

# Some Remarks on Fuzzy Modal Logics

Félix Bou

Artificial Intelligence Research Institute (IIIA - CSIC)  
Barcelona (Spain)  
fbou@iia.csic.es

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# Outline

- The first part of the talk is devoted to survey some recent developments on fuzzy modal logics. We will mostly focus on fuzzy modal logics  $\mathbf{\Lambda}(\text{Fr}, \mathbf{A})$  (which are generalizations of the minimum classical modal logic K).
- The second part will consist on some “philosophical” remarks about these logics.



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Thus, the minimum modal logic (i.e., K) is the closure of  $L$  under  $\vdash_2$ .



# Replacing $\mathbf{2}$ with a residuated lattice $\mathbf{A}$

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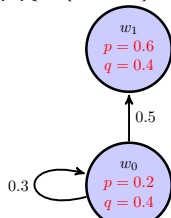
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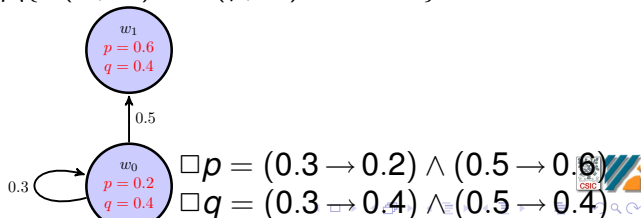
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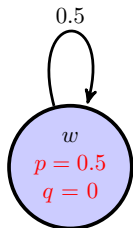
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$$\begin{aligned} \Box(p \rightarrow q) &= 0.5 \rightarrow (0.5 \rightarrow 0) = 1 \text{ (in } \mathbf{L}_3) \\ \Box p &= 0.5 \rightarrow 0.5 = 1 \\ \Box q &= 0.5 \rightarrow 0 = 0.5 \end{aligned}$$



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- CFr is in general not definable by modal axioms even if there are canonical constants. This is so because indeed  $\Lambda(BFr, \mathbf{A}^c) = \Lambda(CFr, \mathbf{A}^c)$  (when  $\mathbf{A}$  is finite).



# How to axiomatize these modal logics?

- The non-modal consequence relation  $\vdash_{\mathbf{A}}$  is defined by

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- It is easy to prove that for every  $K \in \{\text{Fr}, \text{IFr}, \text{CFr}\}$  there is some set  $L$  such that

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So,  $\Lambda(\text{Fr}, [0, 1]_{\mathbf{G}})$  coincides with this set  $L$ .



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So,  $\Lambda(\text{Fr}, [0, 1]_{\mathbf{G}})$  coincides with this set  $L$ . And also  $\Lambda(\text{CFr}, [0, 1]_{\mathbf{G}})$  is the same set.



# The case of a finite MV chain $L_n$

## Theorem (Hansoul-Teheux)

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So,  $\mathbf{\Lambda}(\mathbf{Fr}, \mathbf{L}_3)$  coincides with this set  $L$ . It is not enough to add  $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$  in order to obtain  $\mathbf{\Lambda}(\mathbf{CFr}, \mathbf{L}_3)$ .



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where  $a_2 = \frac{1}{n-1}$ ,  $a_3 = \frac{2}{n-1}$ ,  $\dots$ ,  $a_{n-1} = \frac{n-2}{n-1}$  and  $a_n = 1$ .



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- If moreover we assume that  $\mathbf{A}^c$  has a unique coatom  $k$ , then adding the normality axiom plus  $\Box(\bar{k} \vee \varphi) \rightarrow (\bar{k} \vee \Box\varphi)$  it is enough to axiomatize  $\mathbf{\Lambda}(\text{CFr}, \mathbf{A}^c)$ .



# What is the role of normality in fuzzy modal logics?

## Theorem

If  $\mathbf{A}$  is a complete BL chain, then  $\Lambda(\text{IFr}, \mathbf{A}) = \Lambda(\text{CFr}, \mathbf{A})$ .



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## Theorem

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## Informal Interpretation of the Previous Result

On the BL framework if the normality axioms holds then indeed we are only considering crisp Kripke models.



## Remark 1 (Fitting)

Let us assume that there are two experts  $e_1$  and  $e_2$  such that each of the experts has their own classical Kripke model  $\langle W, R, V_i \rangle$ .



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Then, we can replace both classical Kripke models with just one multi-valued Kripke model  $\langle W, R, V \rangle$  over the three-valued Gödel chain (with universe  $\{\emptyset, \{e_1\}, \{e_1, e_2\}\}$ ).



## Remark 2

From a classical perspective in some contexts it is interesting to work with non-normal modalities (in the sense that  $\Box$  does not distribute over  $\wedge$ ).



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In the modal fuzzy setting we can keep these formulas as consistent ones while remaining in the (simple) Kripke framework. This is due to the fact that normality may fail even in the Kripke semantics (e.g., Łukasiewicz semantics).



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The idea is that the formula

$$\Box(\varphi \leftrightarrow \Delta\varphi)$$

is expressing the fact that “ $\varphi$  is crisp”.

