterization of difficult problems connected with the vagueness phenomenon
- The results are in accordance with the intuitive meaning, great potential for applications
- Higher-order vagueness can be modeled in FL in several ways.

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