# Vagueness and fuzzy logic –

.

Can logicians learn from philosophers?
AND
Can philosophers learn from logicians?

.

Petr Hájek, Inst.of Comp. Sci, Acad. of Sci. Prague

Fermüller: Vagueness and fuzzy logic – Can logicians learn from philosophers? Neural Network World 2003:
Fuzzy logicians should pay more attention to various different types of theories of vagueness... TO judge whether a certain formal model is adequate for a particular application it is not sufficient to refer to mathematical theorems...Formal logic in theorizing about vagueness is not limited to a particular type of theory.(..chalenges of supervaluation or epistemic aspects..)

Here only some remarks and comments on philosophical criticism of the fuzzy approach. First recall the distinction between fuzzy logic in the broad and narrow sense:

Paul P. Wang , Da Ruan, Etienne E. Kerre - eds. Fuzzy logic - a spectrum of theoretical and practical issues. Springer 2007.

"Fuzzy logic" ...either in a narrow sense or in a broad sense. In the narrow sense...formal development of various logical systems of many-valued logic. (Cites Hájek, Běhounek and Cintula.) In the broad sense...extensive agenda whose primary aim is to utilize the apparatus of fuzzy set theory for developing sound concepts, principles and methods for representing and dealing with knowledge expressed by statements. in natural language.

Here: only in the narrow sense – mathematical fuzzy logic.

Reading Rosanna Keefe: Theories of vagueness.

The phenomena of vagueness. Central features of vague expressions. Borderline cases. Vague predicates lack well-defined extensions. (They) are naturally described as having fuzzy, or blurred boundaries. Sorites. Wright: tolerant predicate - there is a notion of degree of change too small to make any difference. We could not operate with a language free of vagueness. Distinguish vagueness from ambiguity.

How to theorize about vagueness. Methodology, aims and constraints.

(Keefe:) Establishing a reflective equilibrium. Theorists should aim to find the best balance between preserving as many as possible of our judgments or opinions of various different kinds and meeting such requirements on theories as simplicity. There is unlikely to be any theory which can be conclusively defended. There can be disputes about what is in the relevant body of opinions. Determining the counter-intuitive consequences of a theory is always a major part of its assessment. And we must be cautious of theories that appear to save the high profile intuitions (e.g. regarding the law of excluded middle) but that do so in a way that requires the denial of equally important intuitions. Discussed: truth degrees, supertruth (and some other approaches).

Fuzzy logic deals with truth degrees – more truth values than (absolute) truth and (absolute (falsity). Comparative notion of truth. Standard choice - the real unit interval $[0, 1]$. (Various other choices.)

Keefe: Perhaps assignment of numbers in degree theory seen merely as a useful instrumental device. But what are we to say about the real truth-value status of borderline case predictions? The modeler's approach is a mere handvawing...surely the assignment of numbers is central to it? Only order is important? In next chapter examined (Keefe says) - final, fatal blows on degree theories of vagueness. ????

Sanford: many truth values multiply the problem

My comment: Yes, mainly a "model"; the task is not assigning concrete numerical values to concrete sentences (formulas); the task is to study the notion of consequence (deduction) in presence of fuzziness.

Some similarity with subjective probability? Saying "Probably I shall come" you assume a concrete value of your subjective probability without feeling obliged to "assign" it to what you say. (By the way, "Probably" is a fuzzy modality.)

*Truth functionality?* The truth value of a compound formula is a function of its components via truth functions of connectives (quantifiers - later). Many-valued logic.

Goguen (Synthese 1969): The logic of inexact concepts.
Gaifman: There is no denying the graded nature of vague predicates - i.e. that the aptness of applying them can be a matter of degree - and there is no denying the gradual decrease in degree (Sorites) More than other approaches degree theory does justice to these facts. But from this to the institution of many-valued logic, where connectives are interpreted as functions over truth degree there is a big jump.
J. Paris (2000) - some support for truth functionality.
N.J.J.Smith: book Vagueness and truth degrees 2008 - positive!!

Now: **what is mathematical fuzzy logic?** (Very short survey.)

*Contiuous t-norms* – possible truth functions of conjunction. A binary operation $*$ on $[0,1]$ is a t-norm if it is commutative ($x*y = y*x$), associative ($x*(y*z) = (x*y)*z$), non-decreasing in both arguments and 1 is the unit element.

| | |
|---|---|
| $x*y = \max(0, x+y-1)$ | (Łukasiewicz's $t$-norm), |
| $x*y = \min(x,y)$ | (Gödel's $t$-norm), |
| $x*y = x \cdot y$ | (product $t$-norm). |

(4) The truth function of *implication* is the *residuum* of the corresponding t-norm:
$x \Rightarrow y = \max\{z | x * z \leq y\}$.

$$x \Rightarrow y = 1 \text{ iff } x \leq y; \text{ for } x > y$$

$x \Rightarrow y = 1 - x + y$ (Łukasiewicz),
$x \Rightarrow y = y$ (Gödel),
$x \rightarrow y = y/x$ (product).

Negation $(-)x = x \Rightarrow 0$ – $(-)x = 1 - x$ for Łukasiewicz,
Gödel and product: $(-)0 = 1, (-)x = 0$ for $x > 0$

Defined connectives: $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$. (Thus two conjunctions: & and $\wedge$.)

**The basic propositional fuzzy logic** BL:

propositional variables $p, q, \ldots$
connectives $\&, \to$, truth constant $\bar{0}$ (falsity)
Pro each given continuous t-norm $*$ (and its residuum $\Rightarrow$), each evaluation of propositional variable by truth values defines the corresponding evaluation of all formulas.

*-tautology: a formula $\varphi$ such that $e_*(\varphi) = 1$ for each evaluation $e$.
t-tautology: $*$-tautology for each continuous t-norm $*$.

Example: $(p\&q) \to (q\&p)$ is a t-tautology. But $p \to (p\&p)$ is not: for $e(p) = 0.5$ and Łukasiewicz conjunction $e(p\&p) = 0$, for product t-norm $e(p\&p) = 0.25$ (for Gödel's t-norm $e(p\&p) = 0.5$).

| | $p$ | $p\&p$ | $p \to (p\&p)$ |
|---|---|---|---|
| Ł | 0.5 | 0 | 0.5 |
| G | 0.5 | 0.5 | 1 |
| Π | 0.5 | 0.25 | 0.75 |

Compare ”I love you” with ”I love you and I love you and I love you”. Clearly the latter implies the former; but conversely??

Moreover: ”Whether I like him? Oh, yes and no”. Doesn't this mean that the truth value of ”I like him” is one half (0.5)??

Also compare ”I shall phone to him and (I shall) go there” with ”I shall go there and (I shall) phone to him” - non-commutative conjunction? Also possible and studied, but I do not discuss it here.

*Axioms for connectives:*

| | |
|---|---|
| (A1) | $(\varphi \to \psi) \to ((\psi \to \chi) \to (\varphi \to \chi))$ |
| (A2) | $(\varphi \& \psi) \to \varphi$ |
| (A3) | $(\varphi \& \psi) \to (\psi \& \varphi)$ |
| (A4) | $(\varphi \& (\varphi \to \psi)) \to (\psi \& (\psi \to \varphi))$ |
| (A5a) | $(\varphi \to (\psi \to \chi)) \to ((\varphi \& \psi) \to \chi)$ |
| (A5b) | $((\varphi \& \psi) \to \chi) \to (\varphi \to (\psi \to \chi))$ |
| (A6) | $((\varphi \to \psi) \to \chi) \to (((\psi \to \varphi) \to \chi) \to \chi)$ |
| (A7) | $\bar{0} \to \varphi$ |

Deduction rule: modus ponens (from $\varphi$ a $\varphi \to \psi$ derive $\psi$).

Łukasiewicz's logic Ł: BL + $\neg\neg\varphi \to \varphi$
Gödel'c logic G: BL + $\varphi \to (\varphi \& \varphi)$
product logic Π: BL + $(\varphi \to \neg\varphi) \to \neg\varphi$ +
$\neg\neg\chi \to (((\varphi \& \chi) \to (\psi \& \chi)) \to (\varphi \to \psi))$

*Completeness:*
Ł proves exactly all $[0,1]_{Ł}$-tautologies. (Rose-Rosser 1958)
G proves exactly all $[01,]_G$-tautologies. (Dummett 1959)
Π proves exactly all $[0,1]_\Pi$-tautologies. (Hajek-Esteva-Godo 1996)
BL proves exactly all t-tautologies. (Cignoli-Esteva-Godo-Torrens 2000).

General semantics: BL-algebras as general algebras of truth functions. Strong completeness.

Fuzzy predicate calculi - well developed.
This mathematical fuzzy logic is not well known by philosophers; deductive aspects ignored.

Keefe objects: the truth value of $\neg(p \wedge \neg p)$ can be less than 1. YES: e.g. for $p$ being half-true. BUT $\neg(p \& \neg p)$ has always the value 1 (is fully true).
For $p$ half-true, $e(p \to p) = e(p \to \neg p)$. YES, since for this truth value $p$ is equivalent to its negation!
Modus ponens fails? $p = 0.5, (p \to q) = 0.5, q = 0$?? OK, but NO FAILURE: the modus ponens for partially true formulas says: if $e(A) = x$ and $e(A \to B) = y$ then $e(B) \geq e(A) * e(A \to B)$, here $0.5 * 0.5 = 0$.

Sanford: truth value of $(A \& \neg A)$ different from 0 is absurd, BUT his & is our $\wedge$ (interpreted by minimum), thus NOT ABSURD: on Ł just the value of $A$ is neither 0 or 1 ($A$ is at least a little true and so is $\neg(A)$. (For Gödel and product logic and the corresponding negation $A \wedge \neg A$ has always value 0.)

Smith (book Vagueness and degrees of truth, 2008. Chapter 5: Who's afraid of degrees of truth? pp.221-222: classical semantics for fuzzy logic? $\Gamma$ entails $A$ if each evaluation making all members of $\Gamma > 0.5$-true makes $A > 0.5$-true. This consequence is claimed to coincide with classical consequence; this is OK for formulas using only connectives $\wedge, \vee, \neg$ interpreted by min, max, $1 - x$, but not for (Łukasiewicz) implication (neither for strong Łukasiewicz conjunction).
Counterexample: $p, p \rightarrow q \vdash q$ classically, but for
$e(p) = 0.6$, $e(q) = 0.4$ we get $e(p \rightarrow q) = 1 - 0.6 + 0.4 = 0.8$, thus $e(p) > 0.5, e(p \rightarrow q) > 0.5$, but $e(q) < 0.5$.

Keefe – serious objections:

Diagnosing the error.(??) Assigning numbers which respect certain truths about e.g. comparative relations is no more than a measure of an related attribute underlying the vague predicate. There is a sense in which (a vague predicate) comes in degrees - whenever there is a measure of the attribute and things have different degrees by having more or less the attribute. Heat comes in degrees but "has heat" is a predicate and its predication is never anything but definitely true or false. (**BUT WHY??"...is hot" - more or less true**)

(Keefe continued) If, as I am suggesting, the numerical assignments are nothing more than measurements of attributes, then the lack of any underlying unified attribute corresponding to our intuitive (unprecisified) concept of intelligence explains the inability to assign numbers effectively in representing the vague predicate "intelligent". (**Who wants effectively assign??** Degree theories fail to provide an acceptable account of vagueness. Moreover, they are forced to make an implausible commitment to a unique numerical assignment for each sentence. **WHY?!!!** *We do not assign; but in each interpretation (model) each sentence has a unique truth value [numerical or more abstract - recall the BL-algebras of truth functions].*

Apparently there are various kinds of conjunction (as example above seems to show). In each fuzzy logic we have at least two :$\&, \wedge$ (being the same only in Gödel logic). I would expect that each kind of conjunction satisfies the assumptions put on a t-norm. Note that in a sufficiently rich fuzzy logic (e.g. called $\text{Ł}\Pi\frac{1}{2}$) we may work with several continuous t-norm conjunctions at the same time. Doesn't this make the truth functionality fully acceptable??

Sorites (bald man paradox). J.L.King: Two ways: to deny that such statements have truth values or suppose that truth is a matter of degree. Smith: Given the existence of a Sorites series for a predicate $F$, there is no way to accomodate the claim that $F$ conforms to Closeness without accepting the idea that truth comes in degrees.

Hedges - relatively, kind of, ...(G. Lakoff). P.H.+ V.Novák - analysis of Sorites. Hedge $At$ - almost true. Axioms: $bold(0)$, $(\forall n)(bold(n) \to At(bold(n+1)))$ (and further reasonable axioms on $At$).

An alternative to fuzzy logic: theory of supertruth.

S. Shapiro: Vagueness in context. Clarendon Press Oxford 2006.

Partial interpretation (also predicate calc.) Three values: 1, 0 and $i$ (unknown). Truth tables of connectives: Kleene: e.g.

| $\Phi \& \Psi$ | 1 | 0 | $i$ |
|---|---|---|---|
| 1 | 1 | 0 | $i$ |
| 0 | 0 | 0 | 0 |
| $i$ | $i$ | 0 | $i$ |

Sharpening: some $i$'s in the interpretation of predicates replaced by 0 or 1. Notation: $M_1 \preceq M_2$.

$i$ - borderline cases (for some speakers it is yes, for some others it is no).

Theorem 1: If $M_1 \preceq M_2$ then for each (closed) $\Phi$ true in $M_1$, $\Phi$ is true in $M_2$ and for each $\Phi$ false in $M_1$, $\Phi$ is false in $M_2$.

Penumbral connections in metatheory: e.g. if $a$ is bald and $b$ has less hair than $a$ then $b$ is bald. $M$ is acceptable if it satisfies all penumbral connections.

$\Phi$ is supertrue in $M$ if is true in at least one acceptable sharpening $M'$ and is not false in any acceptable sharpening $M'$. Dually superfalse. $\Gamma \models_S \Phi$ if for each partial interpretation $M$ with all sentences from $\Gamma$ supertrue in $M$ $\Phi$ is supertrue in $M$.

Theorem 2. Assume that each acceptable partial interpretation has an acceptable crisp sharpening. Then for each set $\Gamma$ of sentences and each sentence $\Phi$, $\Gamma \models_S \Phi$ iff $\Phi$ is a classical consequence of $\Gamma$.

Supervaluationist's slogan: truth is supertruth.
Theorem 2 $\models_S \Phi$ iff $\Phi$ is a classical tautology.

OUR COMMENTS: Can we ask analogous questions as above? Where are frames from? Unique space? Is it only a model?

Fuzzifying!!! (Sanford 1973, Keefe p.116, P.H.: On vagueness, truth values and fuzzy logics, Studia Logica 2008)

**Interval-valued fuzzy logics**

For each BL-chain (say) $\mathcal{L}$, let $T\mathcal{L}$ be the set of closed intervals $[u, v]$ with $u \leq v$ from $\mathcal{L}$. Each element $u$ of $\mathcal{L}$ is identified with $[u, u]$.

First propositional logic: An interval evaluation of propositional variables from a set $Var$ is a mapping $w : Var \to T\mathcal{L}$. To be interpreted that the truth value of a variable $p$ can be any of the elements of $[u, v]$. Observe that this generalizes Shapiro's use of Kleene's three values if the third value is interpreted as $\{0, 1\} = [0, 1]$ in the two-valued Boolean chain.

Two ways of extending to non-atomic formulas (Esteva-Godo) and generalizing to an interval-valued predicate fuzzy logic. A model $\mathcal{M}'$ is a sharpening of $\mathcal{M}$ if the $\mathcal{M}'$-truth value of each instance of each atomic formula is a subinterval of the corresponding $\mathcal{M}$-value.

$\mathbf{M}$ is **fully sharp** if the truth value of each instance of each atomic formula is a singleton interval (identified with the corresponding element of $\mathcal{L}$), hence it is an usual interpretation. Let us say **full sharpening** instead of "fully sharp sharpening".

Assume that a class of models, called acceptable models, is given. Assume it is non-empty and that each acceptable model has an acceptable full sharpening.

A sentence $\varphi$ is **supertrue** in an (interval-valued) model $\mathbf{M}$) if it is true in each acceptable full sharpening of it. Finally define, for a set $\Gamma$ of sentences and a sentence $\varphi$, $\Gamma \models_S \varphi$ (say, $\varphi$ **superfollows from** $\Gamma$) if for each acceptable $\mathbf{M}$ in which all elements of $\Gamma$ are supertrue, also $\varphi$ is supertrue.

Theorem. Assume that each fully sharp model $\mathbf{M}$ in which $\Gamma$ is true is acceptable. Then $\Gamma \models_S \varphi$ iff $\varphi$ is provable from $\Gamma$ over our logic $BL\forall$.

WHAT WE HAVEN'T DISCUSS: fuzzy logic in the broad sense and so-called soft computing: many successful concrete applications of "Fuzzy inference" using "Fuzzy if-then rules" (e.g. If the pressure is high then turn the device slightly to right" with some fuzzy interpretation (NOT as fuzzy implication). The principle of soft computing (Zadeh) is not to look for optimal solution but to a satisfactorily good one.

To close let us quote a critical remark of R. Parikh ons supertruth and fuzzy logic (Vagueness and utility: the semantics of common nouns, in Linguistics and Philosophy 1994):
Both approaches have hidden assumptions about some underlying consensus among individuals. They may be satisfactory as providing semantics for a single person but they fail to explain how communication among people is possible. Super-truth theory fails because it is likely to be disagreement of base points. Similarly, fuzzy logic fails because people disagree on the fuzzy value of sentences. Instead of trying to find shared semantics I propose pragmatism. There is no point in trying to hide the differences of meaning of words between different people; But as long as these meanings are close enough, communication may be useful.

Returning to the two questions from the title slide

Can logicians learn from philosophers?

AND

Can philosophers learn from logicians?

**I believe that the answer is YES.**