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On Deductive Fuzzy Logics as Logics of Gradual Properties

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Motivation for fuzzy logic

The original motivation: fuzzy logics = logics of *vague* properties

Disputed by many
$\Rightarrow$ desirable to formulate a less controversial motivation
= avoid references to *vagueness* and degrees of *truth*

Thesis:
Fuzzy logics = logics of *gradual* (rather than *vague*) properties

Gradual properties

Classical logic deals with *bivalent* properties

Either possessed by an individual or not

E.g.: *pregnant, dead,* > 5, ...

But: many real-world properties are *gradual*

Attributed to some individuals more than to others

E.g.: *tall, warm, steep, green, expensive, difficult,* ...

Usually expressed by adjectives that have a comparative

Often reducible to a real-valued quantity (eg, height in cm)

(Programmer terminology: attributes of type `Boolean` resp `real`)

Classical treatment of gradual properties

Boolean logic and classical mathematics evades direct treatment of gradual properties, by:

(a) Idealization (treating them as if they were bivalent)

(b) Representation by functions (assigning values to individuals)

Gradual properties treated as classical functions = (b)

If the gradual property is based on some *measured quantity*
(eg, *tall* on height in cm),
we can speak about the *values* of the quantity

The proposition *John is 168 cm tall* is bivalent
and the function assigning the heights to people is classical
$\Rightarrow$ Boolean logic and classical mathematics can be used
when speaking of numerical values of gradual properties

Even if such a numerical function is hard to find (*steep, green*)
or even impossible to find (*difficult, beautiful, relaxed*),
we can just stipulate some function or assume its existence
(a common practice in mathematical modelling,
eg, in physics, engineering, . . .)

The conceptual mismatch of the functional treatment

But: Treating gradual properties as functions

- conflicts with our understanding them as *properties*
- denies the role of *properties* they play in arguments

Example:

This apple is *red.*
Red apples are *ripe.*
Therefore, this apple is *ripe.*

The surface structure of the argument speaks about *properties* rather than functions

The conceptual mismatch for gradual collections

Similarly for gradual collections (of *red apples, tall people, . . .*)
= *extensions* of gradual properties

We want to work with their intersections, inclusion, etc
(eg: *tall dark-haired people, small apples are green, . . .*)

But observe:
Intersection or inclusion of *functions* is a different operation

⇒ The functions just *represent* them, but they should be treated as (non-bivalent, ergo perhaps non-classical) properties/sets

An attempt to remedy the mismatch for gradual families

Zadeh's fuzzy sets = conceptual shift:
Treat the functions as if they indeed delimited gradual families

Minimum of functions = Intersection of fuzzy sets

Comparison of functions = Inclusion of fuzzy sets, etc

The conceptual shift has proved to be very fruitful (the analogy between classical and fuzzy sets could be exploited)

Fuzzy set theory has mainly been applied to *imprecise* concepts (which may be disputable!), but its simplest interpretation is just in terms of *gradual* properties

Normalization of values into $[0,1]$

It is convenient to normalize the values of all gradual properties into $[0,1]$, in order to have a common scale—as we want to make intersections (etc) of incommensurable gradual properties (eg: *big red apples, tall young men, . . .*)

Of course, this can be done by various functions

- A particular function is part of the (conventional) semantics of a given attribute
- Similarly dependent are already the values of the measured quantity itself (eg, height in cm / feet / log scale / . . .)
- We use a *conventional* (technical, agreed) meaning of *tall, warm, . . .*
- In natural language, the function is rarely fixed
  $\Rightarrow$ ambiguity (linguistic vagueness)
  $=$ a concern for *formal semantics*, but not for *formal logic*
  (in *logic* we assume that the extensions are given)

Nothing more than the quantities?

Cf. Keefe, Vagueness by numbers (*Mind*, 1998):

> *Take the vague predicate 'tall': I claim that any numbers assigned in an attempt to capture the vagueness of 'tall' do no more than serve as another measure of* height. *More generally, in so far as it is possible to assign numbers which respect certain truths about, for example, comparative relations, this is no more than a measure of an attribute related to, or underlying, the vague predicate.*

In our present approach, we agree completely!

Still, from the point of view of *formal logic* (rather than just *formal semantics*), the quantities generate specific logical calculi that remove certain semantic paradoxes

This argument was not raised in the follow-up exchange between Keefe and Smith (*Mind*, 2003)

Transferring the remedy from sets to logic

The first step towards gradual logic = Goguen 1969:

intersection ... conjunction

complementation ... negation, etc

Eg, Zadeh's operations yield: $\wedge$ = min, $\vee$ = max, $\neg = 1 - x$

Then

$A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$, like in Boolean logic

$(A \wedge B) \vee (A \wedge \neg B) \not\equiv A$ (just $\leq$), unlike in Boolean logic

= the laws of 'fuzzy logic' = logic of gradual properties

= logical laws for gradual properties,

different from classical logical laws for bivalent properties

(expectable—richer models)

Degrees of gradual properties

The role played by truth values in classical logic is here played by the *degrees* = values normalized into $[0,1]$

$\Rightarrow$ It would be natural (for a scientist) to call them *degrees of truth* (with bivalent truth as a special case)

But since many philosophers object to the conception of gradual truth, we shall call them just *degrees* (of the gradual property)

(Similarly we can speak just of gradual *attributes* if gradual *properties* may seem controversial)

Degree-functionality

By a design choice,
the connectives of ‘fuzzy logic’ are *degree-functional*

(eg, the degree of conjunction is a function of the degrees of conjuncts)

Often objected, but:

- This is how we *define* our *logical* conjunction
- It need not coincide with *linguistic* conjunction
  (studied by formal semantics, not formal logic)
- Logical analysis of natural language needs a more advanced, non-degree-functional apparatus (eg, gradual modalities)
- Degree-functionality ensures intersubstitutivity
  = contents-independence
  (we are not interested in *which* property’s degree it is)

The role of logic

Fuzzy logic should be to gradual properties
what classical logic is to bivalent properties
= axiomatize the laws of inference for such properties

What does *inference* mean in gradual properties?

Bivalent properties . . . truth
$\Rightarrow$ laws of inference preserving *truth*
(salva veritate)

Gradual properties . . . degrees
$\Rightarrow$ laws of inference preserving *degrees*
(salvo gradu ~~veritatis~~)

$\Rightarrow$ Def.: $A \models B$ iff for all models M: $\|A\|_M \leq \|B\|_M$
(consequence *local* wrt degrees)

Internalization of the logical apparatus

A useful-for-reasoning logic of gradual properties should also internalize the salvo-gradu consequence relation (as implication):

$$\|A\|_M \leq \|B\|_M \text{ iff } \|A \to B\|_M \in D_M$$

= Cintula's (weakly) implicative fuzzy logics:

The *full degree* of implication $A \to B$ ensures that the degree of B is at least as large as the degree of A

The match of implication as a connective and as internalized inference *salvo gradu* also for *partial degrees*

$\Rightarrow$ deductive fuzzy logics = logics of linear residuated lattices (on $[0,1]$: t-norm logics)

Běhounek: On the difference between traditional and deductive fuzzy logic, *Fuzzy Sets and Systems* 2006

Logical connectives of deductive fuzzy logics

Conjunction as internalization of comma between premises
. . . residuation, $A \to (B \to C) \equiv A \mathbin{\&} B \to C$

Then tautologies $A_1 \mathbin{\&} \ldots \mathbin{\&} A_n \to B$ express sound rules of gradual inference

On the $[0,1]$ scale, the requirements entail that:
$\&$ = a left-continuous t-norm $*$ (eg, Ł = max$((x+y-1),0)$)
$\to$ = its residuum = sup$\{z \mid z * x \leq y\}$

NB: our 'preservation of degrees' differs from Font's—
premises are combined by non-idempotent conjunction!—
and my usage of 'gradual' differs from Dubois'

Clear meaning of connectives:
$\&$ = combination of degrees (ie, values of gradual properties)
$\to$ = the lacking value (in the sense of $\&$)
$\neg$ = reductio ad ~~absurdum~~ the null degree
$\forall$ = guaranteed degree of any instance, etc

Deductive fuzzy logics = logics of inference salvo gradu

MTL = logic of *all* left-continuous t-norms
= of all reasonable choices of functions for connectives
$\Rightarrow$ the most general laws of the logic of gradual properties

Particular choice of functions representing the connectives
is left to the user

Semantical analysis may suggest a more particular choice of t-norm connectives (eg, strict negation, nilpotent t-norm, etc); then a stronger logic (extending MTL) can be used
(eg, Ł, Π, G, IMTL, SMTL, BL, Bool, ...)

Example

$Pa \mathbin{\&} (\forall x)(Px \to Qx) \to Qa$ is a tautology of MTL

$\Rightarrow$ The argument

This apple is *red.*
Red apples are *ripe.*
Therefore, this apple is *ripe.*

is a sound argument for the gradual properties,

no matter which functions are chosen for *red, ripe, and, then:*
For any choice, the degree of Ripe(a) will be at least
the &-combination of the degrees of Red(a) and $(\forall x)(\text{Red}(x) \to \text{Ripe}(x))$

Observe: If the functions for Ripe and Red are suitably chosen
(so that the general premise has the full degree),
we will always have $\|\text{Ripe}(a)\| \geq \|\text{Red}(a)\|$
But even if not, the argument is still sound—only the general premise may have a very small degree and ensure almost nothing

Idealization of gradual properties as bivalent $= (a)$

The logic of gradual properties does not differ much from the logic of bivalent properties (MTL and Bool are quite similar)

$\Rightarrow$ The bivalent idealization of gradual properties often works
(eg, the above argument would work
even if Red and Ripe were assumed to be bivalent)

But not always—in some cases the idealization is inappropriate and leads to semantical paradoxes

Sorites

In particular, many instances of the Sorites paradox arise by an inappropriate application of bivalent logic to gradual properties

(We need not provide a full story here—we are just doing a *logical* analysis of gradual properties, *not* a theory of vagueness)

Assuming that *tall* is bivalent works most of the time,
even when making *short* arguments about a sorites series: eg,

$$Ta_{1000} \mathbin{\&} (Ta_{1000} \rightarrow Ta_{999}) \mathbin{\&} (Ta_{999} \rightarrow Ta_{998}) \rightarrow Ta_{998}$$

is fine (998th man is still tall),

but it leads to a paradox in a *long* series,

because *tall* is in fact gradual

$\Rightarrow$ the logic of gradual properties should have been used

A model of the Sorites in the logic of gradual properties

The premises $Ta_n \to Ta_{n-1}$ do not have the full degree:
There *is* a small difference in height,
which gets converted to a small lapse of degree

Since the lapse is nevertheless very small,

- The inductive premise looks plausible
- The (idealized) assumption that the degree is full
works for short arguments,

as the (non-idempotent!) conjunction will accumulate the lapse
only in a sufficiently long series

In a long series the lapse of degree nevertheless accumulates
and finally invalidates the premise

$\Rightarrow$ no paradox under gradual logic

(NB: non-idempotent conjunction needed here, eg Ł; min would not work;
note also that in Ł with *strong* connectives, $A \to B \equiv \neg A \veebar B \equiv \neg(A \mathbin{\&} \neg B)$
$\Rightarrow$ it does not matter which way we formulate the Sorites)

Summary

- Fuzzy logics (MTL, Ł, G, . . .) can be motivated as logics of gradual properties (independent of vagueness)
- Fuzzy logics describe the laws of inference salvo gradu between gradual propositions, and internalize their deductive apparatus $\Rightarrow$ logics of linear residuated lattices
- Functions representing gradual properties and logical connectives are chosen by the user (model maker) (formal logic $\neq$ logical analysis of natural language)
- Many instances of the Sorites paradox arise by applying bivalent logic to gradual properties