A new resolution calculus for the infinite-valued propositional logic of Łukasiewicz

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1 Introduction and Notation

Motivated by McNaughton's theorem ([7]) we define in the following a resolution calculus for the infinite-valued propositional logic of Łukasiewicz (short IPL) in a new and compared to existing resolution calculi very different way. Our approach is based on McNaughton's characterization of the truth value functions in Łukasiewicz's infinite-valued propositional logic. Grounded on this characterization we introduce the concept of a McNaughton normal form, which, however, is semantically defined in contrast to other normal forms. This McNaughton normal form will form the basis for our resolution calculus. In classical resolution calculi the basic constituents of a clause are literals (or weighted literals). In our approach literals are replaced by linear expressions. Concerning (weak) satisfiability these linear expressions define linear inequalities, which represent cutting planes. Our resolution rule is then based on well-known transformation rules for such linear inequalities. Therefore our resolution calculus can be regarded as a natural generalization of the cutting plane method for the classical 2-valued propositional calculus ([1]) to infinite-valued propositional logic.

Let $V = \{X_i \mid i \in \mathbb{N}\}$ be an infinite set of propositional variables. The set of formulas is defined inductively: each propositional variable is a formula, $\mathbf{0}$ and $\mathbf{1}$ are formulas, and whenever A and B are formulas, then so are $\neg A$, $(A \oplus B)$, $(A \wedge B)$, $(A \vee B)$. For a formula A we will also write $A(X_1, \ldots, X_n)$, indicating that at most the propositional variables $X_1, \ldots, X_n$ occur in A. It is well known that $(A \vee B)$ and $(A \wedge B)$ could have been defined by $(A \vee B) := \neg(\neg A \oplus B) \oplus B$ and $(A \wedge B) := \neg(\neg A \vee \neg B)$. $(A \to B)$ and $(A \leftrightarrow B)$ are introduced as abbreviations: $(A \to B) := (\neg A \oplus B)$ and $(A \leftrightarrow B) := (A \to B) \wedge (B \to A)$.

We adopt the usual conventions for omitting parentheses. So for instance we also write $A \oplus B$ instead of $(A \oplus B)$ and $A_1 \wedge \ldots \wedge A_n$ instead of $(\ldots((A_1 \wedge A_2) \wedge A_3) \wedge \ldots \wedge A_n)$.

For defining the semantics we use the interval $[0,1]$ (this is the set of all reals r with $0 \leq r \leq 1$) as the set of logical values. The semantics of our logical language is based on the concept of interpretation. An *interpretation* I is a mapping $I : V \to [0,1]$. We usually denote an interpretation I with $I(X_k) = x_k$ by the ω-sequence $(x_0, x_1, \ldots)$. Let $\mathfrak{I}$ denote the set of all interpretations. The *value* of a propositional variable X_k under an interpretation $I = (x_0, x_1, \ldots)$ is x_k, the values of $\mathbf{0}$ and $\mathbf{1}$ are always 0 and 1 respectively, and if the values of the formulas A and B under interpretation I are respectively a and b, then under interpretation I $\neg A$ has the value $1 - a$, $(A \oplus B)$ has the value $\min(1, a+b)$, $(A \vee B)$ has the value $\max(a,b)$ and $(A \wedge B)$ has the value $\min(a,b)$. We denote the value of a formula A under an interpretation I by $\mathrm{val}(A, I)$. By this definition it is obvious that to each formula $A = A(X_1, \ldots, X_n)$ there exists a unique function $f_A(x_1, \ldots, x_n)$ in the variables $x_1,\dots,x_n$ such that the value of A under the interpretation $I = (y_0, y_1, \dots)$ equals $f_A(y_1, \dots, y_n)$.

A formula A is *satisfiable* if and only if there is an interpretation I such that A has the value 1 under this interpretation. A is *weak satisfiable* if and only if there is an interpretation I such that A has a value > 0 under I. A is *valid* if and only if $\neg A$ is not weak satisfiable, i.e. if and only if for all interpretations I A has the value 1. Finally two formulas A and B are equivalent if and only if $(A \leftrightarrow B)$ is valid.

2 McNaughton Normal Form

Let us call a function $f : [0,1]^n \to [0,1]$ in n variables $x_1,\dots,x_n$ definable if and only if there is a formula A of IPL such that $f = f_A$. In [7] McNaughton proved the following deep result concerning definability.

Theorem 1 *(McNaughton's characterization theorem)*
A function $f = f(x_1, \dots, x_n) : [0,1]^n \to [0,1]$ is definable if and only if

1. *f is a continuous function.*
2. *There exists a finite number of distinct linear functions $p_1,\dots,p_k$ of the form*

$$p_j = m_1 x_1 + \ldots + m_n x_n - m$$

where $m_1, \dots, m_n$ and m are integers such that for all $x_1,\dots,x_n \in [0,1]$ there is a j, $1 \le j \le k$, such that

$$f(x_1, \dots, x_n) = p_j(x_1, \dots, x_n).$$

While the only-if direction is very easy to prove, McNaughton proved the other direction using a non-constructive argument. Only recently Mundici ([9]) gave a constructive proof of this direction.

McNaughton's characterization will form the basis for a normal form, which we will introduce now. By McNaughton's theorem, if $p = m_1 x_1 + \ldots + m_n x_n - m$ is a linear function in $x_1,\dots,x_n$ and $m_1,\dots,m_n$ and m are integers (henceforth all linear functions are assumed to have such a form), then the function f_p defined by

$$f_p(x_1, \dots, x_n) = \min(\max(0, p), 1)$$

is definable by a formula A_p of IPL. This formula A_p is called *associated to p*.

Definition 2 *A formula A is in* McNaughton normal form *if and only if it is of the form $A = A_1 \wedge \ldots \wedge A_k$, where each $A_i = A_{i1} \vee \ldots \vee A_{in_i}$ and each A_{ij} is associated to a linear function ($1 \le i \le k$, $1 \le j \le n_i$).*

A similar proof as in the easy part of McNaughton's theorem shows that every formula can be transformed into an equivalent McNaughton normal form.

Theorem 3 *For every formula of IPL there is an equivalent formula A in McNaughton normal form.*

3 A resolution calculus

The McNaughton normal form, we introduced in the last section, will be the basis for the following resolution calculus. Formulas A, which are associated to linear functions g_A, correspond with respect to weak satisfiability to cutting planes $g_A > 0$. In this way we can represent a McNaughton normal form by a finite set of clauses, where each clause corresponds to a finite set of such cutting planes. So our resolution calculus is not formulated in a pure logical language. Rather it is a generalization of the cutting plane proof method in classical 2-valued logic obtained by using a modified resolution principle.

We begin by defining the clauses and formulas of our resolution calculus.

3.1 Syntax

We have an infinite set $V' = \{x_i \mid i \in \mathbb{N}\}$ of variables and for each integer $\mathbf{a} \in \mathbb{Z}$ a constant a. Further we have the following symbols: $+, \cdot, (,), \{,\}$. Sometimes we also use the symbols x, y, z possibly with indices for variables.

Definition 4 *1. If a is a constant and x_i is a variable, then a and $a \cdot x_i$ are* linear expressions *with length* $\mathbf{a}$ *and* $1 + |\mathbf{a}| + i \in \mathbb{N}$ *respectively. If E and F are linear expressions with length $\|E\|$ and $\|F\|$ and a is a constant, then $a \cdot E$ and $(E + F)$ are linear expressions with length $|\mathbf{a}| + \|E\|$ and $\|E\| + \|F\|$ respectively.*

2. *A* clause *is a finite set of linear expressions. The* length of a clause *C (short: $\|C\|$) is the sum of the length of the linear expressions occuring in C.*

3. *A* formula *is a finite set of clauses. The* length of a formula *M (short: $\|M\|$) is the sum of the length of the clauses occuring in M.*

Using that $+$ and $\cdot$ are associative and assuming that binding by $\cdot$ is stronger than binding by $+$ we can omit parentheses in linear expressions. Further we write for instance $3 \cdot x_1 - 2 \cdot x_2$ instead of $3 \cdot x_1 + (-2) \cdot x_2$. Also a factor 1 in a linear expression is usually omitted. A constant a is said to be *positive* (*negative*, *nonpositive*) if and only if the integer $\mathbf{a}$ is positive (negative, nonpositive). A variable x_i is *positive (negative) in a linear expression* $a_1 \cdot x_1 + \ldots + a_n \cdot x_n + a$ if a_i is positive (negative) constant.

Definition 5 *A linear expression E is n-*positiv *if and only if there are n positive variables in E. A* Horn clause *is a clause which contains at most one 1-positive linear expression and no n-positive linear expression for any $n > 1$.*

A clause is a Krom clause *if and only if it contains at most two linear expressions.*

A formula is a Horn formula (Krom formula) *if each clause is a Horn clause (Krom clause).*

3.2 Semantic

For defining a satisfiability notion we use the structure

$$R = (\mathbb{R}, +, \cdot, \{\mathbf{a} \in \mathbb{R} \mid \mathbf{a} \text{ integer}\}).$$

(For binary addition and multiplication we use the same symbols as in linear expressions. In each case the meaning should be clear from the context). An interpretation is a mapping $I : V' \to \mathbb{R}$. As for IPL we denote an interpretation I with $I(x_k) = y_k$ by the ω-sequence

$(y_0, y_1, \dots)$. The value $\mathrm{val}(E, I)$ of a linear expression E under an interpretation I is then defined by

- $\mathrm{val}(a, I) := \mathbf{a}$
- $\mathrm{val}(x_i, I) := I(x_i)$
- $\mathrm{val}(a \cdot E, I) := \mathbf{a} \cdot \mathrm{val}(E, I)$ and
- $\mathrm{val}((E + F), I) := \mathrm{val}(E, I) + \mathrm{val}(F, I)$.

Definition 6 *An interpretation I* weakly satisfies *(*satisfies*) a clause C if and only if there is a linear expression E in C such that* $\mathrm{val}(E, I) > 0$ *(*$\mathrm{val}(E, I) \geq 1$*).*

A clause C is weak satisfiable *(*satisfiable*) if and only if there is an interpretation I weakly satisfying (satisfying) C.*

An interpretation I weakly satisfies *(*satisfies*) a formula M if and only if I weakly satisfies (satisfies) each clause of M.*

A formula M is weak satisfiable *(*satisfiable*) if and only if there is an interpretation I weakly satisfying (satisfying) M.*

To any formula $A \in IPL$ in McNaughton normal form we can assign a formula M_A of our resolution calculus in the following way. To each formula $B \in IPL$, which is associated to a linear function, there is a linear expression E_B, such that for $(y_0, y_1, \dots)$ with $y_i \in [0, 1]$ and $i \in \mathbb{N}$

$$\mathrm{val}(B, (y_0, y_1, \dots)) = \max(0, \min(1, \mathrm{val}(E_B, (y_0, y_1, \dots)))).$$

Assume $A = A_1 \wedge \dots \wedge A_n$, where each A_i is a disjunction. To each $A_i = A_{i1} \vee \dots \vee A_{in_i}$ we assign the clause $C_{A_i} = \left\{E_{A_{i1}}, \dots, E_{A_{in_i}}\right\}$. To A the formula $M_A = \{C_{A_1}, \dots, C_{A_n}\}$ is assigned. Let $x_1, \dots, x_k$ be all the variables occuring in any of the clauses C_{A_i}, $1 \leq i \leq n$. We set $X_A := \{\{x_i\}, \{1 - x_i\} \mid 1 \leq i \leq k\}$ and $X_A^s := \{\{1 + x_i\}, \{2 - x_i\} \mid 1 \leq i \leq k\}$. Weak satisfiability of X_A means that the variables x_i $1 \leq i \leq k$, have values > 0 and < 1. Similarly satisfiability of X_A^s expresses that the variables x_i have values ≥ 0 and ≤ 1.

Lemma 7 *A formula $A \in IPL$ in McNaughton normal form is weak satisfiable if and only if $M_A \cup X_A$ is weak satisfiable.*

A formula $A \in IPL$ in McNaughton normal form is satisfiable if and only if $M_A \cup X_A^s$ is satisfiable.

3.3 A resolution rule

Using similar rules as in the cutting plane method we define a resolution rule in such a way that the resulting resolution calculus can be seen as a generalization of the cutting plane method and in some sense of the resolution calculus for 2-valued classical logic.

Definition 8 *Let C, C_1, C_2 be clauses. C is a* resolvent *of C_1, C_2 (denoted by $C = R(C_1, C_2)$) if and only if there are linear expressions $E_1 \in C_1$, $E_2 \in C_2$ and positive constants a and b such that $C = (C_1 \setminus \{E_1\}) \cup (C_2 \setminus \{E_2\}) \cup \{a \cdot E_1 + b \cdot E_2\}$.*

Remark 1 *For shortness of the exposition we have chosen a very liberal form for the definition of a resolvent. In fact if one is interested in an implementation one can formulate very definite restrictions on the choice of a and b. Later on we shall give an example which will clarify this point.*

Definition 9 *A clause C is obtained from a clause C_1 by* simplification *if and only if there is a linear expression $E_1 \in C_1$, such that either*

1. *$C = C_1 \setminus \{E_1\} \cup \{E\}$, where E is obtained from E_1 by finitely many applications of the following rules for simplification:*

   (a) *$+$ is commutative and associative.*

   (b) *$\cdot$ is commutative, associative and distributive over $+$.*

   (c) *$0 \cdot E' = 0$ for any linear expression E'.*

   (d) *Common expressions may be combined (i.e. $a \cdot E + b \cdot E$ may be replaced by $(a+b) \cdot E$).*

   (e) *Sums and products of constants may be "evaluated" (i.e. if $\mathbf{a}+\mathbf{b} = \mathbf{n}$ then $a+b$ may be replaced by n, similarly for $a \cdot b$).*

   or

2. *$E_1 = n$ for a nonpositive integer $\mathbf{n}$ and $C = C_1 \setminus \{E_1\}$.*

Definition 10 *Let M be a set of clauses.*

A resolution deduction *of C from M is a finite sequence $C_1, \ldots, C_n = C$, such that each C_i is either a member of M or is a simplification of a clause C_j for $1 \leq j < i$ or is a resolvent of clauses C_j, C_k for $1 \leq j, k < i$.*

If there is such a resolution deduction of C from M, we say that C is resolution provable *from M and write $M \vdash_R C$.*

A resolution deduction of the empty clause $\perp$ from M is called a resolution refutation *of M. In this case we say that M is resolution refutable.*

$R(M)$ is the closure *of M under resolution deduction, i.e. $R(M)$ is the set of all clauses C, such that there is a resolution deduction of C from M.*

Example 11 *Consider the following clauses:*

$$\begin{aligned} A_1 &= \{3 \cdot x + 2 \cdot y - 3, 3 \cdot z - 1\} \\ A_2 &= \{-2 \cdot y + 1\} \\ A_3 &= \{-3 \cdot x + 1\} \\ A_4 &= \{-6 \cdot z + 1\} \ . \end{aligned}$$

A resolvent of A_1 and A_2 is $\{3 \cdot x + 2 \cdot y - 3 - 2 \cdot y + 1, 3 \cdot z - 1\}$, which can be simplified to $A_5 = \{3 \cdot x - 2, 3 \cdot z - 1\}$. Similarly we can deduce by resolution and simplifiaction from A_5 and A_3 the clause $A_6 = \{-1, 3 \cdot z - 1\}$, which further can be simplified to $A_7 = \{3 \cdot z - 1\}$.

$A_8 = \{2 \cdot (3 \cdot z - 1) + (-6 \cdot z + 1)\}$ is a resolvent of A_7 and A_4. Finally A_8 can be simplified to $\{-1\}$, which then can be simplified to $\perp$.

Lemma 12

1. *If C_1 is a satisfiable clause and C is a simplification of C_1, then C is satisfiable.*

2. *If C_1, C_2 are satisfiable clauses and $C = R(C_1, C_2)$, then C is a satisfiable clause.*

Theorem 13 *(Soundness of resolution) If there is a resolution refutation of a formula M, then M is unsatisfiable.*

Proof: If $C_1, \ldots, C_n = \perp$ is a resolution refutation from M, then, using lemma 12, any interpretation I satisfying M satisfies each C_i, $1 \leq i \leq n$. Especially I satisfies $\perp$. As $\perp$ is not satisfiable, M cannot be satisfiable.■

For proving completeness of our resolution calculus we define the degree of a formula.

Definition 14 *The* degree $\deg(C)$ of a clause C *is the number of linear expressions in C. The* degree $\deg(M)$ of a formula M *is defined by* $\deg(M) = \prod_{C \in M} \deg(C)$.

Lemma 15 *A finite set*

$$\{a_i \cdot x + c_i > 0 \mid k, a_i \in \mathbb{N},\ c_i \in \mathbb{Z},\ 1 \leq i \leq k\}$$
$$\cup \{-b_j \cdot x + d_j > 0 \mid m, b_j \in \mathbb{N}, d_j \in \mathbb{Z},\ 1 \leq j \leq m\}$$

of strict linear inequalities in one variable x has a real valued solution if and only if

$$b_j \cdot c_i + a_i \cdot d_j > 0$$

for all i and j with $1 \leq i \leq k$, $1 \leq j \leq m$.

Theorem 16 *(Completeness of resolution)*
If M is an unsatisfiable set of clauses, then $\perp \in R(M)$.

Proof: By induction on $\deg(M)$ and within this induction by induction on the number n of different variables of M. We omit some trivial cases.
$\deg(M) = 1$ and induction step $n \to n+1$:

Let M be a formula with $n+1$ different variables occuring in it. By the simplification rules we can assume, that any linear expression occuring in some clause of M has the form

$$a_1 \cdot x_1 + \ldots + a_k \cdot x_k$$

with $x_i \neq x_j$ for $i \neq j$. Furthermore not all of the variables are positive resp. negative in M. So there is a variable w.l.o.g x_1 with positive and negative occurence.

Assume that

$$\{a_1 \cdot x_1 + E_1\}$$
$$\vdots$$
$$\{a_k \cdot x_1 + E_k\}$$

$(k > 0)$ are all the clauses with positive occurence of x_1 and that

$$\{-b_1 \cdot x_1 + E'_1\}$$
$$\vdots$$
$$\{-b_m \cdot x_1 + E'_m\}$$

$(m > 0)$ are all the clauses with negative occurence of x_1.

Let M' be the set of the remaining clauses of M.

$M'' := M' \cup \left\{ \left\{ b_j \cdot E_i + a_i \cdot E'_j \right\} \mid 1 \leq i \leq k \; 1 \leq j \leq m \right\}$.
We then have

$$M \text{ is unsatisfiable if and only if } M'' \text{ is unsatisfiable.} \tag{1}$$

Proof of (1):
"Only if" is trivial.
For the "if" case assume that I satisfies M''.

We have to find an interpretation I' which satisfies M' and all the clauses $\{a_i \cdot x_1 + E_i\}$, $\left\{-b_j \cdot x_1 + E'_j\right\}$ for $1 \leq i \leq k$, $1 \leq j \leq m$. By the preceeding lemma the set

$$\left\{ \mathbf{a}_i \cdot x + I(E_i) > 0 \; , \; -\mathbf{b}_j \cdot x + I\left(E'_j\right) > 0 \mid 1 \leq i \leq k, 1 \leq j \leq m \right\} \tag{2}$$

of strict inequalities in one variable x has a solution if and only if

$$\left\{ \mathbf{b}_j \cdot I(E_i) + \mathbf{a}_i \cdot I\left(E'_j\right) > 0 \mid 1 \leq i \leq k, 1 \leq j \leq m \right\}. \tag{3}$$

By assumption I satisfies M'', so $\mathbf{b}_j \cdot I(E_i) + \mathbf{a}_i \cdot I\left(E'_j\right) > 0$ is true for all i and j with $1 \leq i \leq k$, $1 \leq j \leq m$. Therefore (2) has a solution $x = c$. Let I' be the following interpretation

$$I'(x_i) = \begin{cases} c & \text{if } i = 1 \\ I(x_i) & \text{else }. \end{cases}$$

So I' satisfies M' and also all the clauses $\{a_i \cdot x_1 + E_i\}$, $\left\{-b_j \cdot x_1 + E'_j\right\}$ for $1 \leq i \leq k$, $1 \leq j \leq m$. Hence I' satisfies M. This proves (1).

Continuing the case $\deg(M) = 1$:

As the number of different variables of M'' is n, we have by induction hypotheses that $\perp \in R(M'')$. By computing

$$\left\{ b_j \cdot a_i \cdot x_1 + b_j \cdot E_i - a_i \cdot b_j \cdot x_1 + a_i \cdot E'_j \right\}$$

as a resolvent of $\{a_i \cdot x_1 + E_i\}$ and $\left\{-b_j \cdot x_1 + E'_j\right\}$ and some application of the simplification rules, we get, that $\left\{ b_j \cdot E_i + a_i \cdot E'_j \right\} \in R(M)$ for all i and j with $1 \leq i \leq k$, $1 \leq j \leq m$. So $M'' \subseteq R(M)$. Consequently $\perp \in R(M)$.
Therefore the case $\deg(M) = 1$ is proved.

$\deg(M) > 1$:

There exists a clause C in M with $\deg(C) > 1$. Further let $M' := M \setminus \{C\}$ and $C' := C \setminus \{E\}$ for some linear expression $E \in C$. As M is unsatisfiable, both $M' \cup \{C'\}$ and $M' \cup \{\{E\}\}$ are unsatisfiable. The degrees of $M' \cup \{C'\}$ and $M' \cup \{\{E\}\}$ are less $\deg(M)$. So by induction hypotheses $\perp \in R(M' \cup \{C'\})$. From this we can conclude that $\perp \in R(M' \cup C) = R(M)$ or $\{E\} \in R(M)$. For $\{E\} \in R(M)$ we have $M' \cup \{\{E\}\} \subset R(M)$, as $M' \subset M$. Consequently $R(M' \cup \{\{E\}\}) \subset R(M)$. As $M' \cup \{\{E\}\}$ is unsatisfiable and the degree of $M' \cup \{\{E\}\}$ is less $\deg(M)$, we get by induction hypotheses $\perp \in R(M' \cup \{\{E\}\})$, hence $\perp \in R(M)$.■

Remark 2 *It is a consequence of our completeness proof, that the proof complexity of this resolution calculus, i.e. the number of proof steps of a shortest proof, measured in the length of the formula, is* $O\left(2^{p(n)}\right)$ *for some polynomial* p. *For a lower bound: there are already clauses containing only one linear expression which need* $O\left(2^n\right)$ *proof steps. So our resolution calculus is by no means efficient.*

4 Related decision problems

In the following we will consider satisfiability problems for some classes of formulas of our resolution calculus RC_{IPL}. These are given by the decision problem for the corresponding sets. Here we are mainly interested in the complexity of decision algorithms (not in the proof complexity) for formulas, which correspond to formulas of IPL. Therefore we are considering only such formulas, in which every linear expression has the form $a_1 \cdot x_1 + \ldots + a_n \cdot x_n + a$ with $a, a_1, \ldots, a_n$ constants and $x_i \neq x_j$ for $1 \leq i < j \leq n$. Linear expressions of this form are called *simple linear expressions.* We denote the set of formulas in our resolution calculus RC_{IPL} consisting only of such simple linear expressions by $SIPL$.

$SAT_{SIPL} := \{F \in SIPL \mid F \text{ is satisfiable}\}$,
$SAT_{Horn} := SAT_{SIPL} \cap \{F \in SIPL \mid F \text{ Horn formula}\}$ and
$SAT_{Krom} := SAT_{IPL} \cap \{F \in IPL \mid F \text{ Krom formula}\}$.

Here we shall consider only these satisfiability problems. For the corresponding decision problems concerning weak satisfiability the same results can be obtained.

Theorem 17 SAT_{SIPL} *is* NP*-complete.*

Proof:
NP-hard: Obvious.
In NP: This follows immediately by a polynomial translation into the Σ_1-fragment of the theory $\mathrm{TH}(\mathbb{R}, +, <, =, 0, 1)$. By a result of von zur Gathen and Sieveking ([3]) this Σ_1-fragment is NP-complete.■

For classical 2-valued propositional calculus the decision problem for Krom formulas is known to be complete (with respect to logspace-reductions) for the complexity class $NLOGSPACE$. Without any further restriction the corresponding satisfiability problem SAT_{Krom} is NP-complete:

Theorem 18 SAT_{Krom} *is* NP*-complete. In fact we have the stronger result:*
The satisfiability problem for Krom clauses, where at most two different variables are occuring in a Krom clause, is NP*-complete.*

Proof: By the NP-completeness of the unrestricted satisfiability problem we only have to prove the NP-hardness.

In [6] the decision problem for $SLIA$ (simple linear inequalities), which is the following decision problem

Instance: A system Δ of m inequalities in n variables. Each inequality has the form $x_i - x_j \geq 0$, $x_i - x_j \geq -1$ or $|x_i - x_j| \geq 1$.

Question:Does the system Δ have a solution?

is shown to be NP-complete.

We assign clauses to the inequalities in the following way:

$x_i - x_j \geq 0 : \quad \{x_i - x_j + 1\}$

$x_i - x_j \geq -1 : \quad \{x_i - x_j + 2\}$

$|x_i - x_j| \geq 1 : \quad \{x_i - x_j, x_j - x_i\}$.

So there is an obvious polynomial time reduction of $SLIA$ to SAT_{Krom} with the property, that each clause contains at most two different variables. We therefore get the NP-hardness for this satisfiability problem.■

Remark 3 *The satisfiability problem for Krom formulas, where each clause contains at most one variable, is $NLOGSPACE$-complete.*

(In $NLOGSPACE$ follows by a logspace reduction to a satisfiability problem for Krom formulas in classical propositional logic. The hardness property is trivial.)

Unfortunately, we currently don't have so nice completeness results in the case of Horn formulas. Of course we have that SAT_{Horn} is in NP and that SAT_{Horn} is P-hard (with respect to logspace reductions), as the satisfiability problem for Horn formulas in classical 2-valued propositional logic is P-complete. Restricted classes of Horn formulas in IPL have been investigated by Escalada et al. ([2]) and – generalizing the result of [2] – by Hähnle ([5]).

Theorem 19 *The satisfiability problem for Horn clauses, in which every linear expression contains at most one variable, is P-complete*

Containment in P follows by [5]. P-hardness follows by the already mentioned logspace reduction of the satisfiability problem of Horn formulas in classical 2-valued logic to the considered special case of SAT_{Horn}.

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