Truthlikeness, graded similarity and fuzziness: some logic-based approaches

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LoMoReVI Conference

September 2009, Čejkovice (Czech Republic)

Outline

- Introduction: uncertainty, fuzzines and (similarity-based) truthlikeness
- Logical approaches to similarity-based reasoning
- Similarity-based fuzzy modal and conditional approaches
- Combined model: uncertainty + truthlikeness
- Conclusions

Uncertainty, fuzziness, truthlikeness

Possible worlds scenario

Ideal situation: complete information about which is the *real* world w_0
$+$ precise concepts: in any world, either $w \models \varphi$ or $w \models \neg\varphi$

$$Truth = \{\varphi \mid w_0 \models \varphi\} \quad Falsity = \{\psi \mid w_0 \models \neg\psi\}$$

Some more realistic situations:

- Uncertainty: incomplete information about w_0 (precise concepts)

  $\mu(w) \in [0, 1]$ – how likely w is the real world

  how likely is that φ is true? $\mu(\{w \mid w \models \varphi\}) \in [0, 1]$

  - logics of belief (probabilistic, possibilistic, DSS, etc.)
  - non truth-functionality

Uncertainty, fuzziness, truthlikeness

- Fuzziness: (complete information but) imprecise, gradual concepts

  $w(\varphi) \in A \supset \{0, 1\}$ many-valuedness, intermediate degrees of truth

- Fuzzy logics:

  - $A = [0, 1]$ (usual choice, standard semantics)

  - truth-functionality (usual assumption)

  - logics of comparative truth: $w(\varphi \to \psi) = 1$ iff $w(\varphi) \leq w(\psi)$

  - logics of vagueness?

Uncertainty, fuzziness, truthlikeness

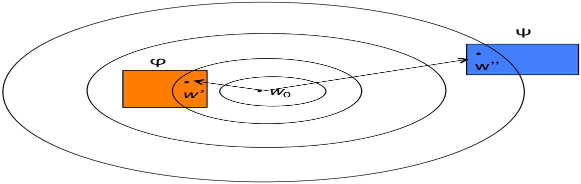
(Similarity-based) Truthlikeness

- assume an ideal scenario and introduce a kind of metric or similarity between possible worlds

$$S : W \times W \to [0,1], \quad S(w, w') := \text{ how similar is } w \text{ to } w'$$

- one can measure a distance or closeness to truth (Niiniluoto, Weston)

$I_S(\varphi \mid w_0) = \sup\{S(w_0, w) \mid w \models \varphi)\} \in [0,1]$



both φ ad ψ are false at w_0, but φ is closer to the truth (more truthlike) than ψ: $I_S(\varphi \mid w_0) \geq I_S(\psi \mid w_0)$

Uncertainty, fuzziness, truthlikeness

Similarity-based truthlikeness

- a more informed scenario (e.g. real world w_0 + similarity S)
- more fine-grained representation and reasoning framework: given a theory (epistemic state), one can identify not only its true, false or undecided consequences, but also which consequences are close to be true, and which are closer than others
- some links to fuzzy sets / vagueness, e.g. fuzzy set = set of prototypes + similarity relation (c.f. talks by Dubois, Prade and Vetterlein),
- able to be combined with uncertainty and fuzziness models

Aim: to show some logical approaches to (degree-based) similarity-based reasoning

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- Introduction: uncertainty, fuzzines and (similarity-based) truthlikeness
- Logical approaches to similarity-based reasoning
  - Graded similarity relations and truthlikeness degrees
  - Some logical readings: basic issues
- Similarity-based fuzzy modal and conditional approaches
- Combined model: uncertainty + truthlikeness
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Graded similarity relations

A similarity relation on D is a mapping $S : D \times D \to [0, 1]$ usually required to satisfy dual properties of those of a (bounded) metric

- Reflexivity: $S(u, u) = 1$
  Separation: $S(u, v) = 1$ only if $u = v$
- Symmetry: $S(u, v) = S(v, u)$
- $\otimes$-Transitivity: $S(u, v) \otimes S(v, w) \leq S(u, w)$

- when $x \otimes y = \max(x + y - 1, 0)$ and S separating, $\delta = 1 - S$ is a distance
- when $x \otimes y = \min(x, y)$, then $\delta = 1 - S$ is an ultrametric

Weaker notions: closeness (Refl), proximity relations (Refl + Sim)

Links to similarity-based semantics for fuzzy sets

Similarity relations and fuzzy sets

1. A fuzzy similarity relation S on D defines, for each crisp subset $E \subseteq D$, a corresponding fuzzy set *approx_E* of those elements which are "*close to E*":

$$\mu_{approx\_E}(u) = \sup\{S(u, v) \mid v \in E\}$$

$\Rightarrow$ if E a set of typical elements satisfying property P, $\mu_{approx\_E}(u)$ reads as a *typicality degree* of u with respect to the property P (Niiniluoto, 87).

2. Conversely, any fuzzy subset A on D, with membership function $\mu_A : D \to [0, 1]$, is indeed the fuzzy set of elements "*close to* E_A", where

(i) $E_A = \{u \in D \mid \mu_A(u) = 1\}$, the set of prototypes of A

(ii) a $\otimes$-similarity S_A

$$S_A(u, v) = \min(\mu_A(u) \Rightarrow_{\otimes} \mu_A(v), \mu_A(v) \Rightarrow_{\otimes} \mu_A(u))$$

Mind: we have one similarity S_A for each fuzzy set A!

Truthlikeness degrees and similarity measures

L: propositional language, W: set of Boolean interpretations of L
Given $S : W \times W \to [0,1]$

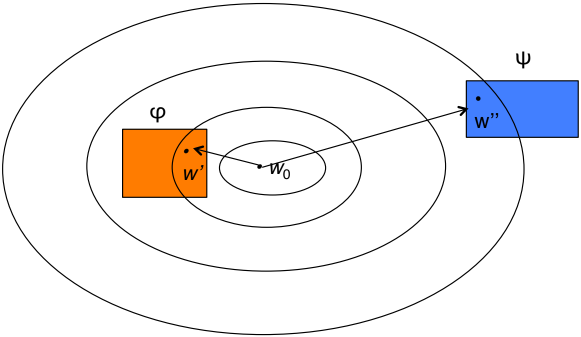
Truthlikeness degree of φ at $w \in W$: $I_S(\varphi \mid w) = \sup_{w' : w' \models \varphi} S(w, w')$

(Ruspini, 91)'s definitions:

Implication measure: $I_S(\varphi \mid \psi) = \inf_{w : w \models \psi} I_S(\varphi \mid w)$

(Hausdorff) Similarity: $E_S(\varphi, \psi) = \min(I_S(\psi \mid \varphi), I_S(\varphi \mid \psi))$

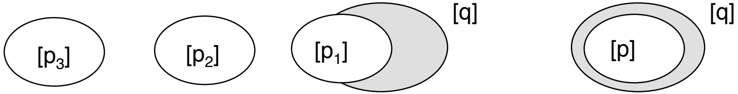
Consistency measure: $C_S(\varphi \mid \psi) = \sup_{w : w \models \psi} I_S(\varphi \mid w)$



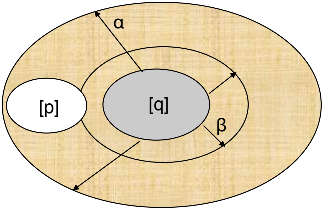
φ is closer to be true than ψ

Classical logic:

$p \not\models q$ $\qquad\qquad$ $p \models q$



$S : W \times W \to [0, 1]$ $\qquad$ $\Rightarrow$ spheres around $[p]$



$I_S(q \mid p) \geq \alpha, \; C_S(q \mid p) \geq \beta$

Some logical readings of I_S

1. Graded Consequence Relation: given $S : W \times W \to [0,1]$, for each $\Gamma \subset L$ define $\mathcal{C}_S(\Gamma) : L \to [0,1]$

$$\mathcal{C}_S(\Gamma)(p) = \inf\{I_S(p \mid w) \mid w \models_S \Gamma\} = I_S(p \mid \Gamma)$$

(i) $\Gamma \subseteq \mathcal{C}_S(\Gamma)$

(ii) if $\Gamma \subseteq \Gamma'$ then $\mathcal{C}_S(\Gamma) \leq \mathcal{C}_S(\Gamma')$

(iii) $\mathcal{C}_S(\mathcal{C}_S(\Gamma)) = \mathcal{C}_S(\Gamma)$

Some logical readings of I_S

2. Approximate entailment: given $S : W \times W \to [0,1]$ define

$w \models^{\alpha}_{S} p$ iff there exists a model w' of p which is α-similar to ω, i. e. $S(w, w') \geq \alpha$

$p \models^{\alpha}_{S} q$ iff $w \models^{\alpha}_{S} q$ for all model w of p, i. e. iff $I_S(q \mid p) \geq \alpha$

(1)	**Nestedness:**	if $p \models^{\alpha} q$ and $\beta \leq \alpha$ then $p \models^{\beta} q$;
(2)	$\otimes$-**Transitivity:**	if $p \models^{\alpha} r$ and $r \models^{\beta} q$ then $p \models^{\alpha \otimes \beta} q$;
(3)	**Reflexivity:**	$p \models^{1} p$;
(4)	**Right weakening:**	if $p \models^{\alpha} q$ and $q \models r$ then $p \models^{\alpha} r$;
(5)	**Left strengthening:**	if $p \models r$ and $r \models^{\alpha} q$ then $p \models^{\alpha} q$;
(6)	**Left OR:**	$p \vee r \models^{\alpha} q$ iff $p \models^{\alpha} q$ and $r \models^{\alpha} q$;
(7)	**Right OR:**	if r has a single model, $r \models^{\alpha} p \vee q$ iff $r \models^{\alpha} p$ or $r \models^{\alpha} q$.
...	...	

(DEGGP,97)

Multi-modal / conditional approaches

Similarity Kripke structures: $M = (W, S, e)$, with
$S : W \times W \to C \subset [0, 1]$

3. Muti-modal systems

$S_\alpha = \{(w, w') \mid S(w, w') \geq \alpha\}$: accesibility relations, α-cuts of S

Language: modal operators $\Diamond_\alpha$, $\alpha \in C$ ($\Diamond^c_\alpha, \Diamond^o_\alpha$)

Semantics:

$(M, \omega) \models \Diamond^c_\alpha \varphi$ if $I_S(\varphi \mid \omega) \geq \alpha$

... $(M, \omega') \models \varphi$ for some ω' s.t. $S(w, w') \geq \alpha$ ...

$(M, \omega) \models \Diamond^o_\alpha \varphi$ if $I_S(\varphi \mid \omega) > \alpha$.

- $\mathcal{M}_S \models p \to \Diamond_\alpha q$ iff $I_S(q \mid p) \geq \alpha$ iff $p \models^\alpha_S q$

Multi-modal / conditional approaches

Modal axioms ($* \in \{c, o\}$) :

K^*: $\Box^*_\alpha(\varphi \to \psi) \to (\Box^*_\alpha\varphi \to \Box^*_\alpha\psi)$
T^*: $\Box^*_\alpha\varphi \to \varphi$
C^c: $\varphi \to \Box^c_1\varphi$
B^*: $\varphi \to \Box^*_\alpha\Diamond^*_\alpha\varphi$
4^*: $\Box^*_{\alpha\otimes\beta}\varphi \to \Box^*_\beta\Box^*_\alpha\varphi$
N^*: $\Box^*_\alpha\varphi \to \Box^*_\beta\varphi$, for $\beta \geq \alpha$,
... ...

Necessitation Rules:

RN^*: From φ infer $\Box^*_\alpha\varphi$, for $\alpha > 0$.

Completeness results for different choices sets of axioms, classes of models and t-norm $\otimes$ (EGGR,97)

Multi-modal / conditional approaches

4. Muti-conditional systems

Aim: encode the graded entailments "$p \models^{\alpha}_{S} q$" at the object level as a conditional-like formula

Language:
- finite set of propositional variables
- propositional formulas are conditional formulas
- if p, q propositional formulas, then $p >_{\alpha} q$ is a conditional formula; no nested conditional formulas!

Semantics: $M = (W, S, e)$

$(M, w) \models p >_{\alpha} q$ if $I_S(q \mid p) \geq \alpha$ (independent of w)

Multi-modal / conditional approaches

Axioms:

N:	$p >_\alpha q \rightarrow p >_\beta q$, for $\beta \leq \alpha$
CS:	$p >_1 q \rightarrow (p \rightarrow q)$
EX:	$p >_0 q$
B:	$r >_\alpha r' \rightarrow r' >_\alpha r$, if r and r' are m.e.c.'s
4:	$(p >_\alpha q) \wedge (q >_\beta r) \rightarrow p >_{\alpha \otimes \beta} r$
LO:	$(p \vee q >_\alpha r) \leftrightarrow (p >_\alpha r) \wedge (q >_\alpha r)$
RO:	$(r >_\alpha p \vee q) \leftrightarrow (r >_\alpha p) \vee (r >_\alpha q)$, if r is a m.e.c.

Rules:

RK:	From $p \rightarrow q$ infer $p >_\alpha q$

Completeness results (Rodriguez, 02)

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Many-valued (fuzzy) modal / conditional approaches

$M = (W, e, S)$

Rather than dealing with formulas with indexed operators $\Diamond_\alpha \varphi$ and $\varphi >_\alpha \psi$, which are classically evaluated, let us allow evaluate them in each world by their corresponding degrees:

- $\Diamond\varphi :=$ "approximately φ"

  $e(\Diamond\varphi, w) = \sup_{w' \in W} S(w, w') \otimes e(\varphi, w') = I_S(\varphi \mid w)$

  $\Rightarrow$ mv/fuzzy modal logics

- $\varphi > \psi :=$ "ψ is an approximate consequence of φ"

  $e(\varphi > \psi, w) = I_S(\psi \mid \varphi)$

  $\Rightarrow$ mv/fuzzy conditional-like logics

More compact representations

Similarity-based fuzzy modal approach

Which logic to consider for $\Diamond$? ... many options!

Several design choices:

- underlying fuzzy (non-modal) logic:
  Gödel, Łukasiewicz, finitely-valued, etc. i.e. which $\otimes$ to take?
- with or without truth-constants (explicit degrees or purely qualitative)
- set of properties for the similarity relations: reflexive, symmetric, $\otimes$-transitive
- constrained by the available results in the literature, e.g. (Hájek, Caicedo & Rodriguez, Bou et al., etc.)

As a suitable example, we can consider the $\Diamond$-modal logic over Gödel fuzzy logic $G_\Diamond$ ([C&R, 09], [M&O, 09]) with min-transitive and reflexive similarity relations: $M = (W, e, S)$ with

$$e(\Diamond\varphi, w) = \sup_{w' \in W} \min(S(w, w'), e(\varphi, w'))$$

Similarity-based fuzzy modal approach: the logic $G_\Diamond$

Axioms and rules of Gödel logic (optionally with fm truth-constants) +

$D_\Diamond$: $\Diamond(\varphi \vee \psi) \to (\Diamond\varphi \vee \Diamond\psi)$
$Z_\Diamond^+$: $\Diamond\neg\neg\varphi \to \neg\neg\Diamond\varphi$
$T_\Diamond$: $\varphi \to \Diamond\varphi$
$4_\Diamond$: $\Diamond\Diamond\varphi \to \Diamond\varphi$

$R1$: $\Diamond\overline{r} \to \overline{r}$
$R2$: $\Diamond(\overline{r} \to \varphi) \to (\overline{r} \to \Diamond\varphi)$
$R3$: $\Diamond((\varphi \to \overline{r}) \to \overline{r}) \to ((\Diamond\varphi \to \overline{r}) \to \overline{r})$

$RN_\Diamond^+$: From $\varphi \to \psi$ infer $\Diamond\varphi \to \Diamond\psi$

Completeness results wrt the class of models (M, S, e) such that S is reflexive and min-transitive (C&R, 09)

Similarity-based mv conditional approach

A simpler framework (no nested conditional formulas!) :
- atomic conditional formulas: $p > q$
- compound conditional formulas built using connectives of a "chosen" fuzzy logic $L_\otimes$ ($\&$, $\rightarrow$, $\neg$)

Semantics: (M, S, e), where S is e.g. reflexive and $\otimes$ transitive

$e(p, w) \in \{0, 1\}$ (classical prop. formulas)
$e(\varphi > \psi, w) = I_S(\psi \mid \varphi)$ (independent of w)

Axioms and rules: those of $L_\otimes$ plus

Bool:	$p \vee \neg p$, for p propositional (non conditional).
4:	$(p > q)\&(q > s) \rightarrow p > s$.
LO:	$(p \vee q) > r \leftrightarrow (p > r) \wedge (q > r)$.
RO:	$(r > p \vee q) \leftrightarrow (r > p) \vee (r > q)$, if r is a m.e.c.
RK:	From $p \rightarrow q$ infer $p > q$

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Introducing uncertainty into the picture

Similarity Kripke models $M = (W, S, e)$ allows us to also account for *approximate truths* in an epistemic state described by a set worlds $K \subseteq W$

- if we know w_0 is the real world, i.e. $K = \{w_0\}$:
  $e(w_0, \Diamond\varphi) = I_S(\varphi \mid w_0) = \sup_{w \models \varphi} S(w, w')$
- if we know $\{w_0\} \subset K \subseteq W$, i.e. incomplete information:

$$e(w_0, \Diamond\varphi) \in [\alpha, \beta]$$

where $\alpha = \inf_{w \in K} I_S(\varphi \mid w) = I_S(\varphi \mid K)$
$\beta = \sup_{w \in K} I_S(\varphi \mid w) = C_S(\varphi \mid K)$

What if we know not all the worlds in K are equally probable or plausible?

Example on probability of fuzzy events (Franco):
"It is likely is that Messi scores a goal at the beginning of the match"
"beginning of the match" = in first quarter (φ) or close to ... $P(\Diamond\varphi)$?

Introducing uncertainty into the picture

This leads to:

- extend the similarity-based Kripke models:
  $M = (W, S, e, \mu)$, where $\pi : W \to [0,1]$ is e.g. a probability or possibility distribution on worlds
- evaluate how probable or plausible are the fuzzy "$\Diamond\varphi$" formulas according to the current results in probability or possibility / necessity of fuzzy events (e.g. states on MV-algebras, etc.)
- Probability:

$$e(w, P\Diamond\varphi) = \Sigma_{w \in W}\, \pi(w) \cdot e(w, \Diamond\varphi) = \Sigma_{w \in W}\, \pi(w) \cdot I_S(\varphi \mid w)$$

  i.e. an average of truthlikeness degrees

- Possibility / Necessity:

$$e(w, \Pi\Diamond\varphi) = \sup_{w \in W} \min(\pi(w), I_S(\varphi \mid w))$$
$$e(w, N\Diamond\varphi) = \inf_{w \in W} \pi(w) \Rightarrow I_S(\varphi \mid w)$$

  aggregation of truthlikeness degress related to Sugeno integrals

Uncertainty and truthlikeness

About axiomatizations:

- Idea is to combine logics for similarity-based $\Diamond$ operators with existing systems axiomatizing e.g.

  - the notion of probability of fuzzy events over finitely-valued Łukasiewicz logic (FG,08)

  - the notion of necessity / possibility over Łukasiewicz (FGM, 08) or over Gödel logic (DGM, 09)

- A simple approach: add one more (fuzzy logic) layer for belief

  - base level: classical logic formulas φ, ψ, ...

  - similarity level: $\Diamond\varphi$ (approximately φ), $\varphi > \psi$ (ψ is an approximate consequence of φ), ...

  - belief level: $P(\Diamond\varphi)$ (It is likely that φ is an approximate truth)

- full development is a current matter of research

Conclusions

- Truthlikeness: *"... classify propositions according to their closeness to the truth, their degree of truthlikeness or verisimilitude ... give an adequate account of the concept and to explore its logical properties and its applications to epistemology and methodology"* (G. Oddie, Stanford Encyclopedia of Philosophy)

  - Popper, Tichý, Hilpinen, Niiniluoto, ...

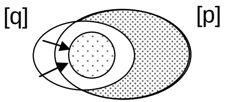
- A graded similarity-based account of truthlikeness enriches representation, reasoning and even decision capabilities and tools
- A further (independent) dimension to be additionally considered to models dealing with imperfect information (uncertainty, fuzziness, nonmonotonicity, ect.)
- Links to some restricited form of fuzziness / vagueness
- As for logical formalization, a lot if items to be further developed (e.g. fuzzy modal logics)

THANKS!

Plausible vs similarity reasoning

- **plausible reasoning: q |~ p**
  "best" models of q are models of p

q, p consistent
non-monotonic



- **similarity reasoning:** q $\models^{\alpha}$ p
  all the models of q are "similar" to some model of p

q, p may be inconsistent
monotonic