GEOMETRY WITHOUT POINTS

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Euclid's Definitions

- A ***point*** is that which has no part.
- A ***line*** is breadthless length.
- A ***surface*** is that which has length and breadth only.

The Basic Questions

- Are these notions too ***abstract*** ? Or too ***idealized*** ?
- Can we develop a theory of ***regions*** without using points ?
- Does it make sense for geometric objects to be only ***solids*** ?

Famous Proponents of Pointlessness

Gottfried Wilhelm von Leibniz (1646 – 1716)
Nikolai Lobachevsky (1792 – 1856)
Edmund Husserl (1859 – 1938)
Alfred North Whitehead (1861 – 1947)
Johannes Trolle Hjelmslev (1873 – 1950)
Edward Vermilye Huntington (1874 – 1952)
Theodore de Laguna (1876 – 1930)
Stanisław Leśniewski (1886 – 1939)
Jean George Pierre Nicod (1893 – 1924)
Leonard Mascot Blumenthal (1901 – 1984)
Alfred Tarski (1901 – 1983)
Karl Menger (1902 – 1985)
John von Neumann (1903 – 1957)
Henry S. Leonard (1905 – 1967)
Nelson Goodman (1906 – 1998)

Two Quotations

Mathematics is a part of physics. Physics is an experimental science, a part of natural science. Mathematics is the part of physics where experiments are ***cheap***.

-- V.I. Arnol’d, in a lecture, Paris, March 1997

I remember once when I tried to add a little seasoning to a review, but I wasn't allowed to. The paper was by ***Dorothy Maharam***, and it was a perfectly sound contribution to abstract measure theory. The domains of the underlying measures were not sets but elements of more general Boolean algebras, and their range consisted not of positive numbers but of certain abstract equivalence classes. My proposed first sentence was:

“The author discusses valueless measures in pointless spaces.”

-- Paul R. Halmos, in ***I want to be a Mathematician***

An Evolution in Thinking

- Older literature emphasized the philosophical areas of:

Metaphysics/ Ontology/ Epistemology/
Logical Foundations

- Newer studies relate to:

Approximate Reasoning/ Artificial Intelligence/
Fuzzy Logic/ Spacial Reasoning/
Practical Geometry/ Computer Graphics

- And the number of publications is expanding rapidly!

Tarski's Regular-Open-Set Geometry

Alfred Tarski, *Les fondements de la géométrie des corps,* ***Annales de la Société Polonaise de Mathématique****,* Kraków 1929, pp. 29–33.

Definition. An open subset of a topological space is said to be ***regular*** iff it is equal to the interior of its closure.

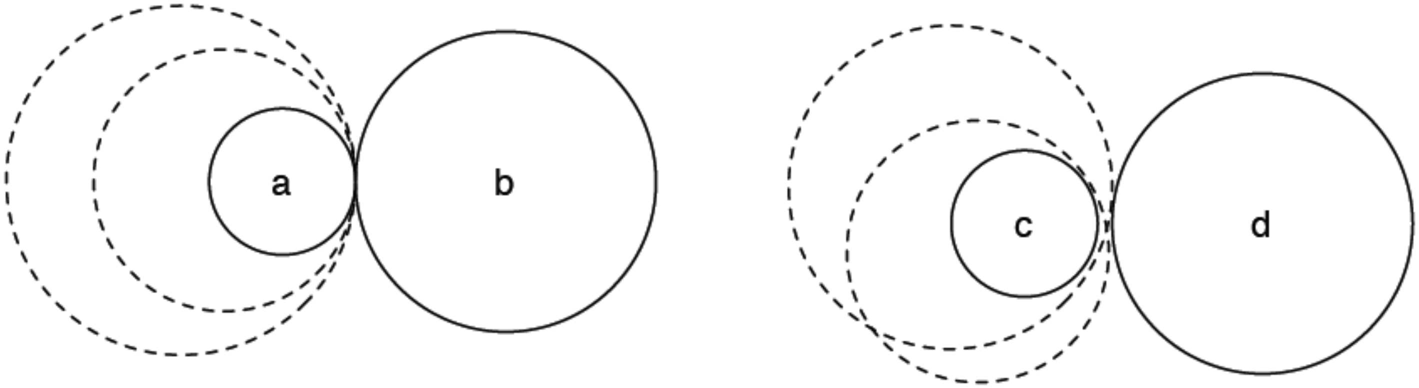
Theorem. As a lattice, the regular open sets of a topological space form a ***complete Boolean algebra****.* Without minimal opens, the algebra is ***atomless****.*

Theorem (Tarski). With the ***addition*** of the primitive notion of being a ***sphere****,* the theory of the regular-open algebra of solids of n-dimensional Euclidean space provides structure equivalent to standard geometry.

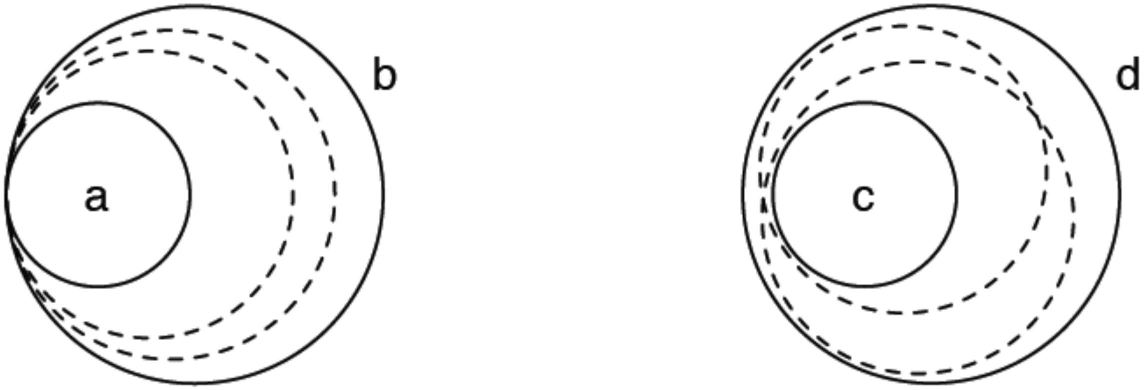
Proof Hint: After defining ***concentric spheres****,* points can be identified with equivalence classes of concentric spheres, and ***equidistance*** can be defined by arrangements of spheres.

Drawback: There is no strictly positive finitely additive measure on the regular-open algebra.

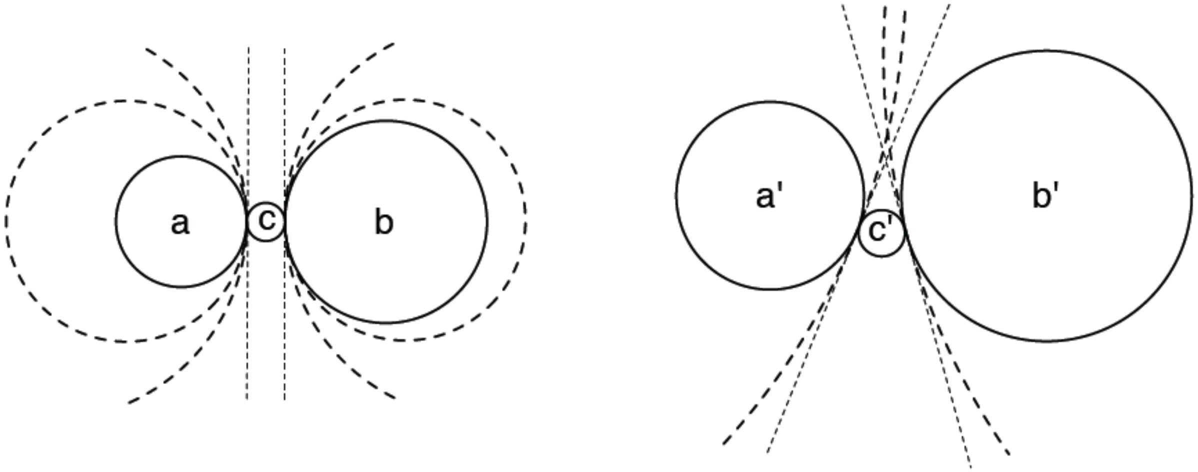
Tarski's Geometry of Disks I



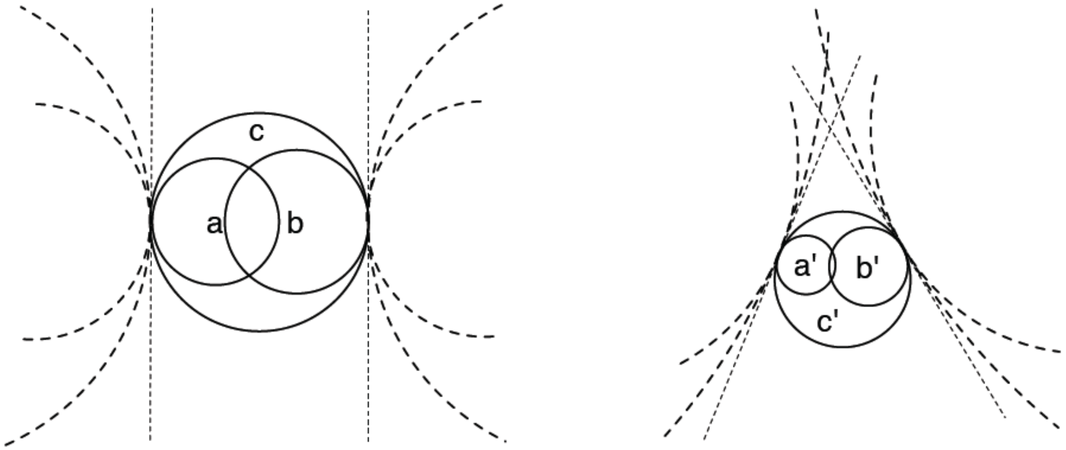
Defining External and Internal Tangency



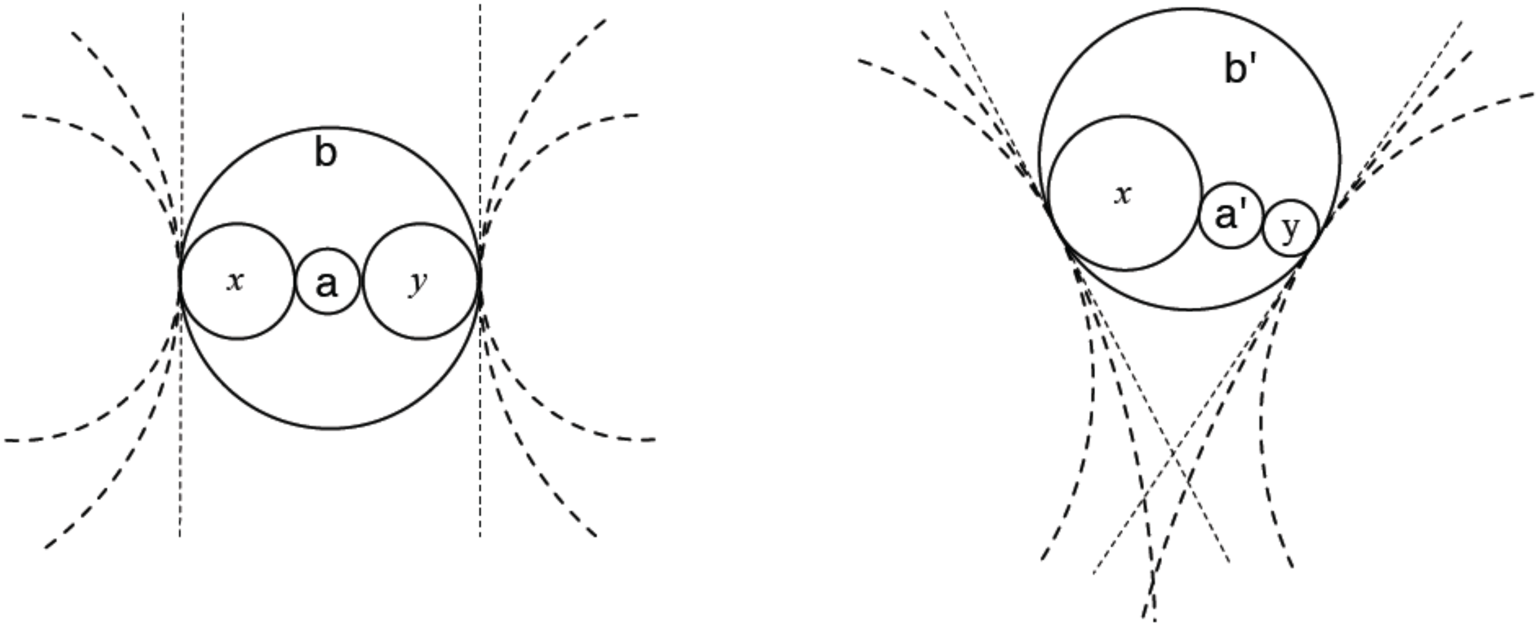
Tarski's Geometry of Disks II



Defining Being Diametrically Opposite



Tarski's Geometry of Disks III



Defining Being Concentric

These figures are from: Stefano Borgo, *Spheres, Cubes and Simplexes in Mereogeometry,* **Logic and Logical Philosophy**, vol. 22 (2013), pp. 255-293.

Gunky-Junky-Hunky Worlds

Definition. A world of solids is called ***gunky*** iff all non-zero solids ***have*** a proper part.

Definition. A world of solids is called ***junky*** iff every solid ***is*** a proper part.

Definition. A world of solids is called ***hunky*** iff it is ***both*** gunky and junky.

Note: Tarski's world is gunky but not junky.

The problem is to define a world of solids/regions which has sufficiently interesting structure but avoids pathological objects with irregular shapes.

Solids as Hunks

Definition. An open region of Euclidean space is a ***hunk*** iff
(a) it is ***regular***,
(b) its closure is ***bounded***, and
(c) it and its closure have the ***same Lebesgue measure***.

Theorem. The hunks of an n-dimensional Euclidean space form an ***atomless*** Boolean ring, $\mathbf{H}_n$, ***without*** a unit element, and carrying a ***finitely additive, finite Lebesgue measure***.

Note: **The ring of hunks can be thought of as an uncountable Boolean subring of the complete Boolean algebra of measurable sets modulo the ideal of sets of measure zero.**

A problem remains, however, of eliminating some unnatural infinite combinations.

Hunky Geometry

Proposition. The ring $\mathbf{H}_n$ of hunks of n-dimensional space is invariant under the Euclidean group $\mathfrak{E}_n$ of ***rigid motions*** of the space.

Definition. Over $\mathbf{H}_n$, define the ***congruence relation*** $X,Y \cong X',Y'$ to mean that there is a rigid motion $\boldsymbol{\rho} \in \mathfrak{E}_n$ where we have $X' = \boldsymbol{\rho}(X)$ and $Y' = \boldsymbol{\rho}(Y)$.

Theorem. For any $\boldsymbol{\rho} \in \mathfrak{E}_n$, there are $A,A' \in \mathbf{H}_n$ where for all X,X' we have

$$X' = \boldsymbol{\rho}(X) \text{ iff } X,A \cong X',A'.$$

Corollary. There is a one-many correspondence between the rigid motions in $\mathfrak{E}_n$ and pairs $A,A' \in \mathbf{H}_n$ such that

(for all X)(there is a unique X′) $X,A \cong X',A'$ and

(for all X′)(there is a unique X) $X,A \cong X',A'$.

Hope: The structure of the Boolean ring $\mathbf{H}_n$ together with the relation $\cong$ should give us enough to recapture geometric notions.

Some Group Properties

Proposition. The group €_n of ***rigid motions*** of n-dimensional space is generated by ***reflections*** $\boldsymbol{\rho} = \boldsymbol{\rho}^{-1}$. Every reflection $\boldsymbol{\rho}$ is uniquely determined by its ***axis***,which is the affine flat of its ***fixed points***. Every affine flat determines a unique reflection using ***orthogonal projection*** of points. Two reflections $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ ***commute***, $\boldsymbol{\rho\,\sigma} = \boldsymbol{\sigma\,\rho}$ iff their axes are ***orthogonal*** or one axis is ***contained in*** the other. Two reflections commute iff the ***product*** is again a reflection.

Reflector subgroups: A subgroup of €_n consisting only of reflections.

Facts: A reflector subgroup is ***commutative*** and has order at most 2^n.

Every reflector subgroup can be ***extended to*** one of order 2^n.

Maximal reflector subgroups are those generated by n-reflections about ***mutually orthogonal hyperplanes***.

Proposition. A maximal reflector subgroup has only ***two elements*** invariant under all inner automorphisms of €_n leaving the the subgroup invariant: the ***identity*** and the ***point reflection*** about the point intersection of the hyperplanes of the n-generators.

From Groups to Geometry

Friedrich Bachmann, *Aufbau der Geometrie aus dem Spiegelungsbegriff*, Springer-Verlag, **Grundlehren der mathematischen Wissenschaften**, vol. 96, 2nd ed. 1973, xvi + 396 pp.

Note: Bachmann, for absolute geometry, used the isometry group along with the subset of line reflections. Points were products of two orthogonal lines in plane geometry. In Euclidean geometry we may define points first.

Definition. In $\mathfrak{E}_n$ a ***point reflection*** is the unique non-identity reflection in some maximal reflector subgroup that is invariant under all inner automorphisms of $\mathfrak{E}_n$ leaving the subgroup invariant.

Definition. In $\mathfrak{E}_n$, given two distinct point reflections $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$, the ***line reflection*** $\boldsymbol{\lambda}$ about the line joining the points is the non-identity reflection invariant under all inner automorphisms leaving $\boldsymbol{\pi}$ and $\boldsymbol{\tau}$ fixed.

Note: In the structure H_n using the relation $\cong$ these definitions can be written out in first-order logic. But simpler definitions are also possible.

An Alternative Point Definition

Theorem. In $\mathfrak{E}_n$ a ***point reflection*** π is a non-identity reflection which ***does not*** commute with any distinct conjugate $\rho \pi \rho^{-1}$ for $\rho \in \mathfrak{E}_n$.

Note: This has the advantage of not depending on the dimension.

Proof. (1) If π is a point reflection, then so is $\tau = \rho \pi \rho^{-1}$. If π and τ are distinct, then $\pi \tau$ is a non-identity ***translation***, while $\tau \pi$ is the distinct ***inverse***.

(2) If π is neither the identity nor a point reflection, then, by a suitable choice of $\rho \in \mathfrak{E}_n$, the flat of π can be moved so that the flat of $\tau = \rho \pi \rho^{-1}$ is orthogonal. But then it will be the case that $\pi \tau = \tau \pi$.

Note: Given three distinct point reflections π_0 π_1 π_2, the three points are collinear iff every inner automorphism fixing two of the points also fixes the third.

From Points to Spheres

Program: For the structure (H_n , ≅), points are transformations not hunks. We have not yet said what it should mean for a point to **belong to** a hunk. To do this we have to determine which hunks are **spheres**, and then when a point lies at the **center** of a sphere.

Definition. For a point reflection $\boldsymbol{\pi}$, let $\text{€}_n[\boldsymbol{\pi}]$ be the subgroup of all those $\boldsymbol{\rho} \in \text{€}_n$ ***commuting*** with $\boldsymbol{\pi}$.

Note: ***This subgroup contains all the rotations about the point of*** $\boldsymbol{\pi}$***.***

Theorem. Given a non-zero hunk $X \in \mathbf{H}_n$ and a point reflection $\boldsymbol{\pi}$, there is a ***fusion*** (least upper bound) called $\text{€}_n[\boldsymbol{\pi}](X) \in \mathbf{H}_n$ of all the images $\boldsymbol{\rho}(X)$ over all the $\boldsymbol{\rho} \in \text{€}_n[\boldsymbol{\pi}]$.

Problem: ***This fusion is not a sphere but a possibly infinite union of solid spherical shells centered around the point of*** $\boldsymbol{\pi}$***. We have to rotate around a different point to get finally a solid spherical fusion.***

Finding Points of Tangency

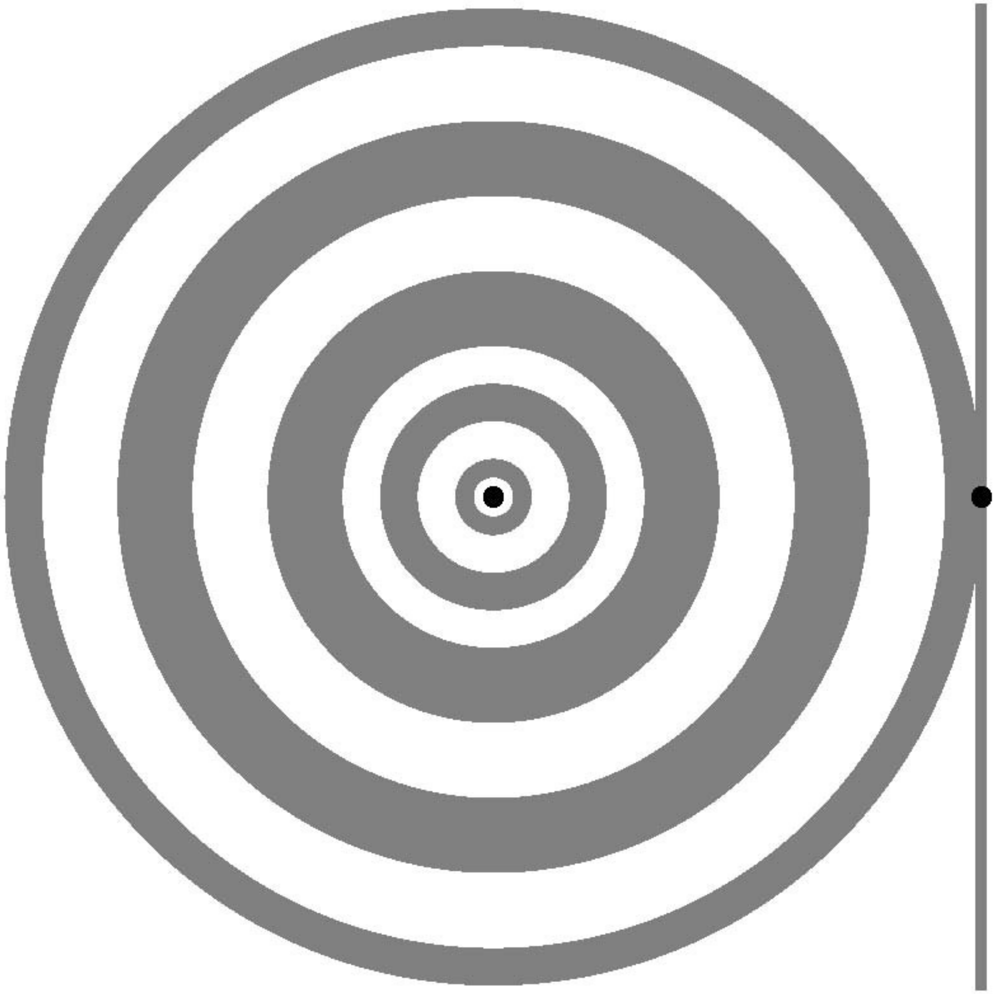
Note: ***A point π belongs to the flat of a reflection ρ just in case π commutes with ρ. And reflections about hyperplanes can be characterized as those non-identity reflections in maximal reflector subgroups with the largest flats.***

Definition. For a non-zero hunk $X \in \mathbf{H}_n$ and a point reflection $\boldsymbol{\pi}$, another point $\boldsymbol{\tau}$ is said to be ***tangent*** to the fusion $€_n[\boldsymbol{\pi}](X)$ iff there is a ***unique*** hyperplane reflection $\boldsymbol{\sigma}$ commuting with $\boldsymbol{\tau}$ such that $€_n[\boldsymbol{\pi}](X)$ and $\boldsymbol{\sigma}(€_n[\boldsymbol{\pi}](X))$ are ***disjoint***.

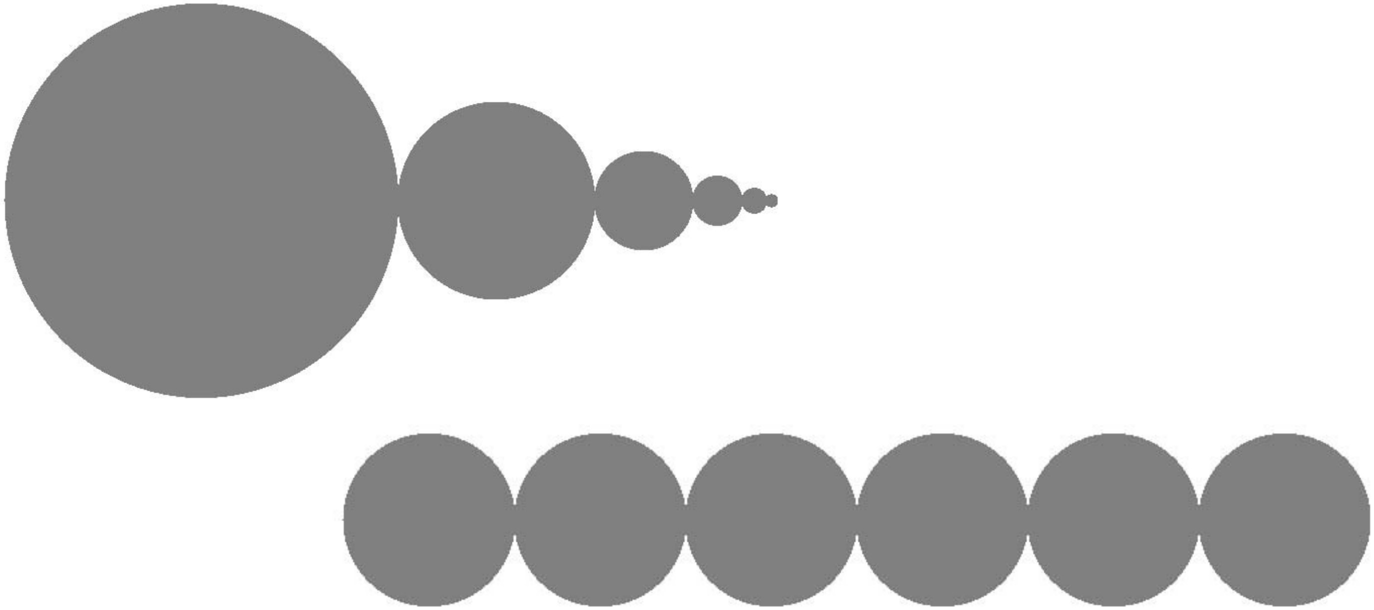
Note: ***The existence of tangency points can be confirmed by taking a line through π and then finding — in view of compactness — a distant point τ on the line with a hyperplane orthogonal to the line at τ so that the corresponding reflection σ makes*** $€_n[\boldsymbol{\pi}](X)$ ***and*** $\sigma(€_n[\boldsymbol{\pi}](X))$ ***disjoint***. ***Then the closest point between π and τ with such a hyperplane is the desired point of tangency.***

Definition. A ***sphere*** around a point $\boldsymbol{\tau}$ is a fusion $€_n[\boldsymbol{\tau}](€_n[\boldsymbol{\pi}](X))$ formed by a non-zero hunk X and a distinct point $\boldsymbol{\pi}$ where $\boldsymbol{\tau}$ is a point of tangency to $€_n[\boldsymbol{\pi}](X)$.

Illustrating Sphere Formation



Strings of Spheres



Theorem. The first-order theory of the structure (H_n , ≅) is as strong as second-order arithmetic. The first-order theory of the structure (FH_n , ≅) of finitary hunks is as strong as first-order arithmetic.

Some Questions for the Future

- Should we allow ***random*** hunks and ***random*** motions?
- Is there an interesting ***axiomatic*** version of the theory of (H_n , ≅)?
- Is the ***Boolean difference*** really needed?
- Should we add the relation $|X| = |Y|$ of having the ***same measure***?
- Should we restrict attention to using ***finitary hunks***?
- Is there a good way of considering ***approximate congruence***?
- Is there a good way of considering ***approximate measure***?
- Can we use relationships between hunks of ***different dimensions***?
- Perhaps we even need ***fractional dimensions***?