Free BL-Algebras, Revisited

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Motivations

About one year ago we presented a representation theorem for free BL-algebras [Aguzzoli-Bova, The Free n-Generated BL-Algebra, submitted].

Our result was stated in the terms of an inductive definition:

The elements of the free n-generated BL-algebra were expressed as a patching combination of elements of the free k-generated BL-algebras, for $k < n$.

We have revised our result and we have managed to eliminate the inductive definition, obtaining a new representation theorem:

Each element of the free n-generated BL-algebra is represented as a finite collection of elements of free Wajsberg hoops, organised in a combinatorial structure based on the ordered partitions of the set of generators.

Free Algebras and Logic

Logic L having as algebraic semantics a variety $\mathbb{V}(L)$.

The free n-generated algebra is (isomorphic to) the Lindenbaum algebra of the classes of L-equivalent formulas (over the first n variables).

Example: The free n-generated Boolean algebra $\mathcal{B}_n$ is the Boolean algebra

$$\langle \{[\varphi] \mid Var(\varphi) \subseteq \{X_1, \ldots, X_n\}\}, \wedge, \vee, \neg, \top \rangle \,,$$

where

$$[\varphi] = \{\psi \mid \vdash \varphi \leftrightarrow \psi\} \,,$$

$$[\varphi] \wedge [\psi] = [\varphi \wedge \psi], \quad [\varphi] \vee [\psi] = [\varphi \vee \psi], \quad \neg[\varphi] = [\neg\varphi], \quad \top = \{\varphi \mid \vdash \varphi\}.$$

Equivalently, $\mathcal{B}_n$ is the algebra of all functions

$$f \colon \{0,1\}^n \to \{0,1\} \,,$$

equipped with pointwise defined operations.

Generic and Free algebras in a variety

– An algebra $\mathcal{F}_X$ is said free in $\mathbb{V}$ over the set of (free) generators X if each map $X \to \mathcal{B} \in \mathbb{V}$ uniquely extends to a homomorphism $\mathcal{F}_X \to \mathcal{B}$.
– $|X| = |Y|$ implies $\mathcal{F}_X \cong \mathcal{F}_Y$.

– An algebra $\mathcal{A}$ is said generic in a variety $\mathbb{V}$ if $\mathbb{V} = \mathbb{HSP}(\mathcal{A})$.
– Equivalently, an equation is valid in $\mathcal{A}$ iff it is valid in all algebras in $\mathbb{V}$.

– If $\mathcal{A}$ is generic in $\mathbb{V}$, the free n-generated algebra in $\mathbb{V}$ is the subalgebra of $\mathcal{A}^{\mathcal{A}^n}$ generated by the projections $(a_1, \ldots, a_n) \mapsto a_i$.
– If $\mathcal{A}$ is generic in the subvariety of $\mathbb{V}$ generated by all n-generated algebras of $\mathbb{V}$, then the free n-generated algebra in $\mathbb{V}$ is the subalgebra of $\mathcal{A}^{\mathcal{A}^n}$ generated by the projections $(a_1, \ldots, a_n) \mapsto a_i$.

Hájek's propositional Basic Fuzzy Logic BL

Language: $\{\odot, \rightarrow, \bot\}$

$$\text{Axioms}: \begin{cases} (\varphi \rightarrow \psi) \rightarrow ((\psi \rightarrow \vartheta) \rightarrow (\varphi \rightarrow \vartheta)) \\ (\varphi \odot \psi) \rightarrow \varphi \\ (\varphi \odot \psi) \rightarrow (\psi \odot \varphi) \\ (\varphi \odot (\varphi \rightarrow \psi)) \rightarrow (\psi \odot (\psi \rightarrow \varphi)) \\ ((\varphi \rightarrow (\psi \rightarrow \vartheta)) \rightarrow ((\varphi \odot \psi) \rightarrow \vartheta)) \\ ((\varphi \odot \psi) \rightarrow \vartheta)) \rightarrow ((\varphi \rightarrow (\psi \rightarrow \vartheta)) \\ ((\varphi \rightarrow \psi) \rightarrow \vartheta) \rightarrow (((\psi \rightarrow \varphi) \rightarrow \vartheta) \rightarrow \vartheta) \\ \bot \rightarrow \varphi \end{cases}$$

Rules: *modus ponens*

$$\frac{\varphi \quad \varphi \rightarrow \psi}{\psi}$$

BL and continuous t-norms

t-norm: $*\colon [0,1]^2 \to [0,1]$ such that:

- $x * (y * z) = (x * y) * z$
- $x * y = y * x$
- If $x_1 \leq x_2$ then $x_1 * y \leq x_2 * y$
- $x * 1 = x$ (and $x * 0 = 0$)

Residuum: $x \to_* y = \max\{z \mid x * z \leq y\}$.

Standard algebra: $\mathcal{A}_* = \langle [0,1], *, \to_*, 0 \rangle$ for some t-norm $*$.

[CEGT] BL is the logic of all continuous t-norms and their residua:

$\mathrm{BL} \vdash \varphi(X_1, \ldots, X_n)$ iff $\varphi^{\mathcal{A}_*}(a_1, \ldots, a_n) = 1$ for each continuous t-norm $*$ and for all $(a_1, \ldots, a_n) \in [0,1]$.

Examples of standard algebras

standard MV-algebra $[0,1]_Ł$

$$\left\langle [0,1], \quad \begin{cases} 0 & \text{if } x+y \leq 1 \\ x+y-1 & \text{otherwise} \end{cases}, \quad \begin{cases} 1 & \text{if } x \leq y \\ 1-x+y & \text{otherwise} \end{cases}, \quad 0 \right\rangle$$

standard Gödel algebra $[0,1]_G$

$$\left\langle [0,1], \quad \begin{cases} x & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}, \quad \begin{cases} 1 & \text{if } x \leq y \\ y & \text{otherwise} \end{cases}, \quad 0 \right\rangle$$

standard Product algebra $[0,1]_\Pi$

$$\left\langle [0,1], \quad xy, \quad \begin{cases} 1 & \text{if } x \leq y \\ y/x & \text{otherwise} \end{cases}, \quad 0 \right\rangle$$

Examples of free algebras: the case of MV-algebras

Chang's Algebraic Completeness theorem:
The standard MV-algebra $\langle [0,1], \max(0, x+y-1), \min(1, 1-x+y), 0\rangle$ is generic for the variety of MV-algebras (BL-algebras plus $\neg\neg x = x$).

Consider the MV-algebra $\mathcal{M}_n$ of all functions $f\colon [0,1]^n \to [0,1]$ endowed with the pointwise standard MV-operations: $(f \odot g)(\mathbf{x}) = \max(0, f(\mathbf{x}) + g(\mathbf{x}) - 1)$, $(f \to g)(\mathbf{x}) = \min(1, 1 - f(\mathbf{x}) + g(\mathbf{x}))$, $\bot(\mathbf{x}) = 0$.

McNaughton's Representation Theorem:
The free n-generated MV-algebra is the subalgebra of $\mathcal{M}_n$ of all continuous piecewise linear functions $f\colon [0,1]^n \to [0,1]$ where each one of the finitely many linear pieces has integer coefficients.
i.e., there is a set $\{p_1, \ldots, p_u\}$ of linear polynomials $p_i(\mathbf{x}) = \mathbf{a}_i\mathbf{x} + b_i$ for $(\mathbf{a}_i, b_i) \in \mathbb{Z}^{n+1}$, such that for each $\mathbf{x} \in [0,1]^n$ there is $k \in \{1, \ldots, u\}$ with $f(\mathbf{x}) = p_k(\mathbf{x})$.

Examples of free algebras: the case of Wajsberg hoops

Wajsberg hoops are the $\bot$-free subreducts of MV-algebras.

Wajsberg hoops form a variety ($\mathbb{WH}$).

The standard Wajsberg hoop

$$\langle [0,1], \max(0, x+y-1), \min(1, 1-x+y), 1\rangle$$

is generic for $\mathbb{WH}$.

[AP]: The free n-generated Wajsberg hoop $\mathcal{WH}_n$ is the Wajsberg hoop of all McNaughton functions $f\colon [0,1]^n \to [0,1]$ such that

$$f(1,1,\ldots,1) = 1\,.$$

The variety $\mathbb{BL}$ of BL-algebras

Derived Connectives :
$$\begin{cases} \neg\varphi & = & \varphi \to \bot \\ \top & = & \neg\bot \\ \varphi \wedge \psi & = & \varphi \odot (\varphi \to \psi) \\ \varphi \vee \psi & = & ((\varphi \to \psi)) \to \psi) \wedge ((\psi \to \varphi) \to \varphi) \\ \varphi \leftrightarrow \psi & = & (\varphi \to \psi) \wedge (\psi \to \varphi) \\ \varphi \oplus \psi & = & ((\varphi \to (\varphi \odot \psi)) \to \psi) \vee ((\psi \to (\psi \odot \varphi)) \to \varphi) \end{cases}$$

BL-algebra : $\langle A, \odot, \to, \bot \rangle$ such that:
$$\begin{cases} \langle A, \odot, \top \rangle \quad \text{is a commutative monoid} \\ \langle A, \vee, \wedge, \bot, \top \rangle \quad \text{is a bounded lattice} \\ x \odot y \leq z \quad \text{iff} \quad y \leq x \to z \quad \text{(residuation)} \\ x \wedge y = x \odot (x \to y) \quad \text{(divisibility)} \\ (x \to y) \vee (y \to x) = \top \quad \text{(prelinearity)} \end{cases}$$

The class of all BL-algebras forms a variety ($\mathbb{BL}$). Moreover:

$\mathrm{BL} \vdash \varphi$ iff $\mathbb{BL} \models \varphi \approx 1\,.$

Ordinal sums, Generic BL-algebras

Ordinal sum of BL-chains $\{\mathcal{A}_i = \langle A_i, \odot_i, \to_i, \bot_i \rangle\}_{i\in\alpha}$,
with $A_i \cap A_j = \{\top_i\} = \{\top_j\}$ for all $i, j \in \alpha$:

$$\bigoplus_{i\in\alpha} \mathcal{A}_i = \left\langle \bigcup_{i\in\alpha} A_i, \odot, \to, \bot_0 \right\rangle$$

where:

$$x \odot y = \begin{cases} x & \text{if } x \in A_i, y \in A_j, i < j \\ x \odot_i y & \text{if } x, y \in A_i \\ y & \text{if } x \in A_i, y \in A_j, j < i \end{cases} \qquad x \to y = \begin{cases} \top_0 & \text{if } x \in A_i, y \in A_j, i < j \\ x \to_i y & \text{if } x, y \in A_i \\ y & \text{if } x \in A_i, y \in A_j, j < i \end{cases}$$

[AM]: The ordinal sum $\omega[0,1]_{Ł}$ of ω many copies of the standard MV-algebra is generic in $\mathbb{BL}$.
Given a formula $\varphi(X_1, \ldots, X_n)$, if the equation $\varphi \approx 1$ fails in some BL-algebra, then it already fails in $(n+1)[0,1]_{Ł}$.

$(n+1)[0,1]_{Ł}$ $\quad n \geq 0$

$$(n+1)[0,1]_{Ł} = ([0,n+1], \odot^{[0,n+1]}, \rightarrow^{[0,n+1]}, \perp^{[0,n+1]})$$

$$\perp^{[0,n+1]} = 0$$

$$x \odot^{[0,n+1]} y = \begin{cases} n+1 & \text{if } x = y = n+1, \\ \min\{x,y\} & \text{if } \lfloor x \rfloor \neq \lfloor y \rfloor, \\ (\{x\} \odot^{[0,1]} \{y\}) + i & \text{if } \lfloor x \rfloor = \lfloor y \rfloor = i, \end{cases}$$

$$x \rightarrow^{[0,n+1]} y = \begin{cases} n+1 & \text{if } x \leq y, \\ y & \text{if } \lfloor y \rfloor < \lfloor x \rfloor, \\ (\{x\} \rightarrow^{[0,1]} \{y\}) + i & \text{if } \lfloor x \rfloor = \lfloor y \rfloor = i. \end{cases}$$

($\lfloor x \rfloor$: integer part of x, $\{x\}$: fractional part of x)

The Free 1-generated BL-algebra $\mathcal{BL}_1$

Given a formula $\varphi(X_1)$, if the equation $\varphi \approx 1$ fails in some BL-algebra, then it already fails in $2[0,1]_{Ł}$.

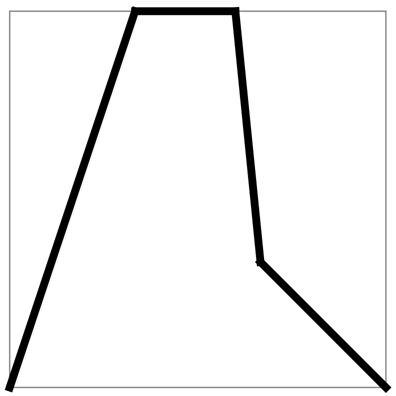
The free BL-algebra $\mathcal{BL}_1$ over 1 generator is the subalgebra of $2[0,1]_{Ł}^{2[0,1]_{Ł}}$ generated by $X_1 \colon a \mapsto a$.

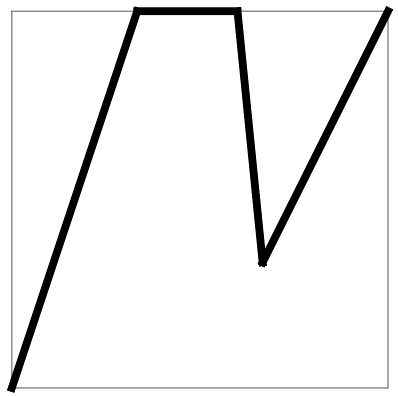
Identify the support of $2[0,1]_{Ł}$ with $[0,2]$.

[M]: $\mathcal{BL}_1$ is the algebra of all functions $f \colon [0,2] \to [0,2]$ such that there exists a pair (g_1, g_2) of two McNaughton functions of 1-variable such that $g_2(1) = 1$ and

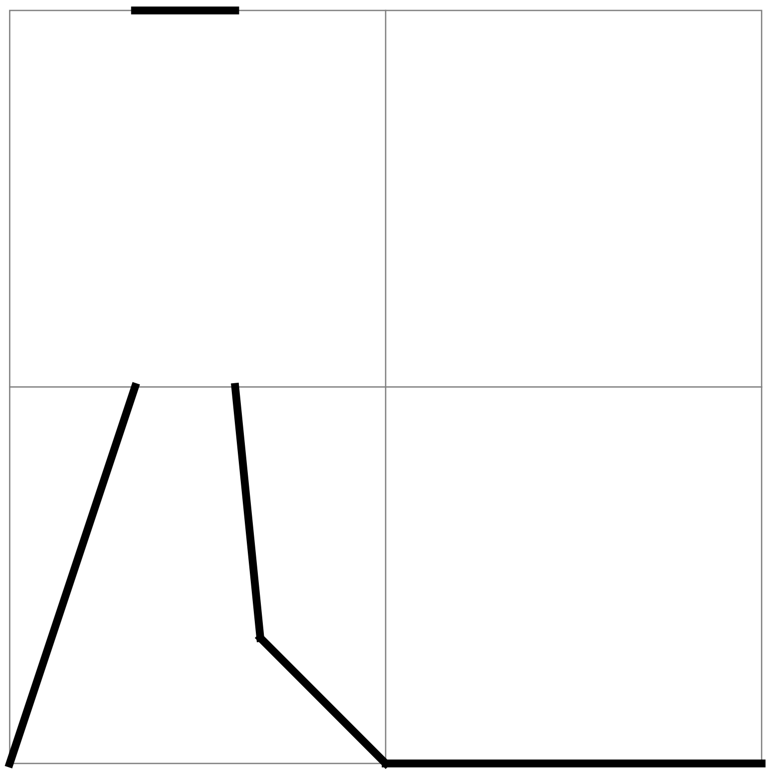
$$f \restriction [0,1)(x) = \begin{cases} g_1(x) & \text{if } g_1(x) < 1 \\ 2 & \text{otherwise} \end{cases}, \quad f \restriction [1,2](x) = \begin{cases} 0 & \text{if } g_1(1) = 0 \\ 1 + g_2(x-1) & \text{otherwise} \end{cases}$$

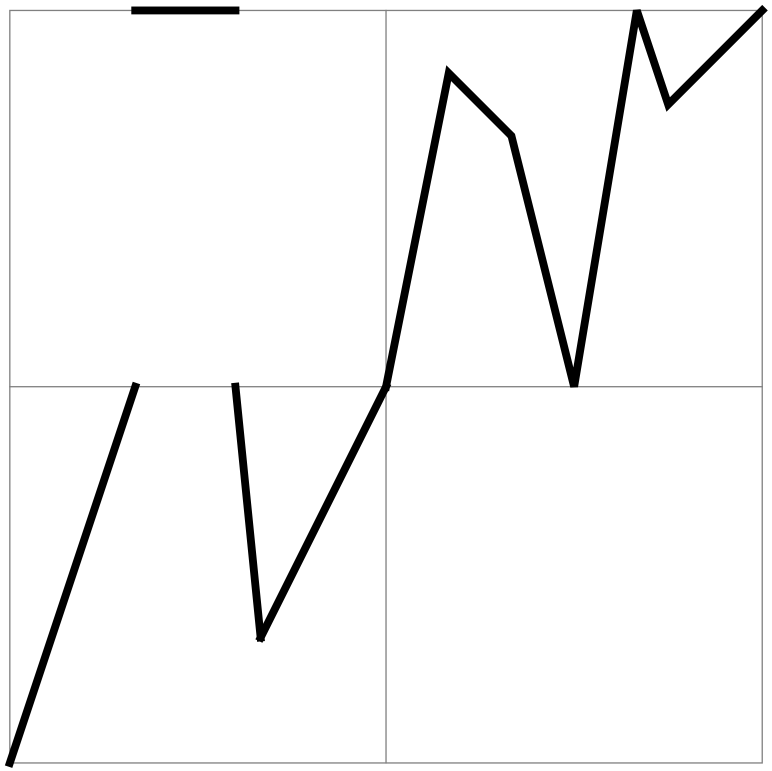
The Free 1-generated BL-algebra $\mathcal{BL}_1$





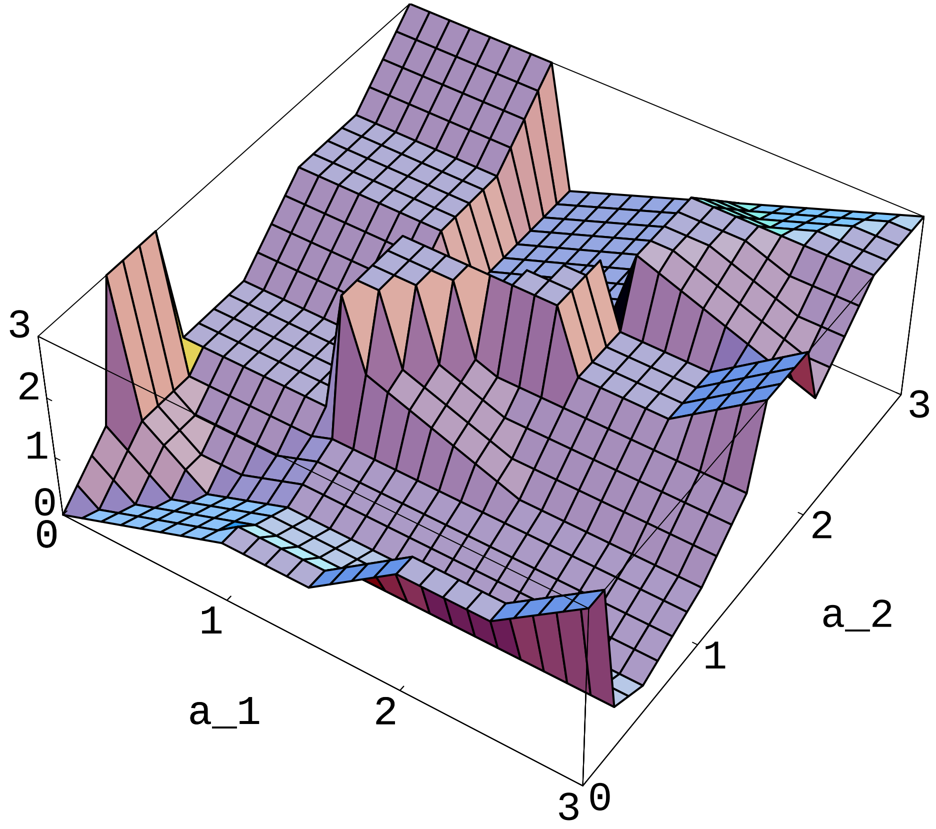
The Free 1-generated BL-algebra $\mathcal{BL}_1$

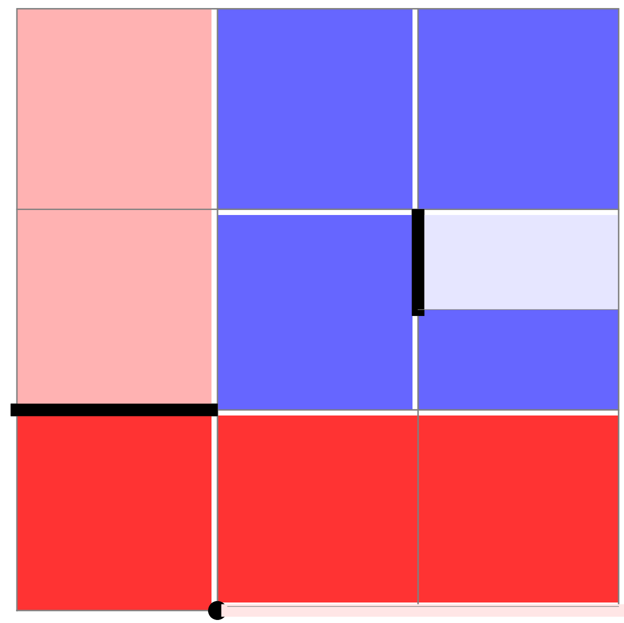




What about $\mathcal{BL}_n$?

An example from $\mathcal{BL}_2$





$\mathcal{BL}_n$ Inductive Definition

To specify a function in $f \in \mathcal{BL}_n$ one needs:

- A McNaughton function $g \in \mathcal{MV}_n$.
- If $g(\mathbf{1}) = 1$, a possibly distinct function $h \in \mathcal{MV}_n$ such that $h(\mathbf{1}) = 1$.
- For each nonempty subset $J \in \{1, \ldots, n\}$, a finite rational polyhedral subdivision $B_J(g) = \{P_1, \ldots, P_u\}$ for g, together with:
  - A finite rational polyhedral subdivision $Q_{i,1}, \ldots, Q_{i,v}$ of each P_i base of an unconstrained prism.
  - a map assigning to each $Q_{i,j}$ a function $g_{i,j} \in \mathcal{BL}_{|J|}$.
- If $g(\mathbf{1}) = 1$, for each $\emptyset \neq J \in \{1, \ldots, n\}$, a f.r.p.s. $B_J(h) = \{P_1, \ldots, P_u\}$ for h, together with:
  - A f.r.p.s. $Q_{i,1}, \ldots, Q_{i,v}$ of each P_i base of an unconstrained prism.
  - a map assigning to each $Q_{i,j}$ a function $h_{i,j} \in \mathcal{BL}_{|J|}$.

$\mathcal{BL}_n$ Inductive Definition

Then these data determine uniquely the function $f\colon [0, n+1]^n \to [0, n+1]$:

- $f \restriction [0,1)^n = (g)_0$
- If $g(\mathbf{1}) = 0$, then $f \restriction [1, n+1]^n = 0$
- If $g(\mathbf{1}) = 1$, for each $i \in \{1, \ldots, n\}$, $f \restriction [i, i+1)^n = (h)_i$ ($f \restriction [i, i+1]^n$ in the case $i = n$)
- For all constrained prisms R_P for g, $f \restriction R_P = (g)_0$.
- If $g(\mathbf{1}) = 1$, for all constrained prisms R_P for h, $f \restriction (R_P + \mathbf{i}^{|J|}) = (h)_i$ for $1 \leq i \leq n$.
- For all $Q_{i,j}$ in the subdivision of a base of an unconstrained prism R_{P_i} for g: $f \restriction Q_{i,j} = (g_{i,j})_0$.
- If $g(\mathbf{1}) = 1$, for all $Q_{i,j}$ in the subdivision of a base of an unconstrained prism R_{P_i} for h: $f \restriction (Q_{i,j} + \mathbf{k}^{|J|}) = (h_{i,j})_k$, for $1 \leq k \leq n$.

Removing the induction from the definition of $\mathcal{BL}_n$

Two main tools:

- The tree of ordered partitions of $\{1, 2, \ldots, n\}$.
- Prismwise Wajsberg functions (one at each node of the tree).

The Free 1-generated BL-algebra $\mathcal{BL}_1$, revisited

Let WH_1 be the universe of $\mathcal{WH}_1$, and let $\bot$ be a new element not belonging to WH_1.

$$\mathcal{BL}_1 \cong (WH_1 \times (WH_1 \cup \{\bot\}), \odot, \rightarrow, 0)\,,$$

$0 = (1, \bot)$

$$(f_1(\mathbf{x}), f_2(\mathbf{y})) \odot (g_1(\mathbf{x}), g_2(\mathbf{y})) = \begin{cases} (f_1(\mathbf{x}) \oplus^{[0,1]} g_1(\mathbf{x}), \bot) & \text{if } f_2 = \bot = g_2\,, \\ (g_1(\mathbf{x}) \rightarrow^{[0,1]} f_1(\mathbf{x}), \bot) & \text{if } f_2 = \bot \neq g_2\,, \\ (f_1(\mathbf{x}) \rightarrow^{[0,1]} g_1(\mathbf{x}), \bot) & \text{if } f_2 \neq \bot = g_2\,, \\ (f_1(\mathbf{x}) \odot^{[0,1]} g_1(\mathbf{x}), f_2(\mathbf{y}) \odot^{[0,1]} g_2(\mathbf{y})) & \text{if } f_2 \neq \bot \neq g_2\,, \end{cases}$$

$$(f_1(\mathbf{x}), f_2(\mathbf{y})) \rightarrow (g_1(\mathbf{x}), g_2(\mathbf{y})) = \begin{cases} (g_1(\mathbf{x}) \rightarrow^{[0,1]} f_1(\mathbf{x}), 1) & \text{if } f_2 = \bot = g_2\,, \\ (f_1(\mathbf{x}) \oplus^{[0,1]} g_1(\mathbf{x}), 1) & \text{if } f_2 = \bot \neq g_2\,, \\ (f_1(\mathbf{x}) \odot^{[0,1]} g_1(\mathbf{x}), \bot) & \text{if } f_2 \neq \bot = g_2\,, \\ (f_1(\mathbf{x}) \rightarrow^{[0,1]} g_1(\mathbf{x}), f_2(\mathbf{y}) \rightarrow^{[0,1]} g_2(\mathbf{y})) & \text{if } f_2 \neq \bot \neq g_2\,. \end{cases}$$

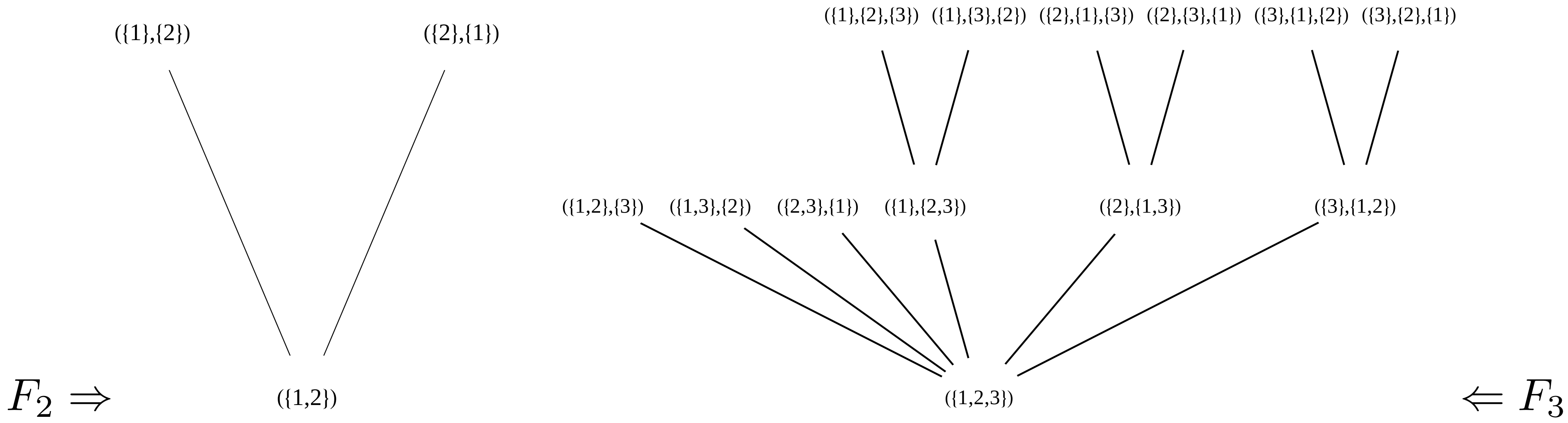
The tree of ordered partitions

An ordered partition $R = (\{B_1, \ldots, B_l\}, \leq)$ of $\{1, 2, \ldots, n\}$ is a partition of $\{1, 2, \ldots, n\}$: $(\bigcup_{i=1}^{l} B_i = \{1, 2, \ldots, n\},\ B_i \cap B_j = \emptyset)$ with a total order $\leq$ on its blocks: $(i \leq j$ iff $B_i \leq B_j)$.

R precedes S, in symbols $R \leq_F S$, if:

$$R = (B_1, \ldots, B_l, B_{l+1}) \qquad S = (B_1, \ldots, B_l, C_{l+1}, \ldots, C_{l+m}) \quad \text{for } m > 1\,.$$

We denote F_n the set of all ordered partitions of $\{1, \ldots, n\}$, ordered by $\leq_F$.



Prismwise Wajsberg functions

Denote $[n] = \{1, 2, \ldots, n\}$.

(Rational) open polyhedral set: finite collection of non-overlapping open polyhedra, whose vertices (of their closure) are rational points.

Let $\emptyset \neq K \subseteq [n]$.
Let $\{\Delta_i \mid i \in [l]\}$ be an open polyhedral set in $[0,1]^{[n]\setminus K}$ s.th. $\Delta_i \subseteq [0,1)^{[n]\setminus K}$

An n-ary prismwise Wajsberg function is a function $f\colon \mathrm{dom}(f) \to [0,1]$ from

$$\mathrm{dom}(f) = \bigcup_{i \in [l]} \Delta_i \times [0,1]^K$$

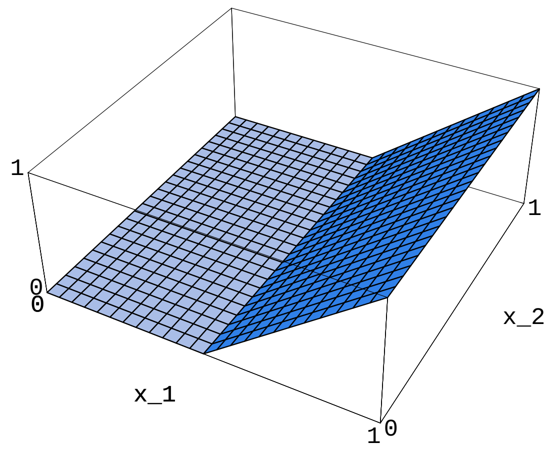
such that for each Δ_i, there exists $g \in \mathbb{WH}_n$, essentially $|K|$-ary such that

$$f \restriction \Delta_i \times [0,1]^K = g.$$

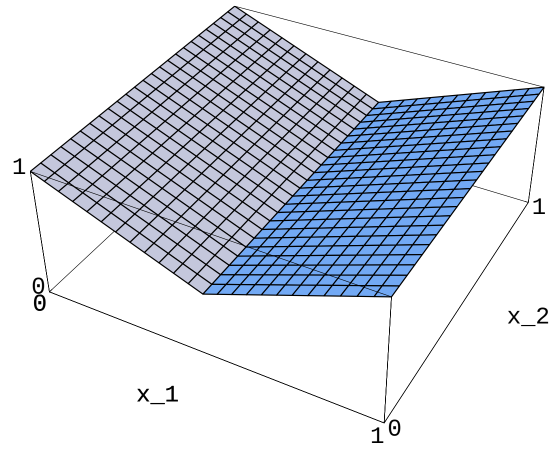
We let $\mathbb{PW}_n$ denote the set of n-ary prismwise Wajsberg functions.

Prismwise Wajsberg functions An example from $\mathbb{PW}_2$

$g_1 \in \mathbb{WH}_2$:

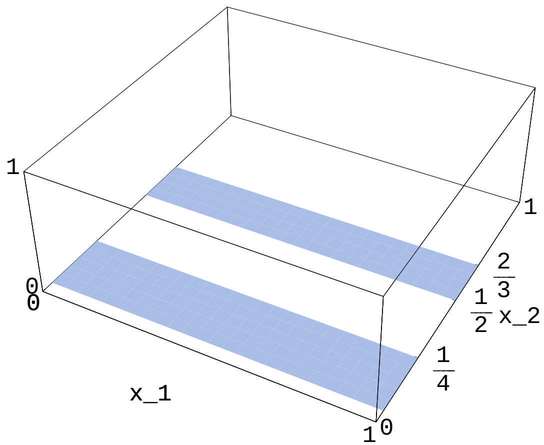


$g_2 \in \mathbb{WH}_2$:

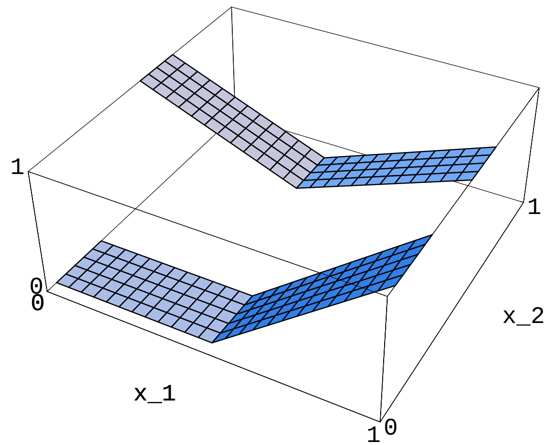


$(g_1, g_2$ essentially 1-ary.)

dom(f):



$f \in \mathbb{PW}_2$:



Encodings

A_n is the set of all pairs of functions $(L\colon F_n \to \mathbb{PW}_n, H\colon F_n \to \mathbb{PW}_n)$ satisfying the following conditions.

1. $\mathrm{dom}(L([n])) = [0,1]^n$ and $L([n]) \in \mathbb{WH}_n$.
2. $\mathrm{dom}(H([n])) = [0,1]^n$ and $H([n]) \in \mathbb{WH}_n$, or $\mathrm{dom}(H([n])) = \emptyset$.
3. Let $I \in \{L, H\}$, $S \prec_F R \in F_n$, K be the maximum block of R,

$$F_K = [0,1)^{[n]\setminus K} \times \{1\}^K ,$$

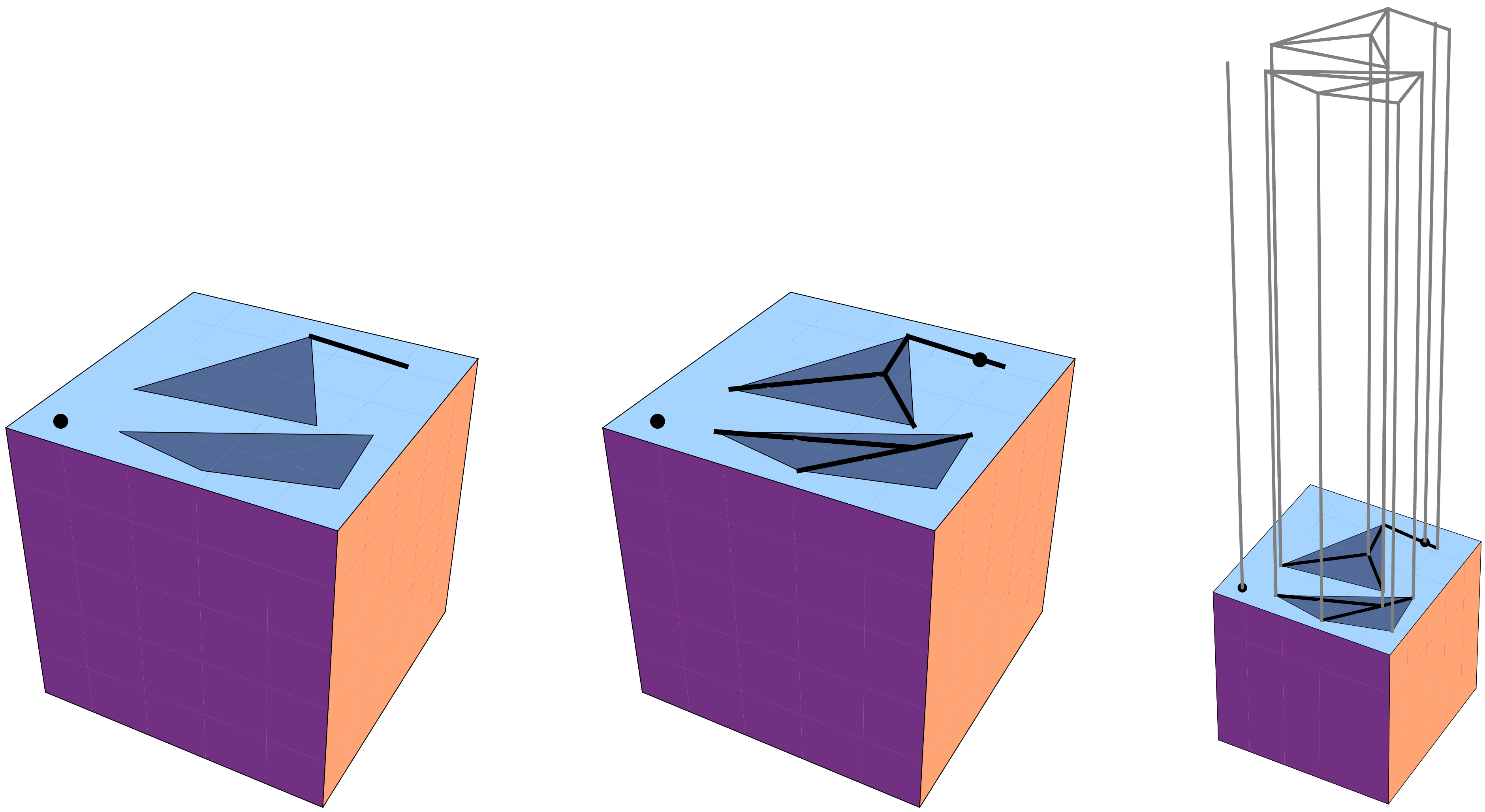
and

$$\Delta_{K,I,S} = \pi_{[n]\setminus K}\left(I(S)^{-1}(b) \cap F_K\right) ,$$

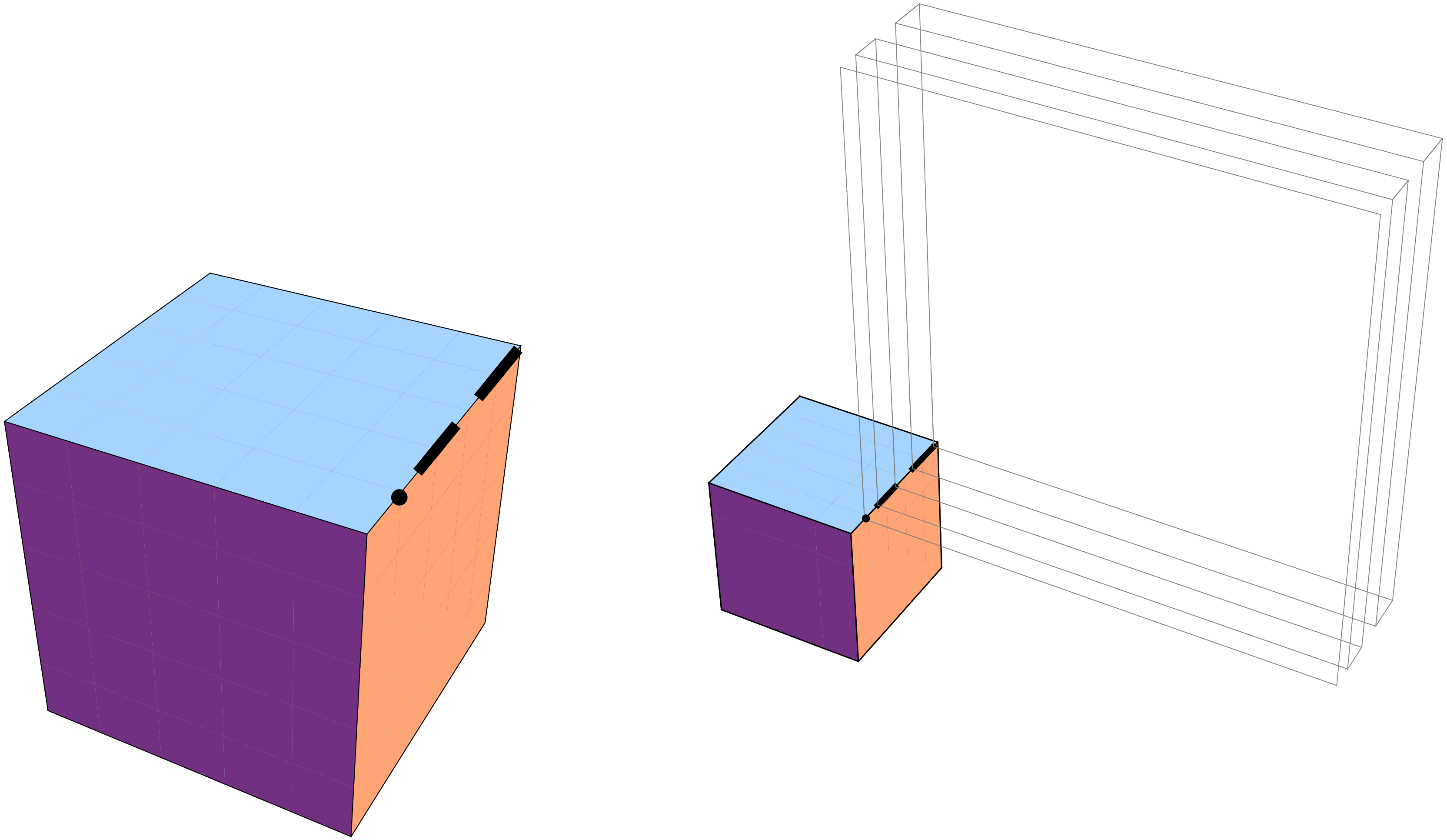
where $b \in \{0,1\}$ is equal to 0 iff: $I = L$, $S = [n]$, $\mathrm{dom}(H(S)) = \emptyset$.
Then the function $I(R)$ is essentially $|K|$-ary and:

$$\mathrm{dom}(I(R)) = \Delta_{K,I,S} \times [0,1]^K .$$

Encodings — Example from A_3

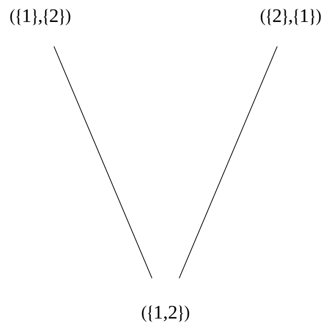


Encodings Example from A_3

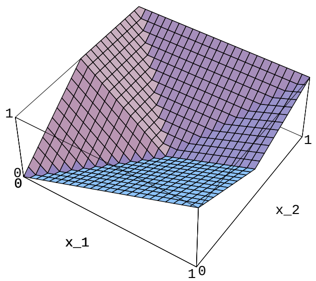


Encodings Example from A_2

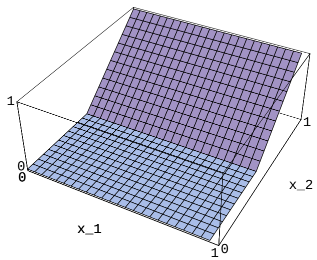
$(L, H) \in A_2$ F_2:



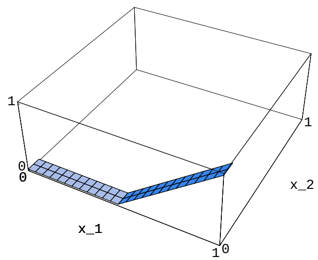
$L(\{1,2\})$:



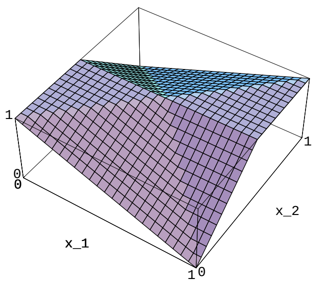
$L(\{1\},\{2\})$:



$L(\{2\},\{1\})$:

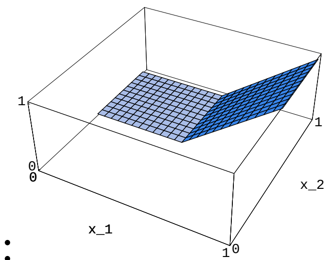


$H(\{1,2\})$:



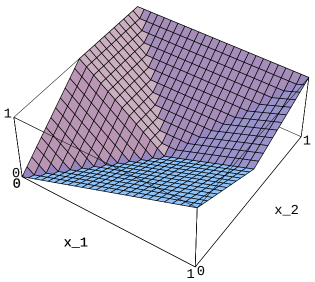
$H(\{1\},\{2\}) = \emptyset$

$H(\{2\},\{1\})$:

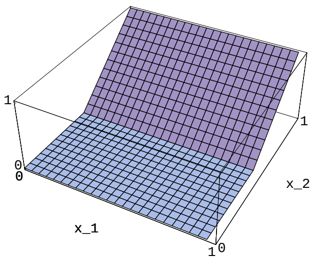


Encodings Example from A_2

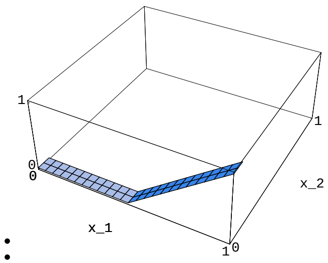
$L(\{1,2\})$:



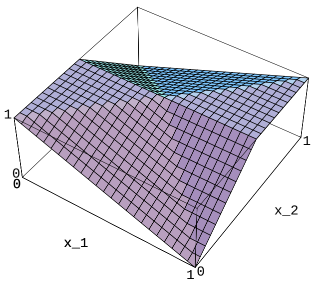
$L(\{1\},\{2\})$:



$L(\{2\},\{1\})$:

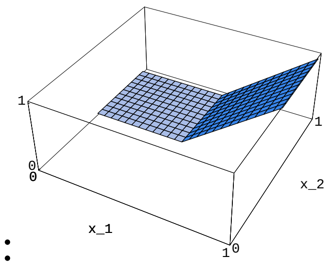


$H(\{1,2\})$:

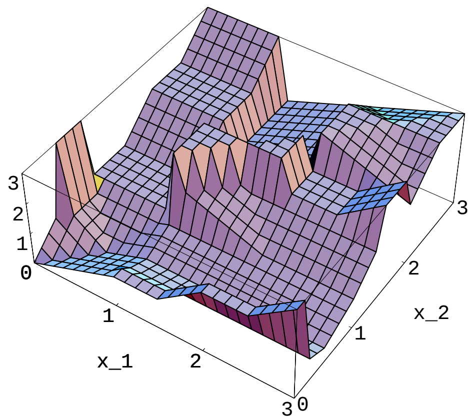


$H(\{1\},\{2\}) = \emptyset$

$H(\{2\},\{1\})$:

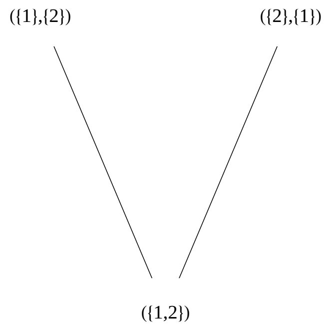


$f\colon [0,3]^2 \to [0,3]$:

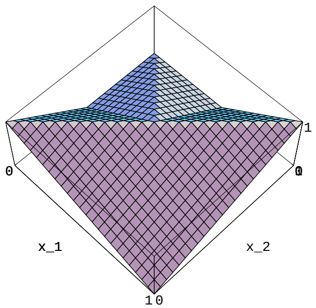


Encodings Example from A_2

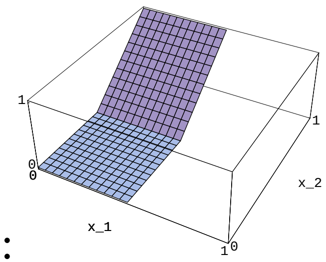
$(L, H) \in A_2$ F_2:



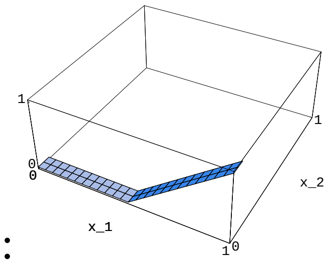
$L(\{1,2\})$:



$L(\{1\},\{2\})$:

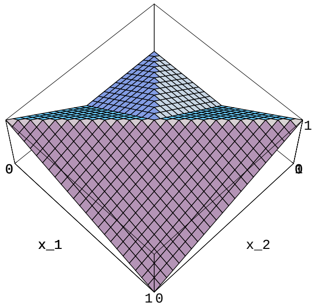


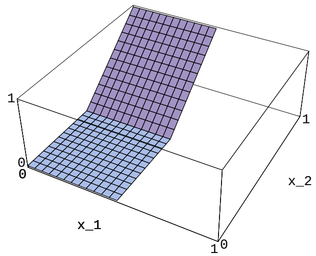
$L(\{2\},\{1\})$:

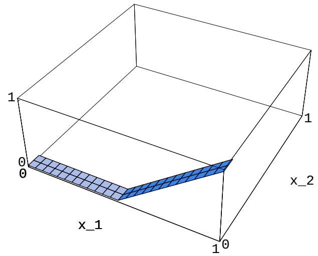


$$\forall R \in F_2 \ : \ H(R) = \emptyset.$$

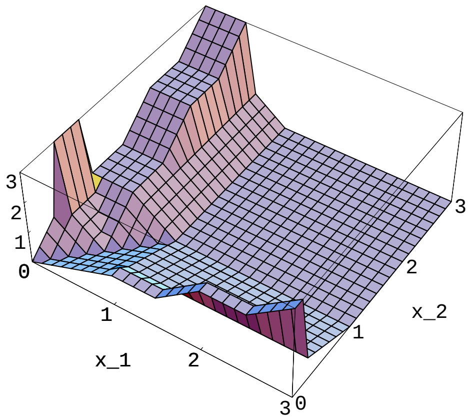
Encodings Example from A_2

$L(\{1,2\})$: 

$L(\{1\},\{2\})$: 

$L(\{2\},\{1\})$: 

$$\forall R \in F_2 \;:\; H(R) = \emptyset.$$

$f : [0,3]^2 \to [0,3]$: 

$\mathcal{BL}_n \cong \langle A_n, \odot, \to, \bot \rangle$

Generators: $(I_{X_i}, I_{X_i}) \in A_n$, where:
For every $R \in F_n$, K maximum block of R:
If $i \in K$, then

$$\mathrm{dom}(I_{X_i}(R)) = [0,1)^{[n]\setminus K} \times [0,1]^K$$

and

$$I_{X_i}(R) = \quad (t_1, \ldots, t_n) \mapsto t_i \quad \in \mathbb{WH}_n;$$

$\mathrm{dom}(I_{X_i}(R)) = \emptyset$, otherwise.

The operations:

$\bot = (L_\bot, H_\bot)$, for
$H_\bot(R) = \emptyset$ for each $R \in F_n$,
$L_\bot(R) = \emptyset$ for each $[n] \neq R \in F_n$, while $L_\bot([n])(\mathbf{x}) = 1$ for all $\mathbf{x} \in [0,1]^n$.

$\mathcal{BL}_n \cong \langle A_n, \odot, \rightarrow, \bot \rangle$

$(I_0'', I_1'') = (I_0, I_1) \odot (I_0', I_1')$
For all $R \in F_n$:
$\mathrm{dom}(I_b''(R)) = \mathrm{dom}(I_b(R)) \cap \mathrm{dom}(I_b'(R))$ for all $b \in \{0,1\}$ and

$$I_0''(R)(\mathbf{x}) = \begin{cases} I_0(R)(\mathbf{x}) * I_0'(R)(\mathbf{x}) & \text{if } R = [n], \\ I_0(R)(\mathbf{x}) \odot^{[0,1]} I_0'(R)(\mathbf{x}) & \text{if } R \neq [n], \end{cases}$$

where

$$I_0(R)(\mathbf{x}) * I_0'(R)(\mathbf{x}) = \begin{cases} I_0(R)(\mathbf{x}) \oplus^{[0,1]} I_0'(R)(\mathbf{x}) & \text{if } I_1(R) = \emptyset \text{ and } I_1'(R) = \emptyset, \\ I_0'(R)(\mathbf{x}) \rightarrow^{[0,1]} I_0(R)(\mathbf{x}) & \text{if } I_1(R) = \emptyset \text{ and } I_1'(R) \neq \emptyset, \\ I_0(R)(\mathbf{x}) \rightarrow^{[0,1]} I_0'(R)(\mathbf{x}) & \text{if } I_1(R) \neq \emptyset \text{ and } I_1'(R) = \emptyset. \\ I_0(R)(\mathbf{x}) \odot^{[0,1]} I_0'(R)(\mathbf{x}) & \text{if } I_1(R) \neq \emptyset \text{ and } I_1'(R) \neq \emptyset, \end{cases}$$

$$I_1''(R)(\mathbf{x}) = I_1(R)(\mathbf{x}) \odot^{[0,1]} I_1'(R)(\mathbf{x}) \,.$$

$\mathcal{BL}_n \cong \langle A_n, \odot, \to, \bot \rangle$

$(I_0'', I_1'') = (I_0, I_1) \to (I_0', I_1')$

For all $R \in F_n$: K maximum block of R, $k = |\{S \in F_n \mid S \leq_F R\}|$. For all $b \in \{0,1\}$:

$$\mathrm{dom}(I_b''(R)) = \mathrm{dom}(I_b'(R)) \cup Q_{b,R}$$

$$Q_{b,R} = \{\mathbf{x} \in [0,1)^{[n]\setminus B_k} \times [0,1]^{B_k} \mid 0 < d_{\mathbf{x}} \leq d'_{\mathbf{x}} < k,\, I_b(R_{d_{\mathbf{x}}})(\mathbf{f}_{\mathbf{x}}) \leq I_b'(R_{d_{\mathbf{x}}})(\mathbf{f}_{\mathbf{x}})\}$$

$$I_b''(R)(\mathbf{x}) = \begin{cases} I_b(R)(\mathbf{x}) \Rightarrow_b I_b'(R)(\mathbf{x}) & \text{if } R = [n], \\ I_b(R)(\mathbf{x}) \to^{[0,1]} I_b'(R)(\mathbf{x}) & \text{if } R \neq [n] \text{ and } \mathbf{x} \in \mathrm{dom}(I_b(R)) \cap \mathrm{dom}(I_b'(R)), \\ 1 & \text{otherwise.} \end{cases}$$

where

$$I_0(R)(\mathbf{x}) \Rightarrow_0 I_0'(R)(\mathbf{x}) = \begin{cases} I_0'(R)(\mathbf{x}) \to^{[0,1]} I_0'(R)(\mathbf{x}) & \text{if } I_1(R) = \emptyset \text{ and } I_1'(R) = \emptyset, \\ I_0(R)(\mathbf{x}) \oplus^{[0,1]} I_0'(R)(\mathbf{x}) & \text{if } I_1(R) = \emptyset \text{ and } I_1'(R) \neq \emptyset, \\ I_0(R)(\mathbf{x}) \odot^{[0,1]} I_0'(R)(\mathbf{x}) & \text{if } I_1(R) \neq \emptyset \text{ and } I_1'(R) = \emptyset. \\ I_0(R)(\mathbf{x}) \to^{[0,1]} I_0'(R)(\mathbf{x}) & \text{if } I_1(R) \neq \emptyset \text{ and } I_1'(R) \neq \emptyset, \end{cases}$$

$\mathcal{BL}_n \cong \langle A_n, \odot, \rightarrow, \bot \rangle$

$(I_0'', I_1'') = (I_0, I_1) \rightarrow (I_0', I_1')$

For all $R \in F_n$: K maximum block of R, $k = |\{S \in F_n \mid S \leq_F R\}|$. For all $b \in \{0, 1\}$:

$$\mathrm{dom}(I_b''(R)) = \mathrm{dom}(I_b'(R)) \cup Q_{b,R}$$

$$Q_{b,R} = \{\mathbf{x} \in [0,1)^{[n]\setminus K} \times [0,1]^K \mid 0 < d_{\mathbf{x}} \leq d'_{\mathbf{x}} < k,\ I_b(R_{d_{\mathbf{x}}})(\mathbf{f}_{\mathbf{x}}) \leq I_b'(R_{d_{\mathbf{x}}})(\mathbf{f}_{\mathbf{x}})\}$$

$$I_b''(R)(\mathbf{x}) = \begin{cases} I_b(R)(\mathbf{x}) \Rightarrow_b I_b'(R)(\mathbf{x}) & \text{if } R = [n], \\ I_b(R)(\mathbf{x}) \rightarrow^{[0,1]} I_b'(R)(\mathbf{x}) & \text{if } R \neq [n] \text{ and } \mathbf{x} \in \mathrm{dom}(I_b(R)) \cap \mathrm{dom}(I_b'(R)), \\ 1 & \text{otherwise.} \end{cases}$$

where

$$I_1(R)(\mathbf{x}) \Rightarrow_1 I_1'(R)(\mathbf{x}) = \begin{cases} 1 & \text{if } I_1(R) = \emptyset, \\ I_1(R)(\mathbf{x}) \rightarrow^{[0,1]} I_1'(R)(\mathbf{x}) & \text{otherwise.} \end{cases}$$

$\mathcal{BL}_n \cong \langle A_n, \odot, \to, \bot \rangle$ $Q_{b,R}$

$(I_0, I_1) \in A_n$, $b \in \{0,1\}$, $R \in F_n$.

Evaluation depth of $\mathbf{x}$ wrt. $I_b(R)$:
the height of the highest element $S \leq_F R$ of F_n such that there is a point $\mathbf{y} \in \mathrm{dom}(I_b(S))$ which is an "ancestor" for $\mathbf{x}$. (that is $\mathbf{y}$ is obtained by projecting to 1 a suitable subset of components of $\mathbf{x}$)

$$Q_{b,R} = \{\mathbf{x} \in [0,1)^{[n]\setminus K} \times [0,1]^K \mid 0 < d_\mathbf{x} \leq d'_\mathbf{x} < k,\ I_b(R_{d_\mathbf{x}})(\mathbf{f_x}) \leq I'_b(R_{d_\mathbf{x}})(\mathbf{f_x})\}$$

$Q_{b,R}$ is the region of all points $\mathbf{x}$ such that the evaluation depth of $\mathbf{x}$ wrt. $I_b(R)$ is lower than the evaluation depth of $\mathbf{x}$ wrt. $I'_b(R)$.