The Model-Theoretic Study of Many-Valued Logics: A Survey

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Logical Models of Reasoning with Vague Information

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1. Model Theory: A Reminder

2. MV Algebras and Łukasiewicz Logics
   - Chang
   - Lacava and Saeli
   - Di Nola
   - Baaz and Veith
   - Caicedo
   - Di Nola, M., and Spada

3. Amalgamation through Quantifier Elimination

4. Further Works

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Model Theory: A Reminder (I)

T has elimination of quantifiers in L if for every formula $\phi(\overline{x})$ there is a quantifier-free formula $\psi(\overline{x})$ that is equivalent to $\phi(\overline{x})$ for all $\mathcal{A} \models \mathrm{T}$

T is model-complete if any embedding between any of its models is elementary, i.e.: for any $\mathcal{A}, \mathcal{B} \models \mathrm{T}$, any embedding $f : \mathcal{A} \to \mathcal{B}$, any L-formula $\phi(x_1, \dots, x_m)$, and $a_1, \dots, a_m \in \mathcal{A}$,

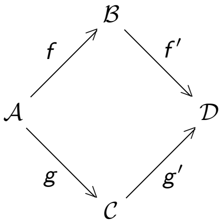
$$\mathcal{A} \models \phi(a_1, \dots, a_m) \text{ IFF } \mathcal{B} \models \phi(f(a_1), \dots, f(a_m)).$$

Two L-structures $\mathcal{A}, \mathcal{B}$ are elementarily equivalent if, for every L-sentence ϕ, $\mathcal{A} \models \phi$ iff $\mathcal{B} \models \phi$.

T is called complete if for any L-sentence ϕ, either $\mathrm{T} \models \phi$ or $\mathrm{T} \models \neg\phi$.

Model Theory: A Reminder (II)

A class $\mathbb{K}$ of structures has the amalgamation property if for every tuple $(\mathcal{A}, \mathcal{B}, \mathcal{C}, f, g)$ such that $\mathcal{A}, \mathcal{B}, \mathcal{C} \in \mathbb{K}$, and $f : \mathcal{A} \to \mathcal{B}$, $g : \mathcal{A} \to \mathcal{C}$ are embeddings, there exist a structure $\mathcal{D} \in \mathbb{K}$ and embeddings $f' : \mathcal{B} \to \mathcal{D}$, $g' : \mathcal{C} \to \mathcal{D}$ such that $f' \circ f = g' \circ g$.



A class $\mathbb{K}$ of structures in the same signature L is said to have the strong amalgamation property, if it has the amalgamation property and, moreover,
$f'[\mathcal{B}] \cap g'[\mathcal{C}] = (f' \circ f)[\mathcal{A}] = (g' \circ g)[\mathcal{A}]$, where for any set X and function h on X, $h[X] = \{h(x) \mid x \in X\}$.

Model Theory: A Reminder (III)

Let T be a theory in a language L, and let $\mathrm{T} \subseteq \mathrm{T}'$.

T' is a model companion of T whenever

(*i*) T' is model-complete

(*ii*) Every model of T' has an extension that is a model of T

(*iii*) Every model of T has an extension that is a model of T'

T' is a model completion of T whenever T' is a model companion of T and T has the amalgamation property

Whenever T' is a model completion of T and T has a universal axiomatization, then T' has quantifier elimination.

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1959

- Chang defined a notion of translation from an MV chain into the related ordered Abelian group:
- To each identity ϵ in the theory of MV algebras, there corresponds a universal sentence ϕ^{ϵ} in the theory of ordered Abelian groups such that for any linearly ordered MV algebra $\mathcal{A}$, ϵ holds in $\mathcal{A}$ if and only if ϕ^{ϵ} holds in $\mathcal{G}_{\mathcal{A}}$.
- Chang used the facts that every ordered Abelian group is embeddable into a divisible one, and that any two ordered Abelian groups are elementarily equivalent to each other to prove that:
- Łukasiewicz logic is complete w.r.t. evaluations over the reals.

1976

- The universal theory of $Ł_3$ is decidable.
- $\mathrm{Th}(Ł_3)$ has the amalgamation property.
- $\mathrm{Th}(Ł_3)$ has a model-completion.
- The same holds for $\mathrm{Th}(Ł_\mathrm{n})$.

1977

- The theory of divisible MV chains is the model-completion of the theory of MV chains.
- The theory of divisible MV chains is complete.

1979

- The theory of MV algebras does not have a model-companion.

Di Nola

1993

Theorem

For any MV algebra $\mathcal{A}$, there exists an ultrapower $[0,1]^*_{\mathrm{MV}}$ of $[0,1]_{\mathrm{MV}}$, such that $\mathcal{A}$ can be embedded into the product $([0,1]^*_{\mathrm{MV}})^{P(\mathcal{A})}$.

1999

- Baaz and Veith study the theory of the MV algebra over $[0,1]$, i.e. $[0,1]_{\mathrm{MV}}$.
- Using the theory of polyhedra and linear programming: $\mathrm{Th}([0,1]_{\mathrm{MV}})$ has QE in the language $\langle \oplus, \neg, 0, < \rangle$.
- $\mathrm{Th}([0,1]_{\mathrm{MV}})$ is model-complete, has the strong amalgamation property, is equivalent to a $\forall\exists$ theory, is not categorical in uncountable powers.
- $\mathrm{Th}_{\forall}([0,1]_{\mathrm{MV}})$ has the amalgamation property.
- $\mathrm{Th}([0,1]_{\mathrm{MV}})$ is *o*-minimal.
- The MV chain over $[0,1] \cap \mathbb{Q}$, is the unique prime model for $\mathrm{Th}([0,1]_{\mathrm{MV}})$.
- $\mathrm{Th}(\mathcal{S}_{\mathrm{n}})$ is ultrahomogeneous and has QE.

2006

- The theory of DMV chains has QE in the language $\langle \oplus, \neg, 0, \delta_n, < \rangle$.
- The theory of DMV chains is complete.

Di Nola, M., and Spada (I)

2009

Theorem

Let $\mathbb{V}$ be a variety of MV algebras, and let $\mathrm{Th}(\mathbb{V}_{lin})$ be the first-order theory of its linearly ordered members in the language $\langle \oplus, \neg, 0, < \rangle$. Then $\mathrm{Th}(\mathbb{V}_{lin})$ admits a model-completion if and only if $\mathbb{V}$ is generated by a single chain.

Corollary

Let $\mathbb{V}$ be a variety of MV algebras. Then, the following statements are equivalent:

1. $\mathbb{V}$ *has the amalgamation property.*
2. $\mathbb{V}$ *is generated by a single chain.*
3. $\mathrm{Th}(\mathbb{V}_{lin})$ *has a model completion.*

Di Nola, M., and Spada (II)

Theorem

Let $\mathbb{V}$ be a variety of MV algebras generated by a single chain, and let T be the model-completion of $\mathrm{Th}(\mathbb{V}_{lin})$. Let $\mathcal{G}$ be any $\mathcal{G} \models \mathrm{T}$. Then, for any MV algebra $\mathcal{A} \in \mathbb{V}$, there exists an ultrapower $\mathcal{G}^*$ of $\mathcal{G}$, such that $\mathcal{A}$ can be embedded into the product $((\mathcal{G})^*)^{P(\mathcal{A})}$.

Corollary

- *Every MV chain is embeddable into an ultrapower $[0,1]^*$ of $[0,1]$*
- *Every MV chain in $\mathbb{V}(\mathcal{S}_n)$ is embeddable into an ultrapower $\mathcal{S}_n^*$ of $\mathcal{S}_n$*
- *Every MV chain in $\mathbb{V}(\mathcal{S}_n^\omega)$ is embeddable into an ultrapower of $\Gamma(\mathbb{Z}\vec{\times}\mathbb{R}, (n,0))$, which corresponds to $\Gamma(\mathbb{Z}\vec{\times}\mathbb{R}^*, (n,0))$.*

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Obtaining Amalgamation from QE (I)

Let $\mathbb{V}$ be a variety of representable commutative residuated lattices.
Suppose that the following conditions hold for some elementary class $\mathbb{K} \subseteq \mathbb{V}_{lin}$:

(*i*) $\mathrm{Th}(\mathbb{K})$ admits quantifier elimination in the language $\langle *, \to, <, e \rangle$ (possibly with additional constant and/or function symbols).

(*ii*) $\mathcal{A} \models \mathrm{Th}_{\forall}(\mathbb{K})$ for all $\mathcal{A} \in \mathbb{V}_{lin}$.

Then, $\mathbb{K}$ enjoys the strong amalgamation property.

Moreover,

$\mathbb{V}$ ENJOYS THE AMALGAMATION PROPERTY.

Obtaining Amalgamation from QE (II)

- If $\mathrm{Th}(\mathbb{K})$ enjoys QE, then $\mathbb{K}$ has the strong amalgamation property.
- If $\mathrm{Th}(\mathbb{K})$ enjoys QE, and every member of $\mathbb{V}_{lin}$ is a model of $\mathrm{Th}_{\forall}(\mathbb{K})$, then $\mathbb{V}_{lin}$ has the amalgamation property.
- If $\mathbb{V}$ is a variety of representable commutative residuated lattices, and $\mathbb{V}_{lin}$ has the amalgamation property, so does $\mathbb{V}$ [Metcalfe, Montagna, Tsinakis (2008)].
- Then we conclude that $\mathbb{V}$ has the amalgamation property.

The basic idea:

- Take a variety $\mathbb{V}$ of representable commutative residuated lattices.
- Look for some elementary class $\mathbb{K} \subseteq \mathbb{V}_{lin}$.
- Prove QE for $\mathrm{Th}(\mathbb{K})$.
- Prove that every $\mathcal{A} \in \mathbb{V}_{lin}$ models $\mathrm{T}_{\forall}(\mathbb{K})$.
- Conclude that $\mathbb{V}$ has the amalgamation property.

Application

	$\mathrm{Th}(\mathbb{K})$	THEORY WITH QE	AMALGAMATION
MV	Div. MV-chains	DOAG	$\mathsf{MV} \hookrightarrow \mathsf{MV}_{div}$
NM	Dense NM-chains with f.p.	DLOEI	$\mathsf{NM} \hookrightarrow \mathsf{NM}_{d+fp}$
Π	Div. Π-chains	DOAG	$\Pi \hookrightarrow \Pi_{div}$
G	Dense G-chains	DLOE	$\mathsf{G} \hookrightarrow \mathsf{G}_{d}$
IUML	Dense IUML-chains	DLOEI	$\mathsf{IUML} \hookrightarrow \mathsf{IUML}_{d}$
RU	Divisible RU-chains	DOAG	$\mathsf{RU} \hookrightarrow \mathsf{RU}_{div}$

Let $\mathbb{K}$ be any of the above classes of algebras whose theory has QE. Then:

(*i*) $\mathrm{Th}(\mathbb{K})$ is model-complete.

(*ii*) $\mathrm{Th}(\mathbb{K})$ is complete.

(*iii*) Any two $\mathcal{A}, \mathcal{B} \in \mathbb{K}$ are elementarily equivalent to each other.

(*iv*) $\mathrm{Th}(\mathbb{K})$ is *o*-minimal.

Notice that the above proves that the variety $\mathbb{V}$ is generated by any member of $\mathbb{K}$.

Finally, the logics related to the above varieties have both the Robinson Property and the Deductive Interpolation Property.

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4. Further Works

Further Works

- BL algebras
- Idempotent bounded residuated lattices
- Model theory of probability

THANKS!