How Complex is a Finite First-Order Sorted Interpretation?

Thierry Boy de la Tour*
LEIBNIZ Laboratory - IMAG
46, Av. Félix Viallet, 38031 Grenoble Cedex, France
Thierry.Boy-de-la-Tour@imag.fr

Abstract

The general problem of testing the isomorphism of two given finite interpretations of first-order logic is known to be isomorphism complete, i.e. polynomially equivalent to the graph isomorphism problem (GI). It is easy to see that this fact still holds when sorts are introduced. However, this isomorphism problem is relevant only for interpretations of a fixed signature, and in some cases, according to the signature, is much simpler than the general problem. We therefore establish exactly for which signatures is the associated isomorphism problem simpler than GI, and for which is it isomorphism complete. It turns out that non-monadic signatures are isomorphism complete just as is the case in unsorted logic, while the classification of monadic signatures is more complex and interesting.

1 Introduction

In the context of model building, it is very common to consider sorts in order to reduce the search space. It is also a trivial thought that things get more complex if we consider a formula with more non-logical symbols than another one. But then why not consider only one sort, and one function symbol encoding all others? Because the corresponding interpretations would poorly represent the objects we are looking for, and the search would browse many meaningless structures. A search can only be efficient if the search space consists of reasonable candidates, not weird mixtures of unsuitable representations. We may question whether the art of finding a suitable, or "searchable" representation can rest on firm ground.

When we search for finite models of a first order sorted formula, the search space is determined by the set of non-logical symbols used in the formula, i.e. the signature. It is clear that some signatures are much simpler than others, for example the interpretations of a signature Σ with only one constant symbol cannot match the rich structure of graphs, while this is possible with a binary predicate symbol. Of course, there may be many ways to represent any kind of objects as finite algebras, but we may obtain negative results by considering the relative complexity of source and target structures of representations: the represented object is necessarily simpler than the structure into which it is encoded.

We will only consider transformations that preserve isomorphisms in order to ensure fair representations. We will also focus on a very elementary measure for the complexity of a structure: the computational complexity of the associated isomorphism problem. The

*This work has been supported by CNRS.

reason is that the general isomorphism problem between finite algebras is known to be isomorphism complete, while it is believed that this class is disjoint from the class P. Hence finite interpretations of simple signatures, i.e. inducing a polynomial isomorphism test, are strictly simpler than those rich enough to embed graphs.

2 Preliminaries

DEFINITION 2.1. Given a finite set $\mathcal{S}$, whose elements are called *sorts*, the set of *first-order $\mathcal{S}$-types* is $\mathfrak{T}_1(\mathcal{S}) = \bigcup_{k\in\mathbb{N}} \mathcal{S}^k \times (\mathcal{S} \uplus \{\mathbf{o}\})$. For $t = \langle d_1, \ldots, d_k, r\rangle \in \mathfrak{T}_1(\mathcal{S})$, if $k \neq 0$ then t is said to be *functional*, and is noted $d_1 \times \ldots \times d_k \to r$; $\operatorname{dom} t$ is $d_1 \times \ldots \times d_k$ and $\operatorname{rng} t$ is r. If $k = 1$, t is said to be *monadic*, and *atomic* if $k = 0$.

A *signature* $\Sigma = \langle \mathcal{S}, \mathcal{F}, \tau\rangle$ is given by a finite set $\mathcal{S}$ of *sorts*, a finite set $\mathcal{F}$ of *symbols* and a function τ from $\mathcal{F}$ to $\mathfrak{T}_1(\mathcal{S})$. $f \in \Sigma$ stands for $f \in \mathcal{F}$, and Σ_f for $\tau(f)$. If Σ_f is functional and $\operatorname{rng} \Sigma_f = \mathbf{o}$, then f is a *predicate* symbol. A signature Σ is *monadic* if $\forall f \in \Sigma$, Σ_f is either monadic or atomic.

A *sort interpretation* $\mathcal{I}$ of $\mathcal{S}$ is a function which associates a finite non empty set to each element of $\mathcal{S}$, such that $\forall s, s' \in \mathcal{S}$, if $s \neq s'$ then $\mathcal{I}(s) \cap \mathcal{I}(s') = \emptyset$ and $\mathcal{I}(s) \cap \{\top, \bot\} = \emptyset$. We extend $\mathcal{I}$ to the set of first order $\mathcal{S}$-types by: $\mathcal{I}(\mathbf{o}) = \{\top, \bot\}$, $\forall s_1, \ldots, s_n \in \mathcal{S}, \mathcal{I}(s_1 \times \ldots \times s_n) = \prod_{i=1}^{n} \mathcal{I}(s_i)$ and for any functional first-order $\mathcal{S}$-type t, $\mathcal{I}(t)$ is the set of functions from $\mathcal{I}(\operatorname{dom} t)$ to $\mathcal{I}(\operatorname{rng} t)$.

A Σ-*interpretation* $\mathcal{I} = \langle \mathcal{D}, v\rangle$ is given by a sort interpretation $\mathcal{D}$ of $\mathcal{S}$ and a function v from $\mathcal{F}$ to $\bigcup_{t\in\mathfrak{T}_1(\mathcal{S})} \mathcal{D}(t)$ such that $\forall f \in \Sigma, v(f) \in \mathcal{D}(\Sigma_f)$. In the sequel, $\mathcal{I}_f$ stands for $v(f)$, and $\mathcal{I}(t)$ for $\mathcal{D}(t)$. ∇

Given two problems $\mathcal{P}$ and $\mathcal{Q}$, we note $\mathcal{P} \propto_P \mathcal{Q}$ when $\mathcal{P}$ polynomially reduces to $\mathcal{Q}$ (see [1]). We note GI the problem of graph isomorphism: given two graphs $G = \langle V, E\rangle$ and $G' = \langle V', E'\rangle$, GI is true of G, G' iff $\exists\alpha$ such that $\alpha : G \cong G'$. We will also consider the usual brands of graphs, directed, labeled, multigraphs. Their isomorphism problem are known to be all polynomially equivalent to GI, i.e. *isomorphism complete* (see e.g. [2]). Other standard notions as paths, connexity, etc. will also be assumed.

We will obviously make extensive use of isomorphisms between interpretations: given a signature Σ and two Σ-interpretations $\mathcal{I}, \mathcal{I}'$, an isomorphism between $\mathcal{I}$ and $\mathcal{I}'$ is a function σ such that $\forall s \in \mathcal{S}$, σ is 1-1 from $\mathcal{I}(s)$ onto $\mathcal{I}'(s)$, σ is the identity on $\mathcal{I}(\mathbf{o}) = \mathcal{I}'(\mathbf{o})$, and $\forall f \in \Sigma$, let $\Sigma_f = d_1 \times \ldots \times d_n \to r$, then $\forall \langle x_1, \ldots, x_n\rangle \in \mathcal{I}(d_1 \times \ldots \times d_n), \mathcal{I}'_f(x_1^\sigma, \ldots, x_n^\sigma) = \mathcal{I}_f(x_1, \ldots, x_n)^\sigma$. This is noted $\sigma : \mathcal{I} \cong \mathcal{I}'$. Finally, we note $\operatorname{Iso}(\Sigma)$ the problem which, given two Σ-interpretations $\mathcal{I}, \mathcal{I}'$, is true iff $\exists\sigma$ such that $\sigma : \mathcal{I} \cong \mathcal{I}'$.

Since we only consider isomorphism problems, we will provide polynomial time transformations from source structures (graphs, interpretations) to target structures, while preserving isomorphisms *in both directions*. When isomorphic source objects are transformed into isomorphic target objects, we say that the transformation is *invariant* (intuitively, *only* their structure is transformed). If source objects are isomorphic whenever their transformed objects are isomorphic, the transformation is *accurate* (*all* the structure is transformed). A transformation both invariant and accurate is said to be *fair*.

As an example, we first prove that things get more complex by adding sorts.

LEMMA 2.1 Let $\Sigma = \langle \mathcal{S}, \mathcal{F}, \tau\rangle$ and $\Sigma' = \langle \mathcal{S} \uplus \{s\}, \mathcal{F}, \tau\rangle$, then $\operatorname{Iso}(\Sigma) \propto_P \operatorname{Iso}(\Sigma')$

Proof. Σ-interpretations $\mathcal{I}$ can be transformed into Σ'-interpretations $\widetilde{\mathcal{I}}$ by taking $\widetilde{\mathcal{I}}_f = \mathcal{I}_f$ and $\widetilde{\mathcal{I}}(t) = \mathcal{I}(t)$ for all $f \in \mathcal{F}$ and $t \in \mathcal{S}$, and $\widetilde{\mathcal{I}}(s) = \{a\}$, where $a \notin \biguplus_{t\in\mathcal{S}} \mathcal{I}(t)$. This transformation is obviously polynomial. It is invariant since any isomorphism σ between two Σ-interpretations $\mathcal{I}, \mathcal{I}'$ can be extended to an isomorphism between $\widetilde{\mathcal{I}}$ and $\widetilde{\mathcal{I}}'$ by $a^\sigma = a$. It is accurate since any isomorphism $\sigma : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}}'$ is 1-1 from $\widetilde{\mathcal{I}}(s)$ onto $\widetilde{\mathcal{I}}'(s)$, hence $a^\sigma = a$, and the restriction of σ to $\biguplus_{t\in\mathcal{S}} \widetilde{\mathcal{I}}(t)$ is an isomorphism between $\mathcal{I}$ and $\mathcal{I}'$. Hence the problem $\mathrm{Iso}(\Sigma)$ can be solved by using $\mathrm{Iso}(\Sigma')$ (whether the answer is yes or no) through this polynomial and fair transformation. Q.E.D.

In the sequel, we will establish properties of specific signatures, the statement of which will be eased by the following notation: for a given $\mathcal{S}$ and any first-order $\mathcal{S}$-types $t_1, \ldots, t_n$, we note $\lceil t_1, \ldots, t_n \rceil$ for any signature $\Sigma = \langle \mathcal{S}, \mathcal{F}, \tau \rangle$ where $\mathcal{F}$ contains exactly n symbols $f_1, \ldots, f_n$ and $\forall i \in \{1 \ldots n\}, \tau(f_i) = t_i$. If $\mathcal{S}$ is not specified, we take the smallest possible one: the set of symbols appearing in the t_i's.

It is easy to see that things get more complex by *adding* arguments to functions.

THEOREM 2.2 $\mathrm{Iso}(\lceil d_1 \times \ldots \times d_n \to r \rceil) \propto_P \mathrm{Iso}(\lceil d_0 \times \ldots \times d_n \to r \rceil)$

Proof. If Σ, Σ' have a unique $f \in \Sigma, f \in \Sigma'$ with $\Sigma_f = d_1 \times \ldots \times d_n \to r$, and $\Sigma'_f = d_0 \times \ldots \times d_n \to r$. We first consider the case where $d_0 \in \{d_1, \ldots, d_n, r\}$.

We transform Σ-interpretations $\mathcal{I}$ into Σ'-interpretations $\widetilde{\mathcal{I}}$ by: $\forall s \in \mathcal{S}, \widetilde{\mathcal{I}}(s) = \mathcal{I}(s)$ and $\forall \langle x_0, \ldots, x_n \rangle \in \widetilde{\mathcal{I}}(d_0 \times \ldots \times d_n), \widetilde{\mathcal{I}}_f(x_0, \ldots, x_n) = \mathcal{I}_f(x_1, \ldots, x_n)$. This transformation is clearly polynomial: the graph of $\mathcal{I}_f$ is duplicated $|\mathcal{I}(d_0)|$ times. Since $\forall \sigma, \widetilde{\mathcal{I}}_f(x_0^\sigma, \ldots, x_n^\sigma) = \mathcal{I}_f(x_1^\sigma, \ldots, x_n^\sigma)$ and $\widetilde{\mathcal{I}}_f(x_0, \ldots, x_n)^\sigma = \mathcal{I}_f(x_1, \ldots, x_n)^\sigma$, it is obviously fair.

If d_0 is a new sort, we first add d_0 to Σ', which yields Σ'' and $\mathrm{Iso}(\Sigma) \propto_P \mathrm{Iso}(\Sigma'')$ by lemma 2.1. The previous case yields $\mathrm{Iso}(\Sigma'') \propto_P \mathrm{Iso}(\Sigma')$. Q.E.D.

It is not as easy to prove that things get more complex by adding *objects* to a signature. More precisely, given two signatures $\Sigma = \langle \mathcal{S}, \mathcal{F}, \tau \rangle$ and $\Sigma' = \langle \mathcal{S}', \mathcal{F}', \tau' \rangle$, we say that $\Sigma \subseteq \Sigma'$ iff $\mathcal{S} \subseteq \mathcal{S}'$, $\mathcal{F} \subseteq \mathcal{F}'$ and $\forall f \in \mathcal{F}, \tau(f) = \tau'(f)$.

DEFINITION 2.2. To any signature Σ we associate a directed multigraph $G_\Sigma = \langle \mathcal{S}, \mathcal{E}_\Sigma, \mathrm{fst}_\Sigma, \mathrm{snd}_\Sigma \rangle$, where $\mathcal{E}_\Sigma$ is the set of $\langle f, i \rangle$ for $f \in \Sigma$ such that Σ_f is functional, with $\mathrm{rng}\, \Sigma_f \neq \mathbf{o}$ and i is an integer between 1 and the arity n of f; then for $\Sigma_f = d_1 \times \ldots \times d_n \to r$, we take $\mathrm{fst}_\Sigma(\langle f, i \rangle) = d_i$ and $\mathrm{snd}_\Sigma(\langle f, i \rangle) = r$ (see figure 1). ∇

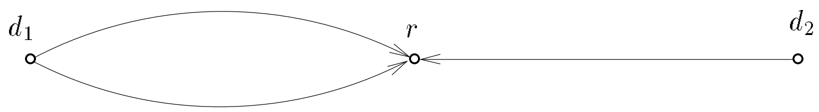

Figure 1: G_Σ for $\Sigma = \lceil d_1 \times d_2 \to r, d_1 \to r, r \to \mathbf{o} \rceil$

We now come to the more difficult task of adding a new function symbol $g : d_1 \times d_n \to t$ to a signature Σ while preserving isomorphisms. The trivial thing to do is to take some constant function for $\mathcal{I}_g$, but this necessarily involves an element of $\mathcal{I}(t)$, therefore disturbing the whole structure of the Σ-interpretation $\mathcal{I}$. The solution is to add a new element a_t to $\mathcal{I}(t)$ in order to hold the "blind" value of $\mathcal{I}_g$. But then for any $f \in \Sigma$ with t among its domain sort, we have to provide a value for $\mathcal{I}_f(a_t)$, and hence to add other elements to other range sets in order to hold the images of these new elements, in an inductive way.

LEMMA 2.3 Let $\Sigma = \langle \mathcal{S}, \mathcal{F}, \tau_{|\mathcal{F}} \rangle$ and $\Sigma' = \langle \mathcal{S}, \mathcal{F} \uplus \{g\}, \tau \rangle$, then $\mathrm{Iso}(\Sigma) \propto_P \mathrm{Iso}(\Sigma')$

Proof. Let $t = \mathrm{rng}\, \tau(g)$ if $\tau(g)$ is functional, and $t = \tau(g)$ otherwise.

If $t = \mathbf{o}$, to every Σ-interpretation $\mathcal{I}$ we associate a Σ'-interpretation $\widetilde{\mathcal{I}}$ defined by: $\forall s \in \mathcal{S}, \widetilde{\mathcal{I}}(s) = \mathcal{I}(s)$, $\forall f \in \Sigma, \widetilde{\mathcal{I}}_f = \mathcal{I}_f$ and $\forall x \in \mathcal{I}(\mathrm{dom}\, \tau(g)), \widetilde{\mathcal{I}}_g(x) = \top$ if $\tau(g)$ is functional, or $\widetilde{\mathcal{I}}_g = \top$ otherwise. This transformation is obviously polynomial and fair.

If $t \in \mathcal{S}$, let $\mathcal{S}_t$ be the set of $s \in \mathcal{S}$ such that there exists a path in G_Σ from t to s, and including t. Given a Σ-interpretation $\mathcal{I}$, to every $s \in \mathcal{S}$ we associate a different a_s such that $a_s \notin \biguplus_{s' \in \mathcal{S}} \mathcal{I}(s')$, and we build the Σ'-interpretation $\widetilde{\mathcal{I}}$ defined by $\forall s \in \mathcal{S} - \mathcal{S}_t, \widetilde{\mathcal{I}}(s) = \mathcal{I}(s)$, $\forall s \in \mathcal{S}_t, \widetilde{\mathcal{I}}(s) = \mathcal{I}(s) \uplus \{a_s\}$ and $\forall f \in \Sigma$, if $\tau(f)$ is atomic then $\widetilde{\mathcal{I}}_f = \mathcal{I}_f$, and if $\tau(f)$ is $d_1 \times \ldots \times d_n \to r$, then $\forall \langle x_1, \ldots, x_n \rangle \in \widetilde{\mathcal{I}}(d_1 \times \ldots \times d_n), \widetilde{\mathcal{I}}_f(x_1, \ldots, x_n) = [\text{if } x_1 = a_{d_1}$ or $\ldots$ or $x_n = a_{d_n}$ then a_r else $\mathcal{I}_f(x_1, \ldots, x_n)]$. Finally, if $\tau(g)$ is functional then $\forall x \in \widetilde{\mathcal{I}}(\mathrm{dom}\, \tau(g)), \widetilde{\mathcal{I}}_g(x) = a_t$, and $\widetilde{\mathcal{I}}_g = a_t$ otherwise (see figure 2). The transformation from $\mathcal{I}$ to $\widetilde{\mathcal{I}}$ is polynomial, and we have to prove that it is fair.

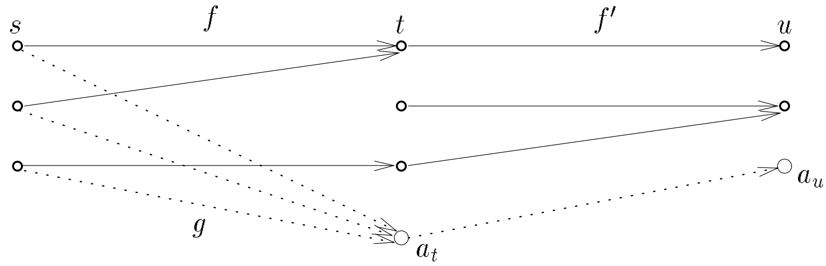

Figure 2: Adding a $g : s \to t$ to $f : s \to t, f' : t \to u$

If $\sigma : \mathcal{I} \cong \mathcal{I}'$, then we extend σ to $\widetilde{\mathcal{I}}(s)$ by: $a_s^\sigma = a'_s$. We have $\widetilde{\mathcal{I}}_g^\sigma = a_t^\sigma = a'_t = \widetilde{\mathcal{I}'}_g$ or $\forall x \in \widetilde{\mathcal{I}}(\mathrm{dom}\, \tau(g)), \widetilde{\mathcal{I}'}_g(x^\sigma) = a'_t = a_t^\sigma = \widetilde{\mathcal{I}}_g(x)^\sigma$. Moreover, $\forall f \in \Sigma$, if $\tau(g) = d_1 \times \ldots \times d_n \to r$, then $\forall \langle x_1, \ldots, x_n \rangle \in \mathcal{I}(d_1 \times \ldots \times d_n), \widetilde{\mathcal{I}'}_f(x_1^\sigma, \ldots, x_n^\sigma) = [\text{if } x_1^\sigma = a'_{d_1}$ or $\ldots$ or $x_n^\sigma = a'_{d_n}$ then a'_r else $\mathcal{I}'_f(x_1^\sigma, \ldots, x_n^\sigma)] = [\text{if } x_1 = a_{d_1}$ or $\ldots$ or $x_n = a_{d_n}$ then a_r^σ else $\mathcal{I}_f(x_1, \ldots, x_n)^\sigma] = \widetilde{\mathcal{I}}_f(x_1, \ldots, x_n)^\sigma$, and obviously $\widetilde{\mathcal{I}}_f^\sigma = \mathcal{I}_f^\sigma = \mathcal{I}'_f = \widetilde{\mathcal{I}'}_f$ if $\tau(f)$ is atomic. The transformation is therefore invariant.

Conversely, let $\sigma : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}}'$, we first prove that $\forall s \in \mathcal{S}_t, a_s^\sigma = a'_s$ by induction on the length of the path form t to s in G_Σ. If this is 0, i.e. $s = t$, we have $\forall x \in \widetilde{\mathcal{I}}(\mathrm{dom}\, \tau(g)), a_t^\sigma = \widetilde{\mathcal{I}}g(x)^\sigma = \widetilde{\mathcal{I}'}_g(x^\sigma) = a'_t$. If this is true of d_i and there is an arrow in G_Σ form d_i to r, i.e. there is a $f \in \Sigma$ with $\tau(f) = d_1 \times \ldots \times d_n \to r$, then $a_r^\sigma = \widetilde{\mathcal{I}}_f(a_{d_1}, \ldots, a_{d_n})^\sigma = \widetilde{\mathcal{I}'}_f(a_{d_1}^\sigma, \ldots, a_{d_n}^\sigma) = a'_r$ since $a_{d_i}^\sigma = a'_{d_i}$. Hence it is clear that $\forall s \in \mathcal{S}$, σ is 1-1 from $\mathcal{I}(s)$ onto $\mathcal{I}'(s)$, and that $\forall f \in \Sigma$ such that $\tau(f)$ is functional, say $d_1 \times \ldots \times d_n \to r$, we have $\forall \langle x_1, \ldots, x_n \rangle \in \mathcal{I}(d_1 \times \ldots \times d_n), \mathcal{I}_f(x_1, \ldots, x_n)^\sigma = \widetilde{\mathcal{I}}_f(x_1, \ldots, x_n)^\sigma = \widetilde{\mathcal{I}'}_f(x_1^\sigma, \ldots, x_n^\sigma) = \mathcal{I}'_f(x_1^\sigma, \ldots, x_n^\sigma)$ since $x_i^\sigma \neq a'_{d_i}$. Hence the transformation is fair. Q.E.D.

COROLLARY 2.4 if $\Sigma \subseteq \Sigma'$ then $\mathrm{Iso}(\Sigma) \propto_P \mathrm{Iso}(\Sigma')$

Proof. If $\Sigma = \langle \mathcal{S}, \mathcal{F}, \tau \rangle$ and $\Sigma' = \langle \mathcal{S}', \mathcal{F}', \tau' \rangle$, let $\Sigma'' = \langle \mathcal{S}', \mathcal{F}, \tau \rangle$, then $\mathrm{Iso}(\Sigma) \propto_P \mathrm{Iso}(\Sigma'')$ by induction with lemma 2.1, and $\mathrm{Iso}(\Sigma'') \propto_P \mathrm{Iso}(\Sigma')$ by induction using lemma 2.3. Q.E.D.

3 Non-monadic signatures

In this section we study the complexity of sorted objects of arity two.

THEOREM 3.1 GI $\propto_P$ Iso($\lceil s \times s \to \mathbf{o} \rceil$) and GI $\propto_P$ Iso($\lceil s \times s' \to \mathbf{o} \rceil$)

Proof. An interpretation $\mathcal{I}$ of $s \times s \to \mathbf{o}$ is a binary relation on $\mathcal{I}(s)$, which is essentially a directed graph with $\mathcal{I}(s)$ as set of vertices. Also, any graph $G = \langle V, E \rangle$ can be considered as an adjacency relation, i.e. an interpretation $\mathcal{I}$ of $R \in \lceil s \times s' \to \mathbf{o} \rceil$ with $\mathcal{I}(s) = V, \mathcal{I}(s') = E$, and $\forall v \in V, \forall e \in E, \mathcal{I}_R(v, e) = \top$ iff $v \in e$. These trivial transformations are fair. Q.E.D.

These two cases will be the base for the five remaining cases of objects of arity two. We begin with the essentially unsorted case.

THEOREM 3.2 Iso($\lceil s \times s \to \mathbf{o} \rceil$) $\propto_P$ Iso($\lceil s \times s \to s \rceil$)

Proof. If $R \in \Sigma, f \in \Sigma'$ with R $\Sigma_R = s \times s \to \mathbf{o}$ and $\Sigma'_f = s \times s \to s$, and given a Σ-interpretation $\mathcal{I}$, we consider two elements which are not in $\mathcal{I}(s)$, say $\mathfrak{t}$ and $\mathfrak{f}$, and we build the Σ'-interpretation $\widetilde{\mathcal{I}}$ by $\widetilde{\mathcal{I}}(s) = \mathcal{I}(s) \uplus \{\mathfrak{t}, \mathfrak{f}\}$ and $\forall x, y \in \mathcal{I}(s), \widetilde{\mathcal{I}}_f(x, y) = \mathfrak{t}$ if $\mathcal{I}_R(x, y) = \top$, and $\mathfrak{f}$ otherwise, $\widetilde{\mathcal{I}}_f(x, \mathfrak{t}) = \widetilde{\mathcal{I}}_f(x, \mathfrak{f}) = \widetilde{\mathcal{I}}_f(\mathfrak{t}, y) = \widetilde{\mathcal{I}}_f(\mathfrak{f}, y) = \widetilde{\mathcal{I}}_f(\mathfrak{t}, \mathfrak{t}) = \widetilde{\mathcal{I}}_f(\mathfrak{f}, \mathfrak{f}) = \mathfrak{t}$ and $\widetilde{\mathcal{I}}_f(\mathfrak{t}, \mathfrak{f}) = \widetilde{\mathcal{I}}_f(\mathfrak{f}, \mathfrak{t}) = \mathfrak{f}$ (see figure 3).

R	a	b
a	$\bot$	$\top$
b	$\bot$	$\bot$

f	a	b	$\mathfrak{t}$	$\mathfrak{f}$
a	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$
b	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$
$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{f}$
$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{t}$

Figure 3: from a $R : s \times s \to \mathbf{o}$ to a $f : s \times s \to s$

This transformation is polynomial and invariant (easy by extending Σ-isomorphisms σ by $\mathfrak{t}^\sigma = \mathfrak{t}'$ and $\mathfrak{f}^\sigma = \mathfrak{f}'$). Suppose now that $\sigma : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}}'$, with $\mathcal{I}, \mathcal{I}'$ two Σ-interpretations. We have $\forall x, y \in \widetilde{\mathcal{I}}(s), \widetilde{\mathcal{I}}_f(x, y)^\sigma = \widetilde{\mathcal{I}}'_f(x^\sigma, y^\sigma) \in \{\mathfrak{t}', \mathfrak{f}'\}$, hence $\{\mathfrak{t}^\sigma, \mathfrak{f}^\sigma\} = \{\mathfrak{t}', \mathfrak{f}'\}$. $\forall z \in \{\mathfrak{t}, \mathfrak{f}\}$, by definition we have $\mathfrak{t}' = \widetilde{\mathcal{I}}'_f(z^\sigma, z^\sigma) = \widetilde{\mathcal{I}}_f(z, z)^\sigma = \mathfrak{t}^\sigma$, and $\mathfrak{f}^\sigma = \mathfrak{f}'$, from which it is easy to conclude that $\sigma : \mathcal{I} \cong \mathcal{I}'$, hence the transformation is fair. Q.E.D.

In the next case, compared with the previous one, we release the constraints by taking one argument of a different sort, which makes things almost easier!

THEOREM 3.3 Iso($\lceil s \times s' \to \mathbf{o} \rceil$) $\propto_P$ Iso($\lceil s \times s' \to s' \rceil$) $\propto_P$ Iso($\lceil s' \times s \to s' \rceil$)

Proof. As in the proof of theorem 3.2, if $\Sigma_R = s \times s' \to \mathbf{o}$ and $\Sigma'_f = s \times s' \to s'$, and given a Σ-interpretation $\mathcal{I}$ we build the Σ'-interpretation $\widetilde{\mathcal{I}}$ by $\widetilde{\mathcal{I}}(s) = \mathcal{I}(s), \widetilde{\mathcal{I}}(s') = \mathcal{I}(s') \uplus \{\mathfrak{t}, \mathfrak{f}\}$, and $\forall \langle x, y \rangle \in \mathcal{I}(s \times s'), \widetilde{\mathcal{I}}_f(x, y) = \mathfrak{t}$ if $\mathcal{I}_R(x, y) = \top$, and $\mathfrak{f}$ otherwise, $\widetilde{\mathcal{I}}_f(x, \mathfrak{t}) = \widetilde{\mathcal{I}}_f(x, \mathfrak{f}) = \mathfrak{t}$ (see figure 4). Invariance is trivial.

R	a'	b'
a	$\bot$	$\top$
b	$\bot$	$\bot$

f	a'	b'	$\mathfrak{t}$	$\mathfrak{f}$
a	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$
b	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$

Figure 4: from a $R : s \times s' \to \mathbf{o}$ to a $f : s \times s' \to s'$

If $\sigma : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}}'$, with $\mathcal{I}, \mathcal{I}'$ two Σ-interpretations, we have $\forall x \in \mathcal{I}(s), \mathfrak{t}^\sigma = \widetilde{\mathcal{I}}_f(x, \mathfrak{t})^\sigma = \widetilde{\mathcal{I}'}_f(x^\sigma, \mathfrak{t}^\sigma) \in \{\mathfrak{t}', \mathfrak{f}'\}$, hence $\mathfrak{t}^\sigma = \widetilde{\mathcal{I}'}_f(x^\sigma, \mathfrak{t}^\sigma) = \mathfrak{t}'$. If $\exists\langle x, y\rangle \in \mathcal{I}(s \times s')$ such that $\mathcal{I}_R(x, y) = \perp$, then $\mathfrak{f}^\sigma = \widetilde{\mathcal{I}'}_f(x^\sigma, y^\sigma) \in \{\mathfrak{t}', \mathfrak{f}'\}$, hence $\mathfrak{f}^\sigma = \mathfrak{f}'$, from which it is easy to prove that $\sigma : \mathcal{I} \cong \mathcal{I}'$. If $\forall\langle x, y\rangle \in \mathcal{I}(s \times s'), \mathcal{I}_R(x, y) = \top$, then $\widetilde{\mathcal{I}'}_f(x^\sigma, y^\sigma) = \mathfrak{t}^\sigma = \mathfrak{t}'$, and hence $\forall\langle x, y\rangle \in \mathcal{I}'(s \times s'), \mathcal{I}'_R(x, y) = \top$, and $\mathcal{I}$ and $\mathcal{I}'$ are also isomorphic. This proves that the transformation is fair. $\mathrm{Iso}(\lceil s \times s' \to s' \rceil) \propto_P \mathrm{Iso}(\lceil s' \times s \to s' \rceil)$ is obvious. Q.E.D.

The next case is a further release of constraints by taking a third sort for the range. This time things get more complex, because the target structure has one more sort than the source, and we have to preclude any unwanted isomorphism on this new sort.

THEOREM 3.4 $\mathrm{Iso}(\lceil s \times s' \to \mathbf{o} \rceil) \propto_P \mathrm{Iso}(\lceil s \times s' \to s'' \rceil)$ and $\mathrm{Iso}(\lceil s \times s \to \mathbf{o} \rceil) \propto_P \mathrm{Iso}(\lceil s \times s \to s'' \rceil)$

Proof. If $R \in \Sigma$, $f \in \Sigma'$ with $\Sigma_R = s \times s' \to \mathbf{o}$ and $\Sigma'_f = s \times s' \to s''$, given a Σ-interpretation $\mathcal{I}$ we build a Σ'-interpretation $\widetilde{\mathcal{I}}$ in the following way. We first consider two sets A, B such that $A, B, \mathcal{I}(s), \mathcal{I}(s')$ are disjoint two by two, and $|A| = |\mathcal{I}(s)| + 1, |B| = |\mathcal{I}(s')| + 1$, and we also consider $\mathfrak{t}, \mathfrak{f}$ as above. Let $\widetilde{\mathcal{I}}(s) = \mathcal{I}(s) \uplus A, \widetilde{\mathcal{I}}(s') = \mathcal{I}(s') \uplus B, \widetilde{\mathcal{I}}(s'') = \{\mathfrak{t}, \mathfrak{f}\}$, and $\forall\langle x, y\rangle \in \widetilde{\mathcal{I}}(s \times s'), \widetilde{\mathcal{I}}_f(x, y) = \mathfrak{t}$ if either $x \in A$ and $y \in B$, or $x \notin A, y \notin B$ and $\mathcal{I}_R(x, y) = \top$; otherwise $\widetilde{\mathcal{I}}_f(x, y) = \mathfrak{f}$ (see figure 5). The transformation from $\mathcal{I}$ to $\widetilde{\mathcal{I}}$ is obviously polynomial and invariant (by extending any Σ-isomorphism $\sigma : \mathcal{I} \cong \mathcal{I}'$ by $\mathfrak{t}^\sigma = \mathfrak{t}', \mathfrak{f}^\sigma = \mathfrak{f}'$, by any bijection from A to A' and from B to B' as well).

R	a'	b'
a	$\perp$	$\top$
b	$\perp$	$\perp$

f	a'	b'	c'	d'	e'
a	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$
b	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}$
c	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$
d	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$
e	$\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{t}$	$\mathfrak{t}$	$\mathfrak{t}$

Figure 5: from a $R : s \times s' \to \mathbf{o}$ to a $f : s \times s' \to s''$, with $A = \{c, d, e\}, B = \{c', d', e'\}$

If $\sigma : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}}'$, we have $\{\mathfrak{t}^\sigma, \mathfrak{f}^\sigma\} = \{\mathfrak{t}', \mathfrak{f}'\}$ as above. Let $n = |\widetilde{\mathcal{I}}(s)|, m = |\widetilde{\mathcal{I}}(s')|$, we can view $\widetilde{\mathcal{I}}_f$ as a (n, m)-matrix; it clearly contains a sub-matrix uniformly equal to $\mathfrak{t}$ (this is $(\widetilde{\mathcal{I}}_f)_{|A \times B}$), hence the (n, m)-matrix $\widetilde{\mathcal{I}'}_f$ contains a $(|A|, |B|)$-matrix uniformly equal to $\mathfrak{t}^\sigma$, and also a $(|A'|, |B'|)$-matrix uniformly equal to $\mathfrak{t}'$. Since $|A'| = |A| > n/2$ and $|B'| = |B| > m/2$, these sub-matrices have to intersect, hence $\mathfrak{t}^\sigma = \mathfrak{t}'$, and $\mathfrak{f}^\sigma = \mathfrak{f}'$ hold.

Suppose there is an $x \in A$ such that $x^\sigma \notin A'$, then $\forall y \in B, \widetilde{\mathcal{I}'}_f(x^\sigma, y^\sigma) = \widetilde{\mathcal{I}}_f(x, y)^\sigma = \mathfrak{t}'$, hence $y^\sigma \notin B'$. Therefore $B^\sigma \cap B' = \emptyset$, hence $B^\sigma \subseteq \mathcal{I}'(s)$, which is impossible since $|B^\sigma| = |B| = |B'| > |\mathcal{I}'(s)|$. We conclude that $\forall x \in A, x^\sigma \in A'$, hence $\mathcal{I}(s)^\sigma = \mathcal{I}'(s)$, and similarly $\mathcal{I}(s')^\sigma = \mathcal{I}'(s')$, hence we easily obtain $\sigma : \mathcal{I} \cong \mathcal{I}'$, which proves that the transformation is fair. This proof holds if $s = s'$ by taking $A = B$. Q.E.D.

COROLLARY 3.5 If Σ is a non-monadic signature, then $\mathrm{Iso}(\Sigma)$ is isomorphism complete.

Proof. Σ contains a f such that Σ_f is not monadic. Let $r = \mathrm{rng}\,\Sigma_f$, s, s' the last two sorts in $\mathrm{dom}\,\Sigma_f$ (we may have $s = s'$), and $t = s \times s' \to r$, by successive applications of theorem 2.2 we obtain $\mathrm{Iso}(\lceil t \rceil) \propto_P \mathrm{Iso}(\lceil \Sigma_f \rceil)$. By corollary 2.4, we also have $\mathrm{Iso}(\lceil \Sigma_f \rceil) \propto_P \mathrm{Iso}(\Sigma)$.

If $r = \mathbf{o}$, theorem 3.1 yields $\mathrm{GI} \propto_P \mathrm{Iso}(\lceil t \rceil)$. If r is a sort, we have three different cases. If $r \notin \{s, s'\}$, we also use theorem 3.4 to get $\mathrm{GI} \propto_P \mathrm{Iso}(\lceil t \rceil)$, if $r = s = s'$, we use theorem

3.2, and if $r \in \{s, s'\}$ with $s \neq s'$, we use theorem 3.3 to get the same result. We therefore have GI $\propto_P$ Iso($\lceil t \rceil$) $\propto_P$ Iso($\lceil \Sigma_f \rceil$) $\propto_P$ Iso(Σ) $\propto_P$ GI (this last fact is well-known, see e.g. [3] [4]). Q.E.D.

4 hard monadic signatures

In this section and the next we only consider monadic signatures. From now on, the term "monadic functions" refers to function symbols which are not predicate symbols. We will prove that the complexity of binary relations can be simulated by pairs of well chosen monadic functions. The criterion for a pair of functions to have this property is purely syntactic: they should have the same domain sort. In graph theoretic language, this means that this domain sort s, as a vertex of G_Σ, has an output degree (number of edges out of s, noted $\mathrm{d}^+(s)$) at least 2. We start with the case where these monadic functions have different domain and range sorts.

THEOREM 4.1 Iso($\lceil s' \times s'' \to \mathbf{o} \rceil$) $\propto_P$ Iso($\lceil s \to s', s \to s'' \rceil$) and Iso($\lceil s' \times s' \to \mathbf{o} \rceil$) $\propto_P$ Iso($\lceil s \to s', s \to s' \rceil$)

Proof. Let Σ be the signature with the unique symbol R and $\Sigma_R = s' \times s'' \to \mathbf{o}$, and Σ' with only the symbols f, g and $\Sigma'_f = s \to s'$ and $\Sigma'_g = s \to s''$. To any Σ-interpretation $\mathcal{I}$ we associate the Σ'-interpretation $\widetilde{\mathcal{I}}$ defined by: $\widetilde{\mathcal{I}}(s') = \mathcal{I}(s'), \widetilde{\mathcal{I}}(s'') = \mathcal{I}(s''), \widetilde{\mathcal{I}}(s) = \{\langle x, y\rangle \in \mathcal{I}(s' \times s'')/\mathcal{I}_R(x, y) = \top\}$ and $\widetilde{\mathcal{I}}_f(\langle x, y\rangle) = x, \widetilde{\mathcal{I}}_g(\langle x, y\rangle) = y$ (see figure 6).

R	a'	b'
a	$\perp$	$\top$
b	$\top$	$\top$

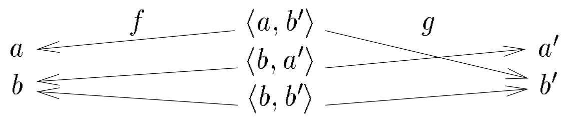

Figure 6: from a $R : s' \times s'' \to \mathbf{o}$ to a $f : s \to s', g : s \to s''$

If $\sigma : \mathcal{I} \cong \mathcal{I}'$, we extend σ to all $\langle x, y\rangle \in \mathcal{I}(s)$ by $\langle x, y\rangle^\sigma = \langle x^\sigma, y^\sigma\rangle$. Since $\forall\langle x, y\rangle \in \mathcal{I}(s' \times s'')$, we have $\langle x, y\rangle \in \widetilde{\mathcal{I}}(s)$ iff $\mathcal{I}_R(x, y) = \top$ iff $\mathcal{I}'_R(x^\sigma, y^\sigma) = \top$ iff $\langle x^\sigma, y^\sigma\rangle = \langle x, y\rangle^\sigma \in \widetilde{\mathcal{I}}'(s)$, then σ is clearly 1-1 from $\widetilde{\mathcal{I}}(s)$ onto $\widetilde{\mathcal{I}}'(s)$. We also have $\forall\langle x, y\rangle \in \widetilde{\mathcal{I}}(s), \widetilde{\mathcal{I}}_f(\langle x, y\rangle)^\sigma = x^\sigma = \widetilde{\mathcal{I}}'_f(\langle x, y\rangle^\sigma)$, and similarly for g, hence $\sigma : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}}'$.

If $\sigma : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}}'$, then $\forall\langle x, y\rangle \in \widetilde{\mathcal{I}}(s)$, we have $\widetilde{\mathcal{I}}'_f(\langle x, y\rangle^\sigma) = \widetilde{\mathcal{I}}_f(\langle x, y\rangle)^\sigma = x^\sigma$, and $\widetilde{\mathcal{I}}'_g(\langle x, y\rangle^\sigma) = y^\sigma$, hence $\langle x, y\rangle^\sigma = \langle x^\sigma, y^\sigma\rangle$. Then $\forall\langle x, y\rangle \in \mathcal{I}(s' \times s'')$, we have $\mathcal{I}_R(x, y) = \top$ iff $\langle x, y\rangle \in \widetilde{\mathcal{I}}(s)$ iff $\langle x, y\rangle^\sigma \in \widetilde{\mathcal{I}}'(s)$ iff $\mathcal{I}'_R(x^\sigma, y^\sigma) = \top$, hence $\sigma : \mathcal{I} \cong \mathcal{I}'$. The transformation is therefore fair, and it is trivially polynomial. This proof holds if $s' = s''$. Q.E.D.

We now turn to the case where monadic functions have the same domain and range sort, which is more difficult than the previous one since we somehow have to "mix" in one set both the domain and the range of a function.

THEOREM 4.2 Iso($\lceil s \times s \to \mathbf{o} \rceil$) $\propto_P$ Iso($\lceil s' \to s', s' \to s'' \rceil$) and Iso($\lceil s \times s \to \mathbf{o} \rceil$) $\propto_P$ Iso($\lceil s' \to s', s' \to s' \rceil$)

Proof. Let Σ be the signature with the unique symbol R and $\Sigma_R = s \times s \to \mathbf{o}$, and Σ' with only the symbols f, g and $\Sigma'_f = s' \to s'$ and $\Sigma'_g = s' \to s''$. To any Σ-interpretation

$\mathcal{I}$ we associate the Σ'-interpretation $\widetilde{\mathcal{I}}$ defined by:

$$\widetilde{\mathcal{I}}(s') = \{\langle x, s'\rangle / x \in \mathcal{I}(s)\} \uplus \{\langle x, y, s'\rangle / x, y \in \mathcal{I}(s), \mathcal{I}_R(x,y) = \top\}$$

$$\widetilde{\mathcal{I}}(s'') = \{\langle x, s''\rangle / x \in \mathcal{I}(s)\} \uplus \{\langle x, y, s''\rangle / x, y \in \mathcal{I}(s), \mathcal{I}_R(x,y) = \top\}$$

and $\widetilde{\mathcal{I}}_f(\langle x, s'\rangle) = \langle x, s'\rangle, \widetilde{\mathcal{I}}_f(\langle x, y, s'\rangle) = \langle x, s'\rangle, \widetilde{\mathcal{I}}_g(\langle x, s'\rangle) = \langle x, s''\rangle, \widetilde{\mathcal{I}}_g(\langle x, y, s'\rangle) = \langle y, s''\rangle$ (see figure 7). Remark that $\widetilde{\mathcal{I}}(s') \cap \widetilde{\mathcal{I}}(s'') = \emptyset$ and $s' = s'' \Rightarrow \widetilde{\mathcal{I}}(s') = \widetilde{\mathcal{I}}(s'')$.

R	a	b
a	$\bot$	$\top$
b	$\top$	$\top$

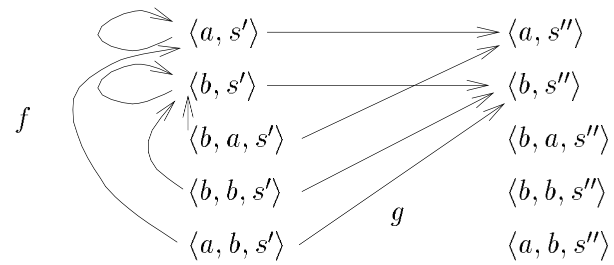

Figure 7: from a $R : s \times s \to \mathbf{o}$ to a $f : s' \to s', g : s' \to s''$

If $\sigma : \mathcal{I} \cong \mathcal{I}'$, we consider the function α from $\widetilde{\mathcal{I}}(s')$ to $\widetilde{\mathcal{I}'}(s')$ and from $\widetilde{\mathcal{I}}(s'')$ to $\widetilde{\mathcal{I}'}(s'')$ defined by $\langle x, s'\rangle^\alpha = \langle x^\sigma, s'\rangle, \langle x, y, s'\rangle^\alpha = \langle x^\sigma, y^\sigma, s'\rangle, \langle x, s''\rangle^\alpha = \langle x^\sigma, s''\rangle, \langle x, y, s''\rangle^\alpha = \langle x^\sigma, y^\sigma, s''\rangle$. Since $\forall x, y \in \mathcal{I}(s)$, we have $\langle x, y, s'\rangle \in \widetilde{\mathcal{I}}(s')$ iff $\mathcal{I}_R(x,y) = \top$ iff $\mathcal{I}'_R(x^\sigma, y^\sigma) = \top$ iff $\langle x^\sigma, y^\sigma, s'\rangle = \langle x, y, s'\rangle^\alpha \in \widetilde{\mathcal{I}'}(s')$, and $\langle x, s'\rangle \in \widetilde{\mathcal{I}}(s')$ iff $x \in \mathcal{I}(s)$ iff $x^\sigma \in \mathcal{I}'(s)$ iff $\langle x^\sigma, s'\rangle = \langle x, s'\rangle^\alpha \in \widetilde{\mathcal{I}'}(s')$, then α is 1-1 from $\widetilde{\mathcal{I}}(s')$ onto $\widetilde{\mathcal{I}'}(s')$, and similarly 1-1 from $\widetilde{\mathcal{I}}(s'')$ onto $\widetilde{\mathcal{I}'}(s'')$. The conclusion $\alpha : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}'}$ comes from:

$$\widetilde{\mathcal{I}}_f(\langle x, s'\rangle)^\alpha = \langle x, s'\rangle^\alpha = \langle x^\sigma, s'\rangle = \widetilde{\mathcal{I}'}_f(\langle x^\sigma, s'\rangle) = \widetilde{\mathcal{I}'}_f(\langle x, s'\rangle^\alpha)$$

$$\widetilde{\mathcal{I}}_f(\langle x, y, s'\rangle)^\alpha = \langle x, s'\rangle^\alpha = \langle x^\sigma, s'\rangle = \widetilde{\mathcal{I}'}_f(\langle x^\sigma, y^\sigma, s'\rangle) = \widetilde{\mathcal{I}'}_f(\langle x, y, s'\rangle^\alpha)$$

$$\widetilde{\mathcal{I}}_g(\langle x, s'\rangle)^\alpha = \langle x, s''\rangle^\alpha = \langle x^\sigma, s''\rangle = \widetilde{\mathcal{I}'}_g(\langle x^\sigma, s'\rangle) = \widetilde{\mathcal{I}'}_g(\langle x, s'\rangle^\alpha)$$

$$\widetilde{\mathcal{I}}_g(\langle x, y, s'\rangle)^\alpha = \langle y, s''\rangle^\alpha = \langle y^\sigma, s''\rangle = \widetilde{\mathcal{I}'}_g(\langle x^\sigma, y^\sigma, s'\rangle) = \widetilde{\mathcal{I}'}_g(\langle x, y, s'\rangle^\alpha)$$

If $\alpha : \widetilde{\mathcal{I}} \cong \widetilde{\mathcal{I}'}$, then $\forall x \in \mathcal{I}(s), \widetilde{\mathcal{I}'}_f(\langle x, s'\rangle^\alpha) = \widetilde{\mathcal{I}}_f(\langle x, s'\rangle)^\alpha = \langle x, s'\rangle^\alpha$, i.e. $\langle x, s'\rangle^\alpha$ is a fix point of $\widetilde{\mathcal{I}'}_f$, hence is of the form $\langle y, s'\rangle$, with $y \in \mathcal{I}'(s)$, and this y is unique (for α is 1-1), we note it x^σ. We also have $\forall x, y \in \mathcal{I}(s), \widetilde{\mathcal{I}'}_f(\langle x, y, s'\rangle^\alpha) = \widetilde{\mathcal{I}}_f(\langle x, y, s'\rangle)^\alpha = \langle x, s'\rangle^\alpha$, hence $\langle x, y, s'\rangle^\alpha$ is not a fixpoint of $\widetilde{\mathcal{I}'}_f$, and should be of the form $\langle x', y', s'\rangle$. Since α is 1-1 from $\widetilde{\mathcal{I}}(s')$ onto $\widetilde{\mathcal{I}'}(s')$, it is therefore also 1-1 form $\{\langle x, s'\rangle / x \in \mathcal{I}(s)\}$ onto $\{\langle y, s'\rangle / y \in \mathcal{I}'(s)\}$, hence σ is also 1-1 from $\mathcal{I}(s)$ onto $\mathcal{I}'(s)$. Moreover, we have $\langle x^\sigma, s''\rangle = \widetilde{\mathcal{I}'}_g(\langle x^\sigma, s''\rangle) = \widetilde{\mathcal{I}'}_g(\langle x, s''\rangle^\alpha) = \widetilde{\mathcal{I}}_g(\langle x, s''\rangle)^\alpha = \langle x, s''\rangle^\alpha$. As noted above, we have $\widetilde{\mathcal{I}'}_f(\langle x, y, s'\rangle^\alpha) = \langle x, s'\rangle^\alpha = \langle x^\sigma, s'\rangle$ and similarly $\widetilde{\mathcal{I}'}_g(\langle x, y, s'\rangle^\alpha) = \langle y, s''\rangle^\alpha = \langle y^\sigma, s''\rangle$, hence $\langle x, y, s'\rangle^\alpha = \langle x^\sigma, y^\sigma, s'\rangle$. We conclude that $\forall x, y \in \mathcal{I}(s), \mathcal{I}(x,y) = \top$ iff $\langle x, y, s'\rangle \in \widetilde{\mathcal{I}}(s')$ iff $\langle x, y, s'\rangle^\alpha = \langle x^\sigma, y^\sigma, s'\rangle \in \widetilde{\mathcal{I}'}(s')$ iff $\mathcal{I}'(x^\sigma, y^\sigma) = \top$, hence that $\sigma : \mathcal{I} \cong \mathcal{I}'$. Hence the transformation is fair, and trivially polynomial. This proof holds if $s' = s''$. Q.E.D.

Corollary 4.3 If Σ is a monadic signature such that $\mathrm{d}^+(G_\Sigma) > 1$ then $\mathrm{Iso}(\Sigma)$ is isomorphism complete.

Proof. If $\mathrm{d}^+(G_\Sigma) > 1$, then $\exists s \in \mathcal{S}, \exists f, g \in \Sigma$ such that $\mathrm{dom}\, f = \mathrm{dom}\, g = s$. If $\mathrm{rng}\, f = s$ or $\mathrm{rng}\, g = s$, we use theorem 4.2, otherwise theorem 4.1, to get $\mathrm{GI} \propto_P \mathrm{Iso}([\tau(f), \tau(g)])$ (together with theorem 3.1). We then proceed as in corollary 3.5. Q.E.D.

5 easy monadic signatures

We now prove that the isomorphism problem for all other signatures, i.e. monadic such that G_Σ has output degree at most one, is polynomial. We first provide the simplest possible representation of the corresponding interpretations.

Definition 5.1. A graph $G = \langle V, E\rangle$ is a *partial function graph* (or PFG), if E is the graph of a partial function from V to V. A *labeled* PFG (or LPFG), is a labeled graph whose underlying graph is a PFG. The isomorphism problem between LPFG's is noted LPFGI. ∇

Lemma 5.1 If Σ is monadic and $\mathrm{d}^+(G_\Sigma) \leq 1$ then $\mathrm{Iso}(\Sigma) \propto_P$ *LPFGI*

Proof. We transform Σ-interpretations $\mathcal{I}$ into graphs. $\forall s \in \mathcal{S}$, let $\mathcal{P}_s$ be the set of predicate symbols of type $s \to \mathbf{o}$ in Σ, and $\forall x \in \mathcal{I}(s)$, let $\mathcal{P}_\mathcal{I}(x) = \{P \in \mathcal{P}_s / \mathcal{I}_P(x) = \top\}$, and $\mathcal{C}_\mathcal{I}(x)$ be the set of constant symbols $c \in \Sigma$ such that $\mathcal{I}_c = x$, then we consider the vertex $v(s,x)$ labeled by $\langle s, \mathcal{P}_\mathcal{I}(x), \mathcal{C}_\mathcal{I}(x)\rangle$. Next, for every function symbol $f \in \Sigma$, say $\Sigma_f = s \to s'$ (with possibly $s = s'$), and every $x \in \mathcal{I}(s)$ we consider the edge $\langle v(s,x), v(s', \mathcal{I}_f(x))\rangle$. Remark that $\forall s \in \mathcal{S}, \forall x \in \mathcal{I}(s)$, there is at most one $f \in \Sigma$ such that $\mathrm{dom}\, f = s$, hence there is at most one edge out of $v(s,x)$. Hence the graph $F_\mathcal{I}$ thus defined is a PFG. The transformation from $\mathcal{I}$ to $F_\mathcal{I}$ is polynomial, we prove that it is fair.

If $\sigma : \mathcal{I} \cong \mathcal{I}'$, let α defined by $\forall s \in \mathcal{S}, \forall x \in \mathcal{I}(s), v(s,x)^\alpha = v'(s, x^\sigma)$, it preserves labels iff $\mathcal{P}_\mathcal{I}(x) = \mathcal{P}_{\mathcal{I}'}(x^\sigma)$ and $\mathcal{C}_\mathcal{I}(x) = \mathcal{C}_{\mathcal{I}'}(x^\sigma)$, which is obvious since $\forall P \in \mathcal{P}_s, \mathcal{I}'_P(x) = \mathcal{I}_P(x^\sigma)$ and $\forall c \in \Sigma, c \in \mathcal{C}_\mathcal{I}(x)$ iff $\mathcal{I}_c = x$ iff $\mathcal{I}'_c = \mathcal{I}^\sigma_c = x^\sigma$ iff $c \in \mathcal{C}_{\mathcal{I}'}(x^\sigma)$. Edges are also preserved by α, since $\langle v(s,x), v(s', \mathcal{I}_f(x))\rangle^\alpha = \langle v(s, x^\sigma), v(s', \mathcal{I}'_f(x^\sigma))\rangle$ is an edge of $F_{\mathcal{I}'}$, hence $\alpha : F_\mathcal{I} \cong F_{\mathcal{I}'}$.

Conversely, if $\alpha : F_\mathcal{I} \cong F_{\mathcal{I}'}$, then $\forall s \in \mathcal{S}, \forall x \in \mathcal{I}(s)$, by the preservation of labels there is a unique $y \in \mathcal{I}'(s)$ such that $v(s,x)^\alpha = v'(s,y)$, and we note it x^σ. For any $f \in \sigma$, say $\Sigma_f = s \to s'$, then $\forall x \in \mathcal{I}(s)$, the unique edge out of $v(s,x)^\alpha$ should be the image of the unique edge out of $v(s,x)$, i.e. $\langle v(s,x), v(s', \mathcal{I}_f(x))\rangle^\alpha = \langle v'(s, x^\sigma), v'(s', \mathcal{I}'_f(x^\sigma))\rangle$, hence $\mathcal{I}_f(x)^\sigma = \mathcal{I}'_f(x^\sigma)$. Moreover, for any $P \in \Sigma$, say $\Sigma_P = s \to \mathbf{o}$, then $\forall x \in \mathcal{I}(s)$, we have $\mathcal{I}_P(x) = \top$ iff $P \in \mathcal{P}_\mathcal{I}(x)$, part of the label of $v(s,x)$, iff (by the preservation of labels) $P \in \mathcal{P}_{\mathcal{I}'}(x^\sigma)$, part of the label of $v'(s, x^\sigma)$, iff $\mathcal{I}'_P(x^\sigma) = \top$. Similarly, for any $c \in \Sigma$, let $x = \mathcal{I}_c$ and $s = \Sigma_c$, we have $c \in \mathcal{C}_\mathcal{I}(x)$, part of the label of $v(s,x)$, hence $c \in \mathcal{C}_{\mathcal{I}'}(x^\sigma)$, part of the label of $v'(s, x^\sigma)$, hence $\mathcal{I}'_c = x^\sigma = \mathcal{I}^\sigma_c$. Hence $\sigma : \mathcal{I} \cong \mathcal{I}'$. Q.E.D.

Remark that not all LPFG's correspond to Σ-interpretations, since the structure of labels is a special one. The following proof analyses the structure of PFG's, hence gives good insight into the structure of "easy" interpretations.

Lemma 5.2 The problem *LPFGI* is polynomial.

Proof. Since testing the isomorphism of two graphs with n connex components each requires $\mathrm{O}(n^2)$ tests of isomorphisms between connex components, we may only consider connex LPFG's. In such a graph $G = \langle V, E\rangle$, there is at least one undirected path between two vertices v_1, v_2. If $\mathrm{d}^+(v_1) = \mathrm{d}^+(v_2) = 0$, then such a path must contain a third vertex v with $\mathrm{d}^+(v) \geq 2$, which is impossible. Hence there is at most one vertex r with $\mathrm{d}^+(r) = 0$. If there is such a r, then the number of vertices exceeds the number of edges by one, hence G is a tree, with edges directed to the root r.

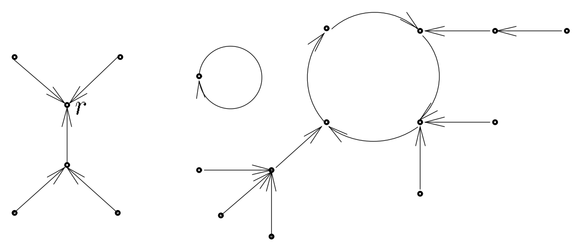

Figure 8: example of a PFG

If there is no root in G, i.e. $\forall v \in V, \mathrm{d}^+(v) = 1$. Let $v_0 \in V$, and $\forall i \in \mathbb{N}, v_{i+1}$ is the unique vertex such that $\langle v_i, v_{i+1}\rangle \in E$. Since V is finite, $\exists i, j, i < j$ and $v_i = v_j$, hence G contains a cycle, of length $c = j - i$. By removing one edge from the cycle we obtain a connex LPFG with a root, hence a tree, which proves that G is a cycle of trees (figure 8).

It is clear that testing the isomorphism of two cycles of c labeled trees requires at most $\mathrm{O}(c^2)$ tests of isomorphism between labeled trees, well-known to be polynomial. Q.E.D.

COROLLARY 5.3 If Σ is monadic and $\mathrm{d}^+(G_\Sigma) \leq 1$ then Iso(Σ) is polynomial.

Therefore, if we agree that GI is not polynomial, we get the result that Iso(Σ) is not isomorphism complete only in the case that Σ is monadic and no two functions have the same domain sort. Monadic predicates have no influence on Iso(Σ).

If we translate this result to standard first order signatures (without sorts), which is equivalent to the sorted case with $|\mathcal{S}| = 1$, we get that Iso(Σ) is not isomorphism complete exactly when Σ is monadic *and* has at most one function symbol. In comparison, the sorted case has a much richer structure, since polynomial cases are obtained with any monadic Σ such that G_Σ is a PFG, and any PFG can be obtained as a G_Σ (more than once since atomic objects and unary predicates are not represented in G_Σ). However, the PFG underlying a Σ-interpretation $\mathcal{I}$ may not be any PFG, and is closely dependent on G_Σ. For instance, $F_\mathcal{I}$ may contain trees as connex components iff this is also the case of G_Σ. Hence our embedding of easy interpretations into LPFG, though fair, is not an exact one.

References

[1] M. Garey and D. S. Johnson. *Computers and intractability: a guide to the theory of* **NP**-*completeness*. Freeman, San Francisco, California, 1979.

[2] C. Hoffmann. *Group-theoretic algorithms and graph isomorphism*. Lecture Notes in Computer Science 136. Springer Verlag, 1981.

[3] Dexter Kozen. Complexity of finitely presented algebras. In *Conference Record of the Ninth Annual ACM Symposium on Theory of Computing*, pages 164–177, Boulder, Colorado, 2–4 May 1977.

[4] Gary L. Miller. Graph isomorphism, general remarks. *Journal of Computer and System Sciences*, 18:128–142, 1979.