# The Coherence of Łukasiewicz Assessments is NP-complete

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# The classical coherence problem

De Finetti's foundation of Probability Theory relies on the *coherence* of betting odds: Let $\phi_1, \ldots, \phi_n$ be classical events and let $\mathbf{a} : \{\phi_1, \ldots, \phi_n\} \to [0, 1]$ be an assessment of $\phi_1, \ldots, \phi_n$. Then $\mathbf{a}$ is said to be *coherent* if and only if there is no system of reversible bets on the events leading to a win independently on the truth of $\phi_1, \ldots, \phi_n$.
Precisely, the assessment $\mathbf{a}$ is coherent if and only if, for every (stake) $\mathbf{b} : \{\phi_1, \ldots, \phi_n\} \to \mathbb{R}$, there exists a Boolean valuation (a possible world) $\mathbf{v}: \{\phi_1, \ldots, \phi_n\} \to \{0, 1\}$ such that

$$\sum_{i=1}^{n} \mathbf{b}(\phi_i)(\mathbf{a}(\phi_i) - \mathbf{v}(\phi_i)) \geq 0. \quad (1)$$

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### Theorem (de Finetti)

*An assessment* $\mathbf{a}$ *is coherent if and only if* $\mathbf{a}$ *coincides with the restriction to* $\{\phi_1, \ldots, \phi_n\}$ *of a finitely additive probability* $P$ *from the free Boolean algebra* $B$ *generated by the* $\phi_i$*'s into* $[0,1]$.

# Introducing MV-algebras

Consider the signature $\Sigma = (\odot, \oplus, \neg, \top, \bot)$ of type $(2, 2, 1, 0, 0)$. Call $T$ the set of formulas over $\Sigma$ and countably many variables $X = \{X_1, \ldots, X_n, \ldots\}$. If we fix a finite set $\{X_1, \ldots, X_m\}$ of variables, the obtained set of formulas in $\Sigma$ is denoted by $T_m$.
The algebra

$$[0,1]_{MV} = ([0,1], \odot^{[0,1]}, \oplus^{[0,1]}, \neg^{[0,1]}, \bot^{[0,1]}, \top^{[0,1]}),$$

where $x \odot^{[0,1]} y = \max(0, x + y - 1)$, $x \oplus^{[0,1]} y = \min(1, x + y)$, $\neg^{[0,1]} x = 1 - x$, $\bot^{[0,1]} = 0$, and $\top^{[0,1]} = 1$, is called the *standard MV-algebra*. $[0,1]_{MV}$ is generic for the variety $\mathbb{MV}$ of MV-algebras.
Call $T^m_{MV}$ the free MV-algebra over $m$ generators $(X_1, \ldots, X_m)$.

For every $k \in \mathbb{N}$, call

$$S^k = \{0, 1/k, \ldots, (k-1)/k, 1\}.$$

The variety of $k+1$-valued MV-algebras (MV$_k$-algebras) is the variety generated by the finite and linearly ordered MV-algebra

$$S^k_{MV} = (S^k, \odot^{S^k}, \oplus^{S^k}, \neg^{S^k}, \bot^{S^k}, \top^{S^k}),$$

where, for every $\star \in \Sigma$, $\star^{S^k}$ denotes the restriction of $\star^{[0,1]}$ to $S^k$.

$T^m_{MV_k}$ denotes the free MV$_k$-algebra over the $m$ generators $X_1, \ldots, X_m$.

## Theorem (McNaughton)

*For every* $m \in \mathbb{N} \cup \{\infty\}$, $T^m_{MV}$ *is isomorphic to the MV-algebra of m-variate McNaughton functions. That is the MV-algebra of all those functions from* $[0,1]^m$ *into* $[0,1]$ *which are continuous, piecewise linear and each piece having integer coefficients.*

Moreover, for every $k \in \mathbb{N}$, and every $m \in \mathbb{N} \cup \{\infty\}$, $T^m_{MV_k}$ is isomorphic to the MV-algebra of all those functions from $(S^k)^m$ into $S^k$ which are the restriction to $(S^k)^m$ of the functions in $T^m_{MV}$.

# From classical to many-valued events

A natural generalization of de Finetti coherence criterion is obtained allowing a many-valued interpretation of events $\phi_1, \ldots, \phi_n$, instead of their classical two-valued interpretation.

**Definition**

(1) For $k \geq 2$, we say that the assessment $\mathbf{a}\colon \{\phi_1, \ldots, \phi_n\} \to [0,1]$ is $k$-coherent if and only if for every $\mathbf{b}\colon \{\phi_1, \ldots, \phi_n\} \to \mathbb{R}$, there exists a valuation $\mathbf{v}\colon \{\phi_1, \ldots, \phi_n\} \to S^k$ satisfying (1).

(2) An assessment $\mathbf{a}\colon \{\phi_1, \ldots, \phi_n\} \to [0,1]$ is $\infty$-coherent if and only if for every $\mathbf{b}\colon \{\phi_1, \ldots, \phi_n\} \to \mathbb{R}$, there exists a valuation $\mathbf{v}\colon \{\phi_1, \ldots, \phi_n\} \to [0,1]$ satisfying (1).

We use the notation: for $\alpha = 2, 3, \ldots, n, \ldots, \infty$

$$\alpha\text{COH-LUK-ASS} = \{\langle \mathbf{a} \rangle \mid \mathbf{a} \text{ is a } \alpha\text{-coherent}\},$$

where $\langle \mathbf{a} \rangle$ denotes the binary encoding of $\mathbf{a} \in ([0,1] \cap \mathbb{Q})^{\{\phi_1, \ldots, \phi_n\}}$.

J. Paris: For every $k \geq 2$, an assessment $\mathbf{a}\colon \{\phi_1, \ldots, \phi_n\} \to [0,1]$ is $k$-coherent if and only if $\mathbf{a}$ extends to a state $\mathbf{s}$ on $T^m_{MV_k}$. Moreover $\mathbf{s}$ is a convex combination of at most $n+1$ homomorphisms $\mathbf{h}_i : T^m_{MV_k} \to S^k_{MV}$.

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In other words $\mathbf{a}$ is coherent iff there are $t \leq n+1$ homomorphisms $\mathbf{h}_1, \ldots, \mathbf{h}_t : T^m_{MV} \to [0,1]_{MV}$ and real numbers $\lambda_1, \ldots, \lambda_t \in \mathbb{R}$ such that

- $\sum_{i=1}^t \lambda_i = 1$,
- For every $j = 1, \ldots, n$, $\mathbf{a}(\phi_j) = \sum_{i=1}^{t} \lambda_i \cdot \mathbf{h}_i(\phi_j)$.

# The complexity of $\alpha$COH-LUK-ASS

J. Paris: The problem 2COH-LUK-ASS is NP-complete.

*

P. Hájek: The problem $k$COH-LUK-ASS is NP-complete. (see [Haj07][1] and [FlaPhD][2])

*

TF.& F. Montagna: The problem $\infty$COH-LUK-ASS is PSPACE.

[1] P. Hájek, Complexity of Fuzzy Probabilistic Logics II. Fuzzy Sets and Systems 158(23), 2605–2611, 2007.

[2] T. Flaminio, A Fuzzy-Modal Approach to Probability: from Crisp to Fuzzy Events. Ph.D. Thesis, University of Siena, 2006.

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**Theorem (Bova-F.)**

*Let $\phi_1, \ldots, \phi_n \in T_m$, and let $\mathbf{a} : \{\phi_1, \ldots, \phi_n\} \to [0,1]$ be an assessment. Then $\mathbf{a} \in \infty$COH-LUK-ASS iff there exist*

- *a unary polynomial $p \colon \mathbb{N} \to \mathbb{N}$,*
- *(at most) $n+1$ homomorphisms $\mathbf{h}_i$ from $T^m_{MV}$ to $[0,1]_{MV}$ satisfying the following.*
    - *For all $1 \leq i \leq n+1$, $\mathbf{h}_i$ ranges over $S^{d_i} = \{0, 1/d_i, \ldots, (d_i - 1)/d_i, 1\}$, where*

$$\log_2 d_i \leq p(\mathrm{size}(\mathbf{a})),$$

    - *$\mathbf{a}$ extends to a convex combination of the $\mathbf{h}_i$'s.*

**LEMMA**

*The problem* $\infty$COH-LUK-ASS *is in NP.*

**Proof** (Sketch) Let $\mathbf{a} : \{\phi_1, \ldots, \phi_n\} \to [0,1] \cap \mathbb{Q}$ be an assessment. We assume without loss of generality that $\phi_1, \ldots, \phi_n \in T_m$.
The algorithm does the following:

(1) For every $1 \leq i \leq n$, guess the denominator $d_i$.

(2) For every $1 \leq i \leq n+1$, guess the homomorphism $\mathbf{h}_i \in (S^{d_i})^m$, and compute the values $\mathbf{h}_i(\phi_j)$ (for $1 \leq i \leq n+1$ and $1 \leq j \leq n$). This computation can be done in polynomial time in $\mathrm{size}(\phi_i)$ and $d_i$ (see [Haj07, Lemma 2])

(3) Check the feasibility of the following linear system:

$$
\begin{aligned}
x_1 + \cdots + x_n + x_{n+1} &= 1 \\
\mathbf{h}_1(\phi_1)x_1 + \cdots + \mathbf{h}_n(\phi_1)x_n + \mathbf{h}_{n+1}(\phi_1)x_{n+1} &= \mathbf{a}(\phi_1) \\
&\vdots \\
\mathbf{h}_1(\phi_n)x_1 + \cdots + \mathbf{h}_n(\phi_n)x_n + \mathbf{h}_{n+1}(\phi_n)x_{n+1} &= \mathbf{a}(\phi_n)
\end{aligned}
$$

The denominator $d_i$ has a polynomial-space encoding. Hence, the restriction of $\mathbf{h}_i$ to $X_1, \ldots, X_m$, as well as the coefficients $\mathbf{h}_i(\phi_1), \ldots, \mathbf{h}_i(\phi_n)$, are in $\{0, 1/d_i, \ldots, (d_i - 1)/d_i, 1\}$. So, the size of the system is polynomial in $\mathrm{size}(\mathbf{a})$, and the algorithm terminates in time polynomial in $\mathrm{size}(\mathbf{a})$.
The linear system is feasible if and only if $\mathbf{a}$ is a convex combination of $\mathbf{h}_1, \ldots, \mathbf{h}_{n+1}$ if and only if $\mathbf{a}$ is coherent.

**Lemma**

*The problem* $\infty$COH-LUK-ASS *is NP-hard.*

**Proof** (Sketch) There exists a logarithmic-space reduction from the NP-hard problem of checking satisfiability in Łukasiewicz logic, to $\infty$COH-LUK-ASS.

## Lemma

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**Proof** (Sketch) There exists a logarithmic-space reduction from the NP-hard problem of checking satisfiability in Łukasiewicz logic, to $\infty$COH-LUK-ASS.

## Theorem

$\infty$COH-LUK-ASS *is NP-complete.*

## On the complexity of Łukasiewicz based probabilistic logics

In [FlGo07] we introduced the following logics to deal with probabilistic sentences over many-valued events. The language is obtained by extending the language of Łukasiewicz logic by a unary modality $P$ for "probably".

- $FP(Ł_k, Ł)$ where $Ł_k$ is the $(k+1)$-valued Łukasiewicz logic and whose formulas are evaluated into $S^k$. This is the logic for events. Ł, Łukasiewicz logic, is the logic for probabilistic sentences.
- $FP(Ł, Ł)$ where Łukasiewicz logic is both the logic for events, and probabilistic sentences.

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$$\mathcal{SFP} \supset \mathcal{FP}$$

The natural algebraic semantics for $SFP(Ł, Ł)$ is the variety of SMV-algebras: Let $A$ be an MV-algebra, and let $\sigma : A \to A$ be a unary operator on $A$ satisfying the following equations:

- $\sigma(\top) = 1$,
- $\sigma(\neg x) = \neg\sigma(x)$,
- $\sigma(\sigma(x) \oplus \sigma(y)) = \sigma(x) \oplus \sigma(y)$,
- $\sigma(x \oplus y) = \sigma(x) \oplus \sigma(y \ominus (x \odot y))$

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The logic $SFP(Ł, Ł)$ is complete with respect to the class of SMV-algebras. In particular $SFP(Ł, Ł)$ is complete with respect to those SMV-algebras such that $\sigma(A)$ is a linearly ordered MV-algebra.

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A formula $\phi \in \mathcal{SFP}$ is said to be:

- SMV-satisfiable if there exists an SMV-algebra $(A, \sigma)$ and a valuation $\mathbf{v}$ such that $\mathbf{v}(\phi) = 1$.
- standard satisfiable if $\phi$ is satisfiable in a $\sigma$-simple SMV-algebra.
- a standard tautology if for every $\sigma$-simple *SMV*-algebra, and every valuation $\mathbf{v}$, $\mathbf{v}(\phi) = 1$.

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**Theorem**

*The set of SMV-satisfiable formulas is NP-complete;*
*The set of standard satisfiable formulas is NP-complete;*
*The set of standard tautologies is coNP-complete.*