Effective properties for some first order intuitionistic modal logics

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Abstract. The indexed sequent calculi are constructed for first order intuitionistic modal logics K, K4, T, S4 with the Barcan axiom as well as for KB, B, and S5, where the Barcan formula is derivable. Effective properties, namely, admissibility of the cut rule, Harrop properties, and the interpolation property for the calculi under consideration are proved using proof-theoretical methods. Basing on the constructed sequent calculi, computer-aided tableaux-like and resolution calculi can be obtained.
Keywords: intuitionistic modal logic, sequent calculus, cut–free calculus, index (prefixed) method, Harrop formula, Harrop theorem, interpolation property

1 Introduction

Modal logics based on logics weaker than the classical logic have been studied by various authors in different aspects. Most of these investigations concern modal logics based on an intuitionistic propositional logic and known as intuitionistic modal logics. Using model-theoretical methods, some results on the model theory for these logics as well as the relation between the intuitionistic and the classical modal logic have been obtained (see, e.g., [10], [14]). Some effective properties for intuitionistic modal logics have been obtained, too. Namely, the cut–free sequent calculi for the $S4$-type intuitionistic propositional logic have been constructed by H. Ono in [10]. There are a lot of studies concerning disjunction and existence properties for intermediate and super-intuitionistic predicate logics using semantic methods. More complex disjunction and existence properties (later on named Harrop disjunction and existence properties) have been investigated for the intuitionistic predicate calculus and fragments of arithmetic in [1] and for some intermediate predicate logics in [9]. Some methods used to prove the Harrop-type properties are based on the proof that the set of provable formulas of the calculus is closed under modus ponens (see [1]), others (see, e.g., [9]) are based on Kripke semantics. The interpolation theorem for $S4$-type and $S5$-type intuitionistic propositional modal logics (investigated in [10]) or some of their extensions has been proved by algebraic methods in [7]. Analogous positive results regarding admissibility of the cut rule, Harrop properties, and the interpolation property are not known for the first order intuitionistic modal logics with the Barcan axiom. The proof-theoretical investigations of first order classical modal logics can be found, for example, in [3]. To obtain a cut–free calculus for classical modal logic $S5$ the indexing has been introduced by S.Kanger in [5]. In [3] M.Fitting extended Kanger's ideas of indexing to construct analytic and semi–analytic tableaux (and called them the prefixed

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tableau systems) for various classical modal logics. The cut–free indexed (prefixed) calculi for first order classical modal logics K, T, $K4$, $S4$ with the Barcan axiom as well as for KB and B (where the Barcan formula is derivable) have been constructed in [11]. In [12] it has been shown that traditional formulations of the Harrop disjunction and existence properties (by the same token, the conventional definition of the Harrop formula) are not suitable to the classical modal logics. Therefore, the notions of extended disjunction and existence properties for predicate modal logics K, $K4$, T, $S4$ as well as these logics with the Barcan axiom have been introduced. Using the proof-theoretical approach, it has been proved that these logics had the properties introduced. It has been shown in [2] that the Craig Interpolation Theorem fails when the constant domain axiom scheme $\forall x \Box A(x) \equiv \Box \forall x A(x)$ is added to $S5$, or, indeed, to any weaker extension of quantified K.

In the paper three effective properties, namely, (a) admissibility of the cut rule, (b) Harrop-type properties, (c) the interpolation-type property for some first order intuitionistic modal logics with the Barcan axiom are investigated using proof-theoretical methods. These properties are very important in the development of theorem proving methods for classical and non-classical logics. The soundness and completeness of various resolution procedures (used as a basis for computer-aided proof systems) can be proved through explicit translations between resolution refutations and a cut–free Gentzen-type calculus (see, e.g., [8]). Basing on the constructed sequent calculi, computer-aided tableaux-like and resolution calculi can be obtained. Harrop-type disjunction and existence properties allow us to get some invertibility of the rules $(\to \vee)$ and $(\to \exists)$. On the other hand, these properties serve as a useful tool to obtain non-derivability results in intuitionistic modal logics. The interpolation-type property can be interpreted as invertibility of the cut rule. In the logics where the cut rule is admissible, the invertibility of the cut rule serves as a measure of efficiency of the logic under consideration.

Sequent cut–free calculi for intuitionistic modal logics K, $K4$, T, $S4$ with the Barcan axiom as well as for KB, B and $S5$, where the Barcan formula is derivable, are constructed here using the index technique from [11]. Relying on the constructed sequent calculi, the Harrop disjunction and existence properties are proved constructively. It is shown that in contrast to classical predicate modal logics, traditional formulation of the Harrop disjunction (existence) property for intuitionistic and intermediate logics is the same as the modal Harrop disjunction (existence) property for the intuitionistic modal logics considered. However, the usual constraints on the shape of the Harrop formulas can be weakened for non-reflexive intuitionistic modal logics K, K4, KB, i.e., an extension of the Harrop-type property is valid for the logics mentioned. An analogy of the interpolation theorem (Lemma 5.2) is proved for the intuitionistic modal logics under consideration. This lemma actually asserts some invertibility of the cut rule. Unlike non-constructive algebraic and model-theoretical methods, the constructive method, used to prove this invertibility, explicitly presents a way of constructing the cut formula. As it is known, the use of the cut rule in derivation can sharply reduce the length of derivation.

2 Gentzen-type calculi for intuitionistic modal logics considered

Let us consider the first order modal logics over K based on the intuitionistic logic, namely, K, T, $K4$, $S4$ with the Barcan axiom (i.e., the formula $\forall x \Box A(x) \supset \Box \forall x A(x)$) and the logics KB, B, $S5$, where the Barcan formula is derivable. The logics considered are in signature $\{\supset, \vee, \&, \forall, \exists, \Box\}$. Formulas are built up by means of logic connectives and modality $\Box$ starting from predicate variables, as usual. We don't use modal operator $\Diamond$ (possibility), though, unlike classical modal logics, $\Diamond A$ cannot be considered as the abbreviation of $\neg\Box\neg A$. The symbol $\neg$ is used for the abbreviation of $A \supset \bot$, where $\bot$ stands for the constant "false". Let HJ be a Hilbert-type calculus for the first order intuitionistic logic without equality (see, e.g., [6]). The intuitionistic modal logics considered are denoted by LJ, where $L \in \{K, T, K4, S4, KB, B, S5\}$, and defined by the postulates of HJ complemented with relevant axioms as follows:

A1. $\Box A \supset A$ (reflexivity);
A2. $A \supset \Box\neg\Box\neg A$ (Brouwerian Axiom or symmetry);
A3. $\Box(A \supset B) \supset (\Box A \supset \Box B)$;
A4. $\Box A \supset \Box\Box A$ (transitivity);
A5. $\forall x \Box A(x) \supset \Box\forall x A(x)$ (Barcan Axiom),

and the rule of inference **R**: $A/\Box A$ (rule of Necessitation).

Then HLJ is a Hilbert-type calculus containing the Barcan formula and corresponding to the first order intuitionistic modal logic LJ defined as follows:

KJ is $A3 + R$	KBJ is $KJ + A2$
TJ is $KJ + A1$	BJ is $KJ + A1 + A2$
$K4J$ is $KJ + A4$	$S5J$ is $KJ + A1 + A2 + A4$
$S4J$ is $KJ + A1 + A4$.	

Calculi HKJ, HTJ, $HK4J$, $HS4J$ under consideration have the Barcan Axiom explicitly. In calculi $HKBJ$, HBJ, $HS5J$ the Barcan Axiom is derivable.

Let us examine the Gentzen-type calculi for the logic under consideration. A sequent is an expression of the form $\Gamma \to \Delta$, where Γ is an arbitrary finite (possibly empty) multiset of formulas (i.e., the order of formulas in Γ is disregarded) and Δ consists of one formula at most. Let us consider only such sequents in which no variable occurs free and bound at the same time. The formula $\mathfrak{A} = (A_1 \& \ldots \& A_n) \supset B$ is called an image formula of the sequent $A_1, \ldots, A_n \to B$. If $n = 0$, then $\mathfrak{A} = B$, and if Δ is empty, then $\mathfrak{A} = \underset{i=1}{\overset{n}{\&}} A_i \supset \bot$, where $\bot$ is false.

Let LJ be the logic considered. Then Gentzen-type sequent calculus GLJ for any logic LJ has the following postulates (see, e.g., [8]).

Axioms: $\Gamma, A \to A$; $\quad \bot, \Gamma \to \Delta$.

Recall that here, in the rules of inference, and below Δ consists of one formula at most.

The rules of inference for logical connectives are defined as follows.

$$\frac{A, \Gamma \to B}{\Gamma \to A \supset B}\ (\to\supset) \qquad \frac{A \supset B, \Gamma \to A; \quad B, \Gamma \to \Delta}{A \supset B, \Gamma \to \Delta}\ (\supset\to)$$

$$\frac{\Gamma \to A; \quad \Gamma \to B}{\Gamma \to A\&B}\ (\to \&) \qquad \frac{A, B, \Gamma \to \Delta}{A\&B, \Gamma \to \Delta}\ (\& \to)$$

$$\frac{\Gamma \to A}{\Gamma \to A \vee B}\ (\to \vee) \qquad \frac{A, \Gamma \to \Delta; \quad B, \Gamma \to \Delta}{A \vee B, \Gamma \to \Delta}\ (\vee \to)$$

$$\frac{\Gamma \to B}{\Gamma \to A \vee B}\ (\to \vee)$$

$$\frac{\Gamma \to A(t)}{\Gamma \to \exists x A(x)}\ (\to \exists) \qquad \frac{A(b), \Gamma \to \Delta}{\exists x\ A(x), \Gamma \to \Delta}\ (\exists \to)$$

$$\frac{\Gamma \to A(b)}{\Gamma \to \forall x A(x)}\ (\to \forall) \qquad \frac{A(t), \forall x\ A(x), \Gamma \to \Delta}{\forall x\ A(x), \Gamma \to \Delta}\ (\forall \to)$$

where the term t in $(\to \exists)$ and $(\forall \to)$ is an arbitrary term free for x in $A(x)$, and b in $(\exists \to)$ and $(\to \forall)$ does not occur in the conclusion of the rule.

The rules of inference for modality $\Box$ are different in the appropriate calculi. There are four different modal rules for the traditional modal logics K, T, $K4$, and $S4$ (without the Barcan formula):

$$\frac{\Gamma \to A}{\Sigma, \Box\Gamma \to \Box A}\ (\Box_1) \qquad \frac{\Gamma, \Box\Gamma \to A}{\Sigma, \Box\Gamma \to \Box A}\ (\Box_2)$$

$$\frac{\Box\Gamma \to A}{\Sigma, \Box\Gamma \to \Box A}\ (\to \Box) \qquad \frac{A, \Box A, \Gamma \to \Delta}{\Box A, \Gamma \to \Delta}\ (\Box \to)$$

These modal rules correspond to the considered modal logics as the following table shows:

KJ	$K4J$	TJ	$S4J$	KB	B	$S5$
$(\Box_1)$	$(\Box_2)$	$(\Box_1)$ $(\Box \to)$	$(\to \Box)$ $(\Box \to)$	$(\Box_1)$	$(\Box_1)$ $(\Box \to)$	$(\to \Box)$ $(\Box \to)$

All the calculi considered have the rule $(cut^{\Box})$ as follows:

$$\frac{\Gamma \to \Box A; \ \Box A, \Gamma \to \Delta}{\Gamma \to \Delta}\ (cut^{\Box}).$$

Besides, calculi GKJ, GTJ, $GK4J$, and $GS4J$ have the following rule of inference corresponding to the Barcan Axiom:

$$\frac{\Gamma \to \forall x \Box A(x)}{\Gamma \to \Box \forall x A(x)}\ (\to \Box\forall).$$

Instead of the rule $(\to \Box\forall)$ corresponding to the Barcan Axiom, calculi $GKBJ$, GBJ, and $GS5J$ have the rule (BA)

$$\frac{A, \Gamma \to}{\Gamma \to \Box\neg\Box A}\ (BA).$$

The rule (BA) corresponds to the Brouwerian Axiom.

It is easy to verify that the structural rules of weakening, contraction and the cut rule with the cut formula of the shape different from $\Box A$ are admissible for the logics under consideration.

Theorem 2.1 *Let A be any formula, then $HL \vdash A \Leftrightarrow GL \vdash\to A$.*

Note that no cut–free Gentzen-type formulation for intuitionistic predicate modal logics with the Barcan axiom is known. To obtain cut–free calculi for the classical modal logics mentioned, indexed calculi have been introduced in [11].

3 Cut–free Indexed Calculi

Let us introduce formulas with indices. Let $\alpha, \beta, \gamma, \ldots$ be variables for the indices. Each predicate symbol P having its own index γ is denoted by P^{γ}. A predicate symbol without an index is regarded as a symbol with the index 1. Two identical predicate symbols with different indices are considered as different symbols.

A formula A with the index β is denoted by $(A)^{\beta}$ and defined in the following way:
1. $(E)^{\beta} = E^{\beta}$, where E is an elementary formula;
2. $(A \odot B)^{\beta} = (A)^{\beta} \odot (B)^{\beta}$, $\odot \in \{\supset, \&, \vee\}$;
3. $(\sigma A)^{\beta} = \sigma(A)^{\beta}$, $\sigma \in \{\Box, \exists x, \forall x\}$.

An index is introduced in different ways for analytic logics LJ, where $LJ \in \{KJ,\ TJ,\ K4J,\ S4J\}$ and for semi-analytic logics KBJ, BJ (see, e.g., [3] for the term of analytic and semi-analytic logics). Note that in spite of the fact that $S5J$ is not an analytic logic, the indexing for this logic is introduced in the same way as for the analytic logics. Let $\gamma_1, \ldots, \gamma_m$ be all different indices entering the sequent $\Gamma \rightarrow \Delta$ as indices of the formulas from Γ, Δ. From now on, let us denote the analytic logics considered by LJ^B (the index B in this notation means that these logics contain the Barcan axiom explicitly) and semi-analytic ones by LJ^S.

An arbitrary natural number i will be called an index (by analogy with [5]) in the logics LJ^B and $S5J$. The relation $\alpha \leq \beta$ for two arbitrary indices α and β is defined in the usual way for the natural numbers. In the logics LJ^S the index σ is a finite sequence of natural numbers (by analogy with [3].

The sequent indexed calculus ILJ for any logic LJ has the same axioms and rules of inference for logic connectives and quantifiers as GLJ.

Instead of the special rules of inference concerning modality $\Box$ in appropriate calculi for logics LJ, namely, $(\Box_1)$, $(\Box_2)$, $(\rightarrow \Box)$, $(\Box \rightarrow)$ and rules $(cut^{\Box})$, $(\rightarrow \Box\forall)$, (BA), two modal rules are added.

The rule of inference for modality in succedent is of the shape:

$$\frac{\Gamma' \rightarrow\ (A)^{\delta}}{\Gamma \rightarrow\ \Box(A)^{\sigma}}\ (\rightarrow \Box^{i}).$$

In the case of logics LJ^B and $S5J$, in the rule $(\rightarrow \Box^{i})$ $\delta = i + 1$ if $\sigma = i$, in addition, if $\sigma = \max(\gamma_1, \ldots, \gamma_m)$, and $\gamma_1, \ldots, \gamma_m$ are all the indices entering the conclusion of this rule, then Γ' is the same as Γ. Otherwise, i.e., if $\sigma < \max(\gamma_1, \ldots, \gamma_m)$, then Γ' is obtained from Γ by deleting all the formulas with the indices greater than σ. In the case of logics LJ^S, in the rule $(\rightarrow \Box^{i})$ $\delta = \sigma i$, where i is any integer and σi is not an initial segment of any index γ from Γ, in addition, Γ' is the same as Γ.

Remark 3.1 Note that in the case of indexed calculi for the first order classical modal logics two succedent rules $(\rightarrow \Box_1^{i})$ and $(\rightarrow \Box_2^{i})$ are formulated (see [11]).

The rule of inference for modality in antecedent has the common shape for all calculi ILJ, but the meaning of the index δ is different in the appropriate calculi:

$$\frac{(A)^{\delta}, \Box(A)^{\sigma}, \Gamma \rightarrow \Delta}{\Box(A)^{\sigma}, \Gamma \rightarrow \Delta}\ (\ \Box^{i} \rightarrow),$$

where $\delta \in \{\gamma_1 \ldots \gamma_m\}$, and $\gamma_1, \ldots, \gamma_m$ are all the indices entering the conclusion of this rule. This is the only restriction to the index δ for logic $S5J$.

Other logics considered need some additional restrictions to the index δ. Namely, for the logics LJ^B $\delta = i + k$, if $\sigma = i$ and the meaning of k corresponds to the modal logics as the following table shows.

K^B	$K4^B$	T^B	$S4^B$
$k = 1$	$k \geq 1$	$k = 0$ or $k = 1$	$k \geq 0$

If such a k does not exist, the rule cannot be applied.

In the case of logics KBJ $\delta \neq \sigma$ and $\delta = \sigma_1 ji$ or $\delta = \sigma_1$, if $\sigma = \sigma_1 j$; for BJ $\delta = \sigma$ or δ is determined in the same way as for KBJ.

Remark 3.2 The choice of index in the rules $(\to \Box^i)$ and $(\Box^i \to)$ of ILJ^S corresponds to the restrictions to the prefix in the prefixed tableau introduced by Fitting [3] for logics KB and B.

Making use of ideas from [11] the admissibility of the cut rule in ILJ can be proved. The presence of uninvertible rules of inference in the logical part of calculi considered requires some additional investigation.

Theorem 3.3 *Let S be an index-free sequent, then $GLJ \vdash S \Leftrightarrow ILJ \vdash S$.*

4 Harrop Properties

We shall deal here with two properties which are analogies of the properties proved by R.Harrop [1] for the intuitionistic propositional logic and intuitionistic number theory. Later on, these properties called by T.Nakamura as the Harrop disjunction property (HDP) and the Harrop existence property (HEP) were investigated for some intermediate predicate logics in [9]. T.Nakamura showed by Kripke semantics that intermediate logic LD, i.e. an intuitionistic predicate logic complemented with $\forall x(P(x) \vee Q) \supset \forall x P(x) \vee Q$, has ($HDP$) and ($HEP$).

Let us recall and modify some notions.

Definition 4.1 Let $A \in LJ$, where $L \in$ {K, T, K4, S4, KB, B, S5}. Then A is the Harrop formula (H-formula) if every occurrence of the disjunction and the existence quantifier is only within the premise of some implication.

Definition 4.2 Let $A \in LJ$, where $L \in$ {K, K4, KB}. Then A is the strong Harrop formula (sH-formula) if every occurrence of the disjunction and the existence quantifier is either within the premise of some implication or in the scope of necessity operator $\Box$.

Definition 4.3 The sequent S of the shape $\Gamma \to A \vee B$ or $\Gamma \to \exists x A(x)$ is said to be of Harrop-type (strong Harrop-type) if each formula of Γ is the H-formula (sH-formula).

It is trivial that any Harrop-type sequent S is a strong Harrop-type sequent, but not vice versa.

Definition 4.4 Logic LJ is said to have the modal Harrop disjunction property ($MHDP$) (strong modal Harrop disjunction property ($sMHDP$)) provided that the Harrop-type (strong Harrop-type) sequent of the shape $\Gamma \to A \vee B$ is derivable in the appropriate sequent calculi ILJ iff $\Gamma \to A$ or $\Gamma \to B$ is derivable in the same ILJ.

Definition 4.5 Logic L is said to have the modal Harrop existence property ($MHEP$) (strong modal Harrop existence property ($sMHEP$)) provided that the Harrop-type (strong Harrop-type) sequent of the shape $\Gamma \to \exists x A(x)$ is derivable in the appropriate sequent calculi ILJ iff there exists a term t such that the sequent $\Gamma \to A(t)$ is derivable in the same ILJ.

Theorem 4.6 (modal Harrop disjunction and existence properties)
a) Intuitionistic modal logics with the Barcan axiom LJ where L ∈ {K, K4, T, S4, KB, B, S5} have the modal Harrop disjunction property and the modal Harrop existence property.

b) Intuitionistic modal logics with the Barcan axiom LJ where L ∈ {K, K4, KB} have the strong modal Harrop disjunction property and the strong modal Harrop existence property.

Proof. The "if" part of the theorem is trivial. In the "only-if" part it is essential that a cut–free derivation can be constructed in calculi ILJ. The proof of part a) is analogous to the syntactical proof of the Harrop theorem for the intuitionistic logic (see, e.g., [13]).

To prove part b), let us consider non-indexed calculi GLJ where $L \in \{K, K4, KB\}$. The proof is carried out by induction on the number of applications of the rules of inference below all the applications of rules for $\vee$ and $\exists$ in the given derivation. From the shape of the rules of calculi considered and the sequent examined we have only two possibilities to start a derivation. If the last step in the given derivation is the application of a logical rule, the case is considered as in part a). Note that the last step cannot be the application of the rule $(\Box_1)$, $(\Box_2)$, $(\to \Box)$, $(\to \Box\forall)$ or (BA), though, it can be the application of the rule $(cut^{\Box})$. In this case, the end of the given derivation is of the form:

$$\frac{S_1\Big\{\Gamma \to \Box A;\ S_2\Big\{\Box A, \Gamma \to \Delta}{\Gamma \to \Delta}\,(cut^{\Box}),$$

where $\Delta \in \{A \vee B,\ \exists x A(x)\}$.

Basing on Definition 4.2 and by the induction hypothesis applied to S_2 we can get the derivation of the sequent $S_3 = \Box A, \Gamma \to \Delta'$, where $\Delta' \in \{A,\ B,\ A(t)\}$. Applying $(cut^{\Box})$ to S_1, S_3 we get the desired derivation. To get (sMHDP) and (sMHEP) for the indexed calculi it suffices to apply Theorem 3.3. $\triangledown$

Remark 4.7 A logic possessing the strong modal Harrop disjunction (existence) property has the modal Harrop disjunction (existence) property, too. Note that in contrast to classical predicate modal logics (see [12]), traditional formulation of the Harrop disjunction (existence) property for intuitionistic and intermediate logics (see, e.g., [1], [9]) is the same as the modal Harrop disjunction (existence) property for the intuitionistic modal logics considered.

Example 4.8 According to Definition 4.3 the sequent $\Box(\Box P \vee \Box Q) \to \Box P \vee \Box Q$ is of strong Harrop-type. It is easy to verify that it is not derivable in KJ, $K4J$, and KBJ. Therefore, basing on (sMHDP) neither $\Box(\Box P \vee \Box Q) \to \Box P$ nor $\Box(\Box P \vee \Box Q) \to \Box Q$ are derivable in KJ, $K4J$, and KBJ. On the other hand, this sequent is derivable in TJ and $S4J$, but it is not of Harrop-type. So, (MHDP) cannot be applied to this sequent.

Example 4.9 According to Definition 4.3 the sequent $\Box(\Box P(a) \vee \Box P(b)) \to \exists x \Box P(x)$ is of strong Harrop-type. It is easy to verify that it is not derivable in KJ, $K4J$, and KBJ. Therefore, indeed, there does not exist any term t such that the sequent $\Box(\Box P(a) \vee \Box P(b)) \to \Box P(t)$ is derivable in ILJ, where $L \in \{K,\ K4,\ KB\}$. The sequent $\Box P(a) \vee \Box P(b) \to \exists x \Box P(x)$ is derivable in KJ, $K4J$, and KBJ, but it is not of strong Harrop-type. Both sequents considered in this example are derivable in T, however, both of them are not of Harrop-type.

5 Analogy of the Interpolation Property

As shown in [2], the Interpolation Theorem does not hold for the first order classical modal logics with the Barcan axiom. The same negative results can be obtained for the intuitionistic version of these modal logics. However, as it follows from [7], the Interpolation Theorem is valid for the intuitionistic propositional $S4$-type and $S5$-type logics considered in [10]. The Craig Interpolation Theorem for the intuitionistic predicate logic with constant domains was first considered in [4] by the semantic approach. We examine here a possibility to construct a cut formula in the cut rule so that the cut rule be invertible. This theorem can be treated as some analogy of the interpolation theorem.

Definition 5.1 Let P^i be any index predicate symbol. Then P is called a basis of P^i. Let Γ be an arbitrary multiset of formulas. Then $V(\Gamma)$ is a set of all different variables, constants (apart from $\perp$), function symbols and bases of predicates with an index entering Γ.

Lemma 5.2 (invertibility of (*cut*) **in** ILJ**)**
Let $ILJ \vdash \Gamma \to \Delta$. Then for any partition $(\Gamma_1,\ \Gamma_2)$ of the multiset Γ there exists a formula C (called an interpolant) such that
1) $ILJ \vdash \Gamma_1 \to C$; $ILJ \vdash C, \Gamma_2 \to \Delta$ (invertibility condition)
2) $V(C) \subseteq V(\Gamma_1 \cup \Delta) \cap V(\Gamma_2 \cup \Delta)$ (intersection condition)
3) the index of C does not exceed the indices from $\Gamma_1,\ \Gamma_2,\ \Delta$ (index condition).

Proof. The lemma is proved by induction on the height h of the derivation, relying on the fact that ILJ has not contain the cut rule. If $h = 0$, then the proof is carried out in the same way as in [13]. Let $h > 0$ and (k) be the rule applied last in the given derivation. Let us consider only the case $(k) = (\to \Box^i)$. Then the end of the derivation is of the form:

$$\frac{\Gamma' \to\ (A)^\delta}{\Gamma \to\ \Box(A)^\sigma}\,(\to \Box^i),$$

where $\delta = i+1$ if $\sigma = i$. Let $\gamma_1, \dots, \gamma_m$ be all the indices entering the conclusion of the rule $(\to \Box^i)$. Then Γ' is obtained from Γ by deleting all the formulas with the indices greater than σ, if $\sigma < \max(\gamma_1, \dots, \gamma_m)$. Otherwise, i.e., if $\sigma = \max(\gamma_1, \dots, \gamma_m)$, then Γ' is the same as Γ. So, $\Gamma' \subseteq \Gamma$ and every partition $(\Gamma_1,\ \Gamma_2)$ of the multiset Γ has corresponding (the induced, in term from [13]) partition $(\Gamma_1',\ \Gamma_2')$ of the multiset Γ'. By the induction assumption $ILJ \vdash S_1 = \Gamma_1' \to C$; and $ILJ \vdash S_2 = C, \Gamma_2' \to (A)^{i+1}$ and the interpolant C satisfies the intersection and index conditions. Let us consider two subcases.

1. Formula C has the index $i+1$. Since $i+1 \notin \Gamma_1'$, we can apply $(\to \Box^i)$ to the sequent S_1 and get $ILJ \vdash S_1' = \Gamma_1' \to \Box(C)^i$. Using the admissibility of $(W \to)$ and relying on the restriction to the indices in the rule $(\Box^i \to)$, having applied $(\Box^i \to)$ to

S_2, we get $ILJ \vdash S_2' = \Box(C)^i, \Gamma_2' \to (A)^{i+1}$. Applying $(\to \Box^i)$ to the sequent S_2' we get $ILJ \vdash S_3' = \Box(C)^i, \Gamma_2' \to \Box(A)^i$. Derivations of the sequents S_1' and S_3' (or the sequents obtained from these sequents having applied $(W \to)$, if Γ' is not the same as Γ) are the desired ones and the formula $\Box(C)^i$ is interpolant in this case.

2. Formula C has the index less than $i+1$. In this case, relying on the index condition we can apply $(\to \Box^i)$ to the sequent S_2 and get $ILJ \vdash S_2' = C, \Gamma_2' \to \Box(A)^i$. Derivations of the sequents S_1 and S_2' or the sequents obtained from these sequents having applied $(W \to)$ are the desired ones and the formula C is interpolant in this case. $\triangledown$

References

[1] Harrop, R.: 1960, Concerning formulas of the types $A \to B \vee C$, $A \to (Ex)B(x)$ in intuitionistic formal systems, *Journal of Symbolic Logic* **25**: 27–32.

[2] Fine K.: 1979, Failures of the interpolation lemma in quantified modal logic, *Journal of Symbolic Logic* **44**: 201–206.

[3] Fitting M.C.: 1983, *Proof methods for modal and intuitionistic Logics*, D. Reidel, Dordrecht.

[4] Gabbay D.: 1977, Craig interpolation theorem for intuitionistic logic and extensions Part III, *Journal of Symbolic Logic* **42**: 269–271.

[5] Kanger S.G.: 1957, *Provability in logic*, Volume 1, Acta Univ. Stock., Stockholm Studies in Philosophy

[6] Kleene S.C.: 1952, *Introduction to metamathematics*, North-Holland. Amsterdam.

[7] Luppi C.: 1996, On the interpolation property of some intuitionistic modal logics, *Arch. Math. Logic* **35**: 173–189.

[8] Mints G.: 1994, Resolution strategies for the intuitionistic logic, *Constraint Programming*, NATO ASI Series, Springer Verlag: 289–311.

[9] Nakamura T.: 1983, Disjunction property for some intermediate predicate logics, *Reports on Mathematical Logic* **15**: 33–39.

[10] Ono, H.: 1977, On Some intuitionistic modal logics, *Publ. R.I.M.S. Kyoto Univ.* **13**: 687–722.

[11] Pliuškevičienė A.: 1997, Cut–free indexed calculi for modal logics containing the Barcan axiom, in M.Kracht et al. (eds.): *Advances in Modal Logic'96*, CSLJ Publications: 155–170.

[12] Pliuškevičienė A.: 1998, Extended disjunction and existence properties for some predicate modal logics, *Journal of the IGPL*, **6(5)**: 775–787.

[13] Takeuti G.: 1975, *Proof theory*, North-Holland. Amsterdam, American Elsevier, inc, New-York.

[14] Wolter, F., Zakharyaschev, M.: 1997, On the relation between intuitionistic and classical modal logics, *Algebra and Logic* **36**: 121-155.